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Several Integrability Formulas of Special Functions

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Summary. In this article, we give several integrability formulas of special and composite functions including trigonometric function, inverse trigonometric function, hyperbolic function and logarithmic function.

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The notation and terminology used here are introduced in the following papers: [21], [20], [7], [12], [6], [25], [3], [8], [26], [24], [5], [22], [18], [19], [17], [9], [16], [11], [14], [1], [15], [23], [13], [10], [2], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: f, f_1, f_2, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, A denotes a closed-interval subset of $\mathbb{R}$, r, x, x_0 denote real numbers, n denotes an element of $\mathbb{N}$, and Z denotes an open subset of $\mathbb{R}$.

The following propositions are true:

(1) $\sin(x + 2 \cdot n \cdot \pi) = \sin x$.

(2) $\sin(x + (2 \cdot n + 1) \cdot \pi) = -\sin x$.

(3) $\cos(x + 2 \cdot n \cdot \pi) = \cos x$.

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(4) $\cos(x + (2 \cdot n + 1) \cdot \pi) = -\cos x$.

(5) If $\sin(\frac{x}{2}) \geq 0$, then $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos x}{2}}$.

(6) If $\sin(\frac{x}{2}) < 0$, then $\sin(\frac{x}{2}) = -\sqrt{\frac{1-\cos x}{2}}$.

(7) $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

(8) $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$.

(9) $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \subseteq]-1, 1[$.

(10) $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

(11) $\arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$.

(12) If $\cos(\frac{x}{2}) \geq 0$, then $\cos(\frac{x}{2}) = \sqrt{\frac{1+\cos x}{2}}$.

(13) $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

(14) $\cos(\frac{3 \cdot \pi}{4}) = -\frac{\sqrt{2}}{2}$.

(15) $\arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

(16) $\arccos(-\frac{\sqrt{2}}{2}) = \frac{3 \cdot \pi}{4}$.

(17) (The function sinh)(1) $= \frac{e^{\mathbf{2}}-1}{2 \cdot e}$.

(18) (The function cosh)(0) $= 1$.

(19) (The function cosh)(1) $= \frac{e^{\mathbf{2}}+1}{2 \cdot e}$.

(20) For every linear function L_1 holds $-L_1$ is a linear function.

(21) For every rest R_1 holds $-R_1$ is a rest.

(22) For all f_1, x_0 such that f_1 is differentiable in x_0 holds $-f_1$ is differentiable in x_0 and $(-f_1)'(x_0) = -f_1{}'(x_0)$.

(23) Let given f_1, Z. Suppose $Z \subseteq \mathrm{dom}(-f_1)$ and f_1 is differentiable on Z. Then $-f_1$ is differentiable on Z and for every x such that $x \in Z$ holds $(-f_1)'_{\restriction Z}(x) = -f_1{}'(x)$.

(24) $-$the function sin is differentiable on $\mathbb{R}$.

(25) $-$the function cos is differentiable in x and ($-$the function cos)$'(x)$ = (the function sin)(x).

(26)(i) $-$the function cos is differentiable on $\mathbb{R}$, and

(ii) for every x such that $x \in \mathbb{R}$ holds ($-$the function cos)$'(x)$ = (the function sin)(x).

(27) (The function sin)$'_{\restriction \mathbb{R}}$ = the function cos.

(28) (The function cos)$'_{\restriction \mathbb{R}}$ = $-$the function sin.

(29) ($-$the function cos)$'_{\restriction \mathbb{R}}$ = the function sin.

(30) (The function sinh)$'_{\restriction \mathbb{R}}$ = the function cosh.

(31) (The function cosh)$'_{\restriction \mathbb{R}}$ = the function sinh.

(32) (The function exp)$'_{\restriction \mathbb{R}}$ = the function exp.

(33) Suppose $Z \subseteq \mathrm{dom}\,(\text{the function tan})$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$ and $(\text{the function cos})(x) \neq 0$. Then

(i) the function tan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\text{the function tan})'_{\restriction Z}(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(34) Suppose that

(i) $Z \subseteq \mathrm{dom}\,(\text{the function cot})$, and

(ii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$ and $(\text{the function sin})(x) \neq 0$.

