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Basic Properties of the Rank of Matrices over a Field

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Summary. In this paper I present selected properties of triangular matrices and basic properties of the rank of matrices over a field.

I define a submatrix as a matrix formed by selecting certain rows and columns from a bigger matrix. That is in my considerations, as an array, it is cut down to those entries constrained by row and column. Then I introduce the concept of the rank of a $m \times n$ matrix A by the condition: A has the rank r if and only if, there is a $r \times r$ submatrix of A with a non-zero determinant, and for every $k \times k$ submatrix of A with a non-zero determinant we have $k \leq r$.

At the end, I prove that the rank defined by the size of the biggest submatrix with a non-zero determinant of a matrix A, is the same as the maximal number of linearly independent rows of A.

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The articles [27], [10], [37], [23], [1], [2], [12], [38], [39], [7], [8], [3], [4], [24], [36], [31], [15], [6], [13], [28], [14], [41], [30], [19], [34], [42], [9], [22], [16], [11], [25], [40], [18], [20], [26], [33], [21], [17], [35], [32], [29], [43], and [5] provide the terminology and notation for this paper.

1. Triangular Matrices

For simplicity, we use the following convention: x, X, Y are sets, D is a non empty set, i, j, k, m, n, m', n' are elements of $\mathbb{N}$, i_0, j_0, n_0, m_0 are non zero elements of $\mathbb{N}$, K is a field, a, b are elements of K, p is a finite sequence of elements of K, and M is a matrix over K of dimension n.

Next we state a number of propositions:

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(1) For every matrix A over D of dimension $n \times m$ holds if $n = 0$, then $m = 0$ iff $\operatorname{len} A = n$ and $\operatorname{width} A = m$.

(2) The following statements are equivalent

(i) M is a lower triangular matrix over K of dimension n,

(ii) M^{T} is an upper triangular matrix over K of dimension n.

(3) The diagonal of $M =$ the diagonal of M^{T}.

(4) Let p_1 be an element of the permutations of n-element set. Suppose $p_1 \neq \operatorname{idseq}(n)$. Then there exists i such that $i \in \operatorname{Seg} n$ and $p_1(i) > i$ and there exists j such that $j \in \operatorname{Seg} n$ and $p_1(j) < j$.

(5) Let M be a matrix over K of dimension n and p_1 be an element of the permutations of n-element set. Suppose that

(i) $p_1 \neq \operatorname{idseq}(n)$, and

(ii) M is a lower triangular matrix over K of dimension n or an upper triangular matrix over K of dimension n.

Then (the product on paths of $M)(p_1) = 0_K$.

(6) Let M be a matrix over K of dimension n and I be an element of the permutations of n-element set. If $I = \operatorname{idseq}(n)$, then the diagonal of $M = I$-Path M.

(7) Let M be an upper triangular matrix over K of dimension n. Then $\operatorname{Det} M =$ (the multiplication of $K) \circledast$ (the diagonal of M).

(8) Let M be a lower triangular matrix over K of dimension n. Then $\operatorname{Det} M =$ (the multiplication of $K) \circledast$ (the diagonal of M).

(9) For every finite set X and for every n holds $\overline{\overline{\{Y; Y \text{ ranges over subsets of } X\colon \operatorname{card} Y = n\}}} = \binom{\operatorname{card} X}{n}$.

(10) $\overline{\overline{2\mathrm{Set}\operatorname{Seg} n}} = \binom{n}{2}$.

(11) Let R be an element of the permutations of n-element set. If $R = \operatorname{Rev}(\operatorname{idseq}(n))$, then R is even iff $\binom{n}{2} \bmod 2 = 0$.

(12) Let M be a matrix over K of dimension n and R be a permutation of $\operatorname{Seg} n$. Suppose $R = \operatorname{Rev}(\operatorname{idseq}(n))$ and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is an upper triangular matrix over K of dimension n.