Then

(iii) the function cot is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds $(\text{the function cot})'_{\restriction Z}(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(35) For every real number r holds $\mathrm{dom}(\mathbb{R} \longmapsto r) = \mathbb{R}$ and $\mathrm{rng}(\mathbb{R} \longmapsto r) \subseteq \mathbb{R}$.

Let r be a real number. The functor $\mathrm{Cst}\,r$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined as follows:

(Def. 1) $\mathrm{Cst}\,r = \mathbb{R} \longmapsto r$.

We now state two propositions:

(36) For all real numbers a, b and for every closed-interval subset A of $\mathbb{R}$ holds $\chi_{A,A} = \mathrm{Cst}\,1{\restriction}A$.

(37) For all real numbers a, b and for every closed-interval subset A of $\mathbb{R}$ such that $A = [a, b]$ holds $\sup A = b$ and $\inf A = a$.

2. Several Integrability Formulas of Special Functions

The following propositions are true:

(38) For all real numbers a, b such that $a \leq b$ holds $\int\limits_a^b \mathrm{Cst}\,1(x)dx = b - a$.

(39) $\int\limits_A (\text{the function cos})(x)dx = (\text{the function sin})(\sup A) - (\text{the function sin})(\inf A)$.

(40) If $A = [0, \frac{\pi}{2}]$, then $\int\limits_A (\text{the function cos})(x)dx = 1$.

(41) If $A = [0, \pi]$, then $\int\limits_A (\text{the function cos})(x)dx = 0$.

(42) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int\limits_A (\text{the function cos})(x)dx = -1$.

(43) If $A=[0,\pi\cdot 2]$, then $\int\limits_A$ (the function cos)$(x)dx=0$.

(44) If $A=[2\cdot n\cdot\pi,(2\cdot n+1)\cdot\pi]$, then $\int\limits_A$ (the function cos)$(x)dx=0$.

(45) If $A=[x+2\cdot n\cdot\pi,x+(2\cdot n+1)\cdot\pi]$, then $\int\limits_A$ (the function cos)$(x)dx=$ $-2\cdot\sin x$.

(46) $\int\limits_A$(−the function sin)$(x)dx=$ (the function cos)$(\sup A)-$ (the function cos)$(\inf A)$.

(47) If $A=[0,\frac{\pi}{2}]$, then $\int\limits_A$(−the function sin)$(x)dx=-1$.

(48) If $A=[0,\pi]$, then $\int\limits_A$(−the function sin)$(x)dx=-2$.

(49) If $A=[0,\frac{\pi\cdot 3}{2}]$, then $\int\limits_A$(−the function sin)$(x)dx=-1$.

(50) If $A=[0,\pi\cdot 2]$, then $\int\limits_A$(−the function sin)$(x)dx=0$.

(51) If $A=[2\cdot n\cdot\pi,(2\cdot n+1)\cdot\pi]$, then $\int\limits_A$(−the function sin)$(x)dx=-2$.

(52) If $A=[x+2\cdot n\cdot\pi,x+(2\cdot n+1)\cdot\pi]$, then $\int\limits_A$(−the function sin)$(x)dx=$ $-2\cdot\cos x$.

(53) $\int\limits_A$ (the function exp)$(x)dx=$ (the function exp)$(\sup A)-$ (the function exp)$(\inf A)$.

(54) If $A=[0,1]$, then $\int\limits_A$ (the function exp)$(x)dx=e-1$.

(55) $\int\limits_A$ (the function sinh)$(x)dx=$ (the function cosh)$(\sup A)-$ (the function cosh)$(\inf A)$.

(56) If $A=[0,1]$, then $\int\limits_A$ (the function sinh)$(x)dx=\frac{(e-1)^{\mathbf{2}}}{2\cdot e}$.

(57) $\int\limits_A$ (the function cosh)$(x)dx=$ (the function sinh)$(\sup A)-$ (the function

sinh)(inf A).