(13) Let M be a matrix over K of dimension n and R be a permutation of $\operatorname{Seg} n$. Suppose $R = \operatorname{Rev}(\operatorname{idseq}(n))$ and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j > n + 1$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is a lower triangular matrix over K of dimension n.

(14) Let M be a matrix over K of dimension n and R be an element of the permutations of n-element set. Suppose that

(i) $R = \operatorname{Rev}(\operatorname{idseq}(n))$, and

(ii) for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$ or for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i+j > n+1$

holds $M_{i,j} = 0_K$.
Then $\operatorname{Det} M = (-1)^{\operatorname{sgn}(R)}$(the multiplication of $K \odot (R\text{-Path}\, M)$).

(15) Let M be a matrix over K of dimension n and M_1, M_2 be upper triangular matrices over K of dimension n. Suppose $M = M_1 \cdot M_2$. Then
  (i) M is an upper triangular matrix over K of dimension n, and
  (ii) the diagonal of $M =$ (the diagonal of M_1) $\bullet$ (the diagonal of M_2).

(16) Let M be a matrix over K of dimension n and M_1, M_2 be lower triangular matrices over K of dimension n. Suppose $M = M_1 \cdot M_2$. Then
  (i) M is a lower triangular matrix over K of dimension n, and
  (ii) the diagonal of $M =$ (the diagonal of M_1) $\bullet$ (the diagonal of M_2).

2. The Rank of Matrices

Let D be a non empty set, let M be a matrix over D, let n, m be natural numbers, let n_1 be an element of $\mathbb{N}^n$, and let m_1 be an element of $\mathbb{N}^m$. The functor $\operatorname{Segm}(M, n_1, m_1)$ yielding a matrix over D of dimension $n \times m$ is defined as follows:

(Def. 1) For all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of $\operatorname{Segm}(M, n_1, m_1)$ holds $(\operatorname{Segm}(M, n_1, m_1))_{i,j} = M_{n_1(i), m_1(j)}$.

For simplicity, we follow the rules: A denotes a matrix over D, A' denotes a matrix over D of dimension $n' \times m'$, M' denotes a matrix over K of dimension $n' \times m'$, n_1, n_2, n_3 denote elements of $\mathbb{N}^n$, m_1, m_2 denote elements of $\mathbb{N}^m$, and M denotes a matrix over K.

Next we state a number of propositions:

(17) If $[\![\operatorname{rng} n_1, \operatorname{rng} m_1]\!] \subseteq$ the indices of A, then $\langle i, j \rangle \in$ the indices of $\operatorname{Segm}(A, n_1, m_1)$ iff $\langle n_1(i), m_1(j) \rangle \in$ the indices of A.

(18) If $[\![\operatorname{rng} n_1, \operatorname{rng} m_1]\!] \subseteq$ the indices of A and $n = 0$ iff $m = 0$, then $(\operatorname{Segm}(A, n_1, m_1))^{\mathrm{T}} = \operatorname{Segm}(A^{\mathrm{T}}, m_1, n_1)$.

(19) If $[\![\operatorname{rng} n_1, \operatorname{rng} m_1]\!] \subseteq$ the indices of A and if $m = 0$, then $n = 0$, then $\operatorname{Segm}(A, n_1, m_1) = (\operatorname{Segm}(A^{\mathrm{T}}, m_1, n_1))^{\mathrm{T}}$.

(20) For every matrix A over D of dimension 1 holds $A = \langle\langle A_{1,1} \rangle\rangle$.

(21) If $n = 1$ and $m = 1$, then $\operatorname{Segm}(A, n_1, m_1) = \langle\langle A_{n_1(1), m_1(1)} \rangle\rangle$.

(22) For every matrix A over D of dimension 2 holds $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$.

(23) If $n = 2$ and $m = 2$, then $\operatorname{Segm}(A, n_1, m_1) =$ $\begin{pmatrix} A_{n_1(1), m_1(1)} & A_{n_1(1), m_1(2)} \\ A_{n_1(2), m_1(1)} & A_{n_1(2), m_1(2)} \end{pmatrix}$.