(58) If $A = [0,1]$, then $\int\limits_A$ (the function cosh)$(x)dx = \frac{e^{\mathbf{2}}-1}{2\cdot e}$.

(59) Suppose that

(i) $A \subseteq Z$,

(ii) dom (the function tan) $= Z$,

(iii) dom (the function tan) $=$ dom f_2,

(iv) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$ and (the function cos)$(x) \neq 0$, and

(v) f_2 is continuous on A.

Then $\int\limits_A f_2(x)dx =$ (the function tan)(sup A) $-$ (the function tan)(inf A).

(60) Suppose that

(i) $A \subseteq Z$,

(ii) dom (the function cot) $= Z$,

(iii) dom (the function cot) $=$ dom f_2,

(iv) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$ and (the function sin)$(x) \neq 0$, and

(v) f_2 is continuous on A.

Then $\int\limits_A f_2(x)dx =$ (the function cot)(sup A) $-$ (the function cot)(inf A).

(61) Suppose dom (the function tanh) $=$ dom f_2 and for every x such that $x \in \mathbb{R}$ holds $f_2(x) = \frac{1}{(\text{the function cosh})(x)^{\mathbf{2}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function tanh)(sup A) $-$ (the function tanh)(inf A).

(62) Suppose $A \subseteq]-1,1[$ and dom((the function arcsin)$'_{\restriction]-1,1[}$) $=$ dom f_2 and for every x holds $x \in]-1,1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^{\mathbf{2}}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function arcsin)(sup A) $-$ (the function arcsin)(inf A).

(63) Suppose $A \subseteq]-1,1[$ and dom((the function arccos)$'_{\restriction]-1,1[}$) $=$ dom f_2 and for every x holds $x \in]-1,1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^{\mathbf{2}}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function arccos)(sup A) $-$ (the function arccos)(inf A).

(64) Suppose that

(i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,

(ii) dom((the function arcsin)$'_{\restriction]-1,1[}$) $=$ dom f_2,

(iii) for every x holds $x \in]-1,1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^2}}$, and

(iv) f_2 is continuous on A.

Then $\int_A f_2(x)dx = \frac{\pi}{2}$.

(65) Suppose that

(i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,

(ii) dom((the function arccos)$'_{\restriction]-1,1[}$) = dom f_2,

(iii) for every x holds $x \in]-1,1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^2}}$, and

(iv) f_2 is continuous on A.

Then $\int_A f_2(x)dx = -\frac{\pi}{2}$.

(66) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A and $g'_{\restriction Z}$ is integrable on A and $g'_{\restriction Z}$ is bounded on A. Then $\int_A (f'_{\restriction Z} + g'_{\restriction Z})(x)dx = ((f(\sup A) - f(\inf A)) + g(\sup A)) - g(\inf A)$.

(67) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A and $g'_{\restriction Z}$ is integrable on A and $g'_{\restriction Z}$ is bounded on A. Then $\int_A (f'_{\restriction Z} - g'_{\restriction Z})(x)dx = f(\sup A) - f(\inf A) - (g(\sup A) - g(\inf A))$.

(68) Suppose f is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A. Then $\int_A (r\, f'_{\restriction Z})(x)dx = r \cdot f(\sup A) - r \cdot f(\inf A)$.

(69) $\int_A$((the function sin) + (the function cos))$(x)dx$ = (((−the function cos) $(\sup A)$ − (−the function cos)$(\inf A)$) + (the function sin)$(\sup A)$) − (the function sin)$(\inf A)$.

(70) If $A = [0, \frac{\pi}{2}]$, then $\int_A$((the function sin) + (the function cos))$(x)dx = 2$.

(71) If $A = [0, \pi]$, then $\int_A$((the function sin) + (the function cos))$(x)dx = 2$.

(72) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int_A$((the function sin) + (the function cos))$(x)dx = 0$.

(73) If $A = [0, \pi \cdot 2]$, then $\int_A$((the function sin) + (the function cos))$(x)dx =$

0.