(24) If $i \in \operatorname{Seg} n$ and $\operatorname{rng} m_1 \subseteq \operatorname{Seg} \operatorname{width} A$, then $\operatorname{Line}(\operatorname{Segm}(A, n_1, m_1), i) = \operatorname{Line}(A, n_1(i)) \cdot m_1$.

(25) If $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $n_1(i) = n_1(j)$, then $\operatorname{Line}(\operatorname{Segm}(A, n_1, m_1), i) = \operatorname{Line}(\operatorname{Segm}(A, n_1, m_1), j)$.

(26) If $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $n_1(i) = n_1(j)$ and $i \neq j$, then $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = 0_K$.

(27) If n_1 is not one-to-one, then $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = 0_K$.

(28) If $j \in \operatorname{Seg} m$ and $\operatorname{rng} n_1 \subseteq \operatorname{Seg} \operatorname{len} A$, then $(\operatorname{Segm}(A, n_1, m_1))_{\square, j} = A_{\square, m_1(j)} \cdot n_1$.

(29) If $i \in \operatorname{Seg} m$ and $j \in \operatorname{Seg} m$ and $m_1(i) = m_1(j)$, then $(\operatorname{Segm}(A, n_1, m_1))_{\square, i} = (\operatorname{Segm}(A, n_1, m_1))_{\square, j}$.

(30) If $i \in \operatorname{Seg} m$ and $j \in \operatorname{Seg} m$ and $m_1(i) = m_1(j)$ and $i \neq j$, then $\operatorname{Det} \operatorname{Segm}(M, m_2, m_1) = 0_K$.

(31) If m_1 is not one-to-one, then $\operatorname{Det} \operatorname{Segm}(M, m_2, m_1) = 0_K$.

(32) Let n_1, n_2 be elements of $\mathbb{N}^n$. Suppose n_1 is one-to-one and n_2 is one-to-one and $\operatorname{rng} n_1 = \operatorname{rng} n_2$. Then there exists a permutation p_1 of $\operatorname{Seg} n$ such that $n_2 = n_1 \cdot p_1$.

(33) For every function f from $\operatorname{Seg} n$ into $\operatorname{Seg} n$ such that $n_2 = n_1 \cdot f$ holds $\operatorname{Segm}(A, n_2, m_1) = \operatorname{Segm}(A, n_1, m_1) \cdot f$.

(34) For every function f from $\operatorname{Seg} m$ into $\operatorname{Seg} m$ such that $m_2 = m_1 \cdot f$ holds $(\operatorname{Segm}(A, n_1, m_2))^{\mathrm{T}} = (\operatorname{Segm}(A, n_1, m_1))^{\mathrm{T}} \cdot f$.

(35) Let p_1 be an element of the permutations of n-element set. If $n_2 = n_3 \cdot p_1$, then $\operatorname{Det} \operatorname{Segm}(M, n_2, n_1) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} \operatorname{Segm}(M, n_3, n_1)$ and $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} \operatorname{Segm}(M, n_1, n_3)$.

(36) For all elements n_1, n_2, n'_1, n'_2 of $\mathbb{N}^n$ such that $\operatorname{rng} n_1 = \operatorname{rng} n'_1$ and $\operatorname{rng} n_2 = \operatorname{rng} n'_2$ holds $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = \operatorname{Det} \operatorname{Segm}(M, n'_1, n'_2)$ or $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = -\operatorname{Det} \operatorname{Segm}(M, n'_1, n'_2)$.

(37) Let F, F_1 be finite sequences of elements of D and given n_1, m_1. Suppose $\operatorname{len} F = \operatorname{width} A'$ and $F_1 = F \cdot m_1$ and $[\!: \operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A'. Let given i, j. If ${n_1}^{-1}(\{j\}) = \{i\}$, then $\operatorname{RLine}(\operatorname{Segm}(A', n_1, m_1), i, F_1) = \operatorname{Segm}(\operatorname{RLine}(A', j, F), n_1, m_1)$.