(74) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2$.

(75) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2 \cdot \cos x - 2 \cdot \sin x$.

(76) $\int_A ((\text{the function sinh}) + (\text{the function cosh}))(x)dx = (((\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A)) + (\text{the function sinh})(\sup A)) - (\text{the function sinh})(\inf A)$.

(77) If $A = [0, 1]$, then $\int_A ((\text{the function sinh}) + (\text{the function cosh}))(x)dx = e - 1$.

(78) $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = (-\text{the function cos})(\sup A) - (-\text{the function cos})(\inf A) - ((\text{the function sin})(\sup A) - (\text{the function sin})(\inf A))$.

(79) If $A = [0, \frac{\pi}{2}]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 0$.

(80) If $A = [0, \pi]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2$.

(81) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2$.

(82) If $A = [0, \pi \cdot 2]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 0$.

(83) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2$.

(84) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2 \cdot \cos x + 2 \cdot \sin x$.

(85) $\int_A (r\,(\text{the function sin}))(x)dx = r \cdot (-\text{the function cos})(\sup A) - r \cdot (-\text{the function cos})(\inf A)$.

(86) $\int_A (r\,(\text{the function cos}))(x)dx = r \cdot (\text{the function sin})(\sup A) - r \cdot (\text{the function sin})(\inf A)$.

(87) $\int_A (r\,(\text{the function sinh}))(x)dx = r \cdot (\text{the function cosh})(\sup A) - r \cdot (\text{the function cosh})(\inf A)$.

(88) $\int_A (r\,(\text{the function cosh}))(x)dx = r \cdot (\text{the function sinh})(\sup A) - r \cdot (\text{the function sinh})(\inf A)$.

(89) $\int_A (r\,(\text{the function exp}))(x)dx = r \cdot (\text{the function exp})(\sup A) - r \cdot (\text{the function exp})(\inf A)$.

(90) $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2} \cdot ((\text{the function cos})(\inf A) \cdot (\text{the function cos})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function cos})(\sup A))$.

(91) If $A = [0, \frac{\pi}{2}]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2}$.

(92) If $A = [0, \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(93) If $A = [0, \pi \cdot \frac{3}{2}]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2}$.

(94) If $A = [0, \pi \cdot 2]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(95) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(96) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(97) $\int_A ((\text{the function sin})\,(\text{the function sin}))(x)dx = ((\text{the function cos})(\inf A) \cdot (\text{the function sin})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function sin})(\sup A)) + \int_A ((\text{the function cos})\,(\text{the function cos}))(x)dx$.

(98) $\int_A ((\text{the function sinh})\,(\text{the function sinh}))(x)dx = (\text{the function cosh})$ $(\sup A) \cdot (\text{the function sinh})(\sup A) - (\text{the function cosh})(\inf A) \cdot (\text{the function sinh})(\inf A) - \int_A ((\text{the function cosh})\,(\text{the function cosh}))(x)dx$.

(99) $\int_A ((\text{the function sinh})\,(\text{the function cosh}))(x)dx = \frac{1}{2} \cdot ((\text{the function cosh})(\sup A) \cdot (\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A) \cdot (\text{the function cosh})(\inf A))$.

(100) $\int_A ((\text{the function exp})\,(\text{the function exp}))(x)dx = \frac{1}{2} \cdot ((\text{the function exp})(\sup A)^{\mathbf{2}} - (\text{the function exp})(\inf A)^{\mathbf{2}})$.

(101) $\int_A ((\text{the function exp})\,((\text{the function sin}) + (\text{the function cos})))(x)dx = ((\text{the function exp})\ (\text{the function sin}))(\sup A) - ((\text{the function exp})\ (\text{the function sin}))(\inf A)$.

(102) $\int_A ((\text{the function exp})\,((\text{the function cos}) - (\text{the function sin})))(x)dx = ((\text{the function exp})\ (\text{the function cos}))(\sup A) - ((\text{the function exp})\ (\text{the function cos}))(\inf A)$.

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