(38) Let F be a finite sequence of elements of D and given i, n_1. If $i \notin \operatorname{rng} n_1$ and $[\!: \operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A', then $\operatorname{Segm}(A', n_1, m_1) = \operatorname{Segm}(\operatorname{RLine}(A', i, F), n_1, m_1)$.

(39) If $i \in \operatorname{Seg} n'$ and $i \in \operatorname{rng} n_1$ and $[\!: \operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A', then there exists n_2 such that $\operatorname{rng} n_2 = (\operatorname{rng} n_1 \setminus \{i\}) \cup \{j\}$ and $\operatorname{Segm}(\operatorname{RLine}(A', i, \operatorname{Line}(A', j)), n_1, m_1) = \operatorname{Segm}(A', n_2, m_1)$.

(40) For every finite sequence F of elements of D such that $i \notin \operatorname{Seg} \operatorname{len} A'$ holds $\operatorname{RLine}(A', i, F) = A'$.

Let n, m be natural numbers, let K be a field, let M be a matrix over K of dimension $n \times m$, and let a be an element of K. Then $a \cdot M$ is a matrix over

K of dimension $n \times m$.

We now state two propositions:

(41) If $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1:\!] \subseteq$ the indices of M, then $a \cdot \mathrm{Segm}(M, n_1, m_1) = \mathrm{Segm}(a \cdot M, n_1, m_1)$.

(42) If $n_1 = \mathrm{idseq}(\mathrm{len}\, A)$ and $m_1 = \mathrm{idseq}(\mathrm{width}\, A)$, then $\mathrm{Segm}(A, n_1, m_1) = A$.

Let us observe that there exists a subset of $\mathbb{N}$ which is empty, without zero, and finite and there exists a subset of $\mathbb{N}$ which is non empty, without zero, and finite.

Let us consider n. Observe that $\mathrm{Seg}\, n$ is without zero.

Let X be a without zero set and let Y be a set. One can verify that $X \setminus Y$ is without zero and $X \cap Y$ is without zero.

One can prove the following proposition

(43) For every finite without zero subset N of $\mathbb{N}$ there exists k such that $N \subseteq \mathrm{Seg}\, k$.

Let N be a finite without zero subset of $\mathbb{N}$. Then $\mathrm{Sgm}\, N$ is an element of $\mathbb{N}^{\mathrm{card}\, N}$.

Let D be a non empty set, let A be a matrix over D, and let P, Q be without zero finite subsets of $\mathbb{N}$. The functor $\mathrm{Segm}(A, P, Q)$ yields a matrix over D of dimension $\mathrm{card}\, P \times \mathrm{card}\, Q$ and is defined by:

(Def. 2) $\mathrm{Segm}(A, P, Q) = \mathrm{Segm}(A, \mathrm{Sgm}\, P, \mathrm{Sgm}\, Q)$.

Next we state two propositions:

(44) $\mathrm{Segm}(A, \{i_0\}, \{j_0\}) = \langle\langle A_{i_0, j_0} \rangle\rangle$.

(45) If $i_0 < j_0$ and $n_0 < m_0$, then $\mathrm{Segm}(A, \{i_0, j_0\}, \{n_0, m_0\}) = \begin{pmatrix} A_{i_0,n_0} & A_{i_0,m_0} \\ A_{j_0,n_0} & A_{j_0,m_0} \end{pmatrix}$.

In the sequel P, P_1, P_2, Q, Q_1, Q_2 are without zero finite subsets of $\mathbb{N}$.

The following propositions are true:

(46) $\mathrm{Segm}(A, \mathrm{Seg}\,\mathrm{len}\, A, \mathrm{Seg}\,\mathrm{width}\, A) = A$.

(47) If $i \in \mathrm{Seg}\,\mathrm{card}\, P$ and $Q \subseteq \mathrm{Seg}\,\mathrm{width}\, A$, then $\mathrm{Line}(\mathrm{Segm}(A, P, Q), i) = \mathrm{Line}(A, (\mathrm{Sgm}\, P)(i)) \cdot \mathrm{Sgm}\, Q$.

(48) If $i \in \mathrm{Seg}\,\mathrm{card}\, P$, then $\mathrm{Line}(\mathrm{Segm}(A, P, \mathrm{Seg}\,\mathrm{width}\, A), i) = \mathrm{Line}(A, (\mathrm{Sgm}\, P)(i))$.

(49) If $j \in \mathrm{Seg}\,\mathrm{card}\, Q$ and $P \subseteq \mathrm{Seg}\,\mathrm{len}\, A$, then $(\mathrm{Segm}(A, P, Q))_{\Box, j} = A_{\Box, (\mathrm{Sgm}\, Q)(j)} \cdot \mathrm{Sgm}\, P$.

(50) If $j \in \mathrm{Seg}\,\mathrm{card}\, Q$, then $(\mathrm{Segm}(A, \mathrm{Seg}\,\mathrm{len}\, A, Q))_{\Box, j} = A_{\Box, (\mathrm{Sgm}\, Q)(j)}$.

(51) $\mathrm{Segm}(A, \mathrm{Seg}\,\mathrm{len}\, A \setminus \{i\}, \mathrm{Seg}\,\mathrm{width}\, A) = A_{\restriction i}$.

(52) $\mathrm{Segm}(M, \mathrm{Seg}\,\mathrm{len}\, M, \mathrm{Seg}\,\mathrm{width}\, M \setminus \{i\}) =$ the deleting of i-column in M.

(53) $(\mathrm{Sgm}\, P)^{-1}(X)$ is a without zero finite subset of $\mathbb{N}$.

(54) If $X \subseteq P$, then $\mathrm{Sgm}\, X = \mathrm{Sgm}\, P \cdot \mathrm{Sgm}((\mathrm{Sgm}\, P)^{-1}(X))$.

(55) $[\!:(\mathrm{Sgm}\, P)^{-1}(X), (\mathrm{Sgm}\, Q)^{-1}(Y)\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P, Q)$.

(56) If $P \subseteq P_1$ and $Q \subseteq Q_1$ and $P_2 = (\mathrm{Sgm}\, P_1)^{-1}(P)$ and $Q_2 = (\mathrm{Sgm}\, Q_1)^{-1}(Q)$, then $[\!:\mathrm{rng}\,\mathrm{Sgm}\, P_2, \mathrm{rng}\,\mathrm{Sgm}\, Q_2\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P_1, Q_1)$ and $\mathrm{Segm}(\mathrm{Segm}(A, P_1, Q_1), P_2, Q_2) = \mathrm{Segm}(A, P, Q)$.

(57) Suppose $P = \emptyset$ iff $Q = \emptyset$ and $[\!:P, Q\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P_1, Q_1)$. Then there exist P_2, Q_2 such that $P_2 \subseteq P_1$ and $Q_2 \subseteq Q_1$ and $P_2 = (\mathrm{Sgm}\, P_1)^{\circ} P$ and $Q_2 = (\mathrm{Sgm}\, Q_1)^{\circ} Q$ and $\mathrm{card}\, P_2 = \mathrm{card}\, P$ and $\mathrm{card}\, Q_2 = \mathrm{card}\, Q$ and $\mathrm{Segm}(\mathrm{Segm}(A, P_1, Q_1), P, Q) = \mathrm{Segm}(A, P_2, Q_2)$.

(58) For every matrix M over K of dimension n holds $\mathrm{Segm}(M, \mathrm{Seg}\, n \setminus \{i\}, \mathrm{Seg}\, n \setminus \{j\}) =$ the deleting of i-row and j-column in M.

(59) Let F, F_2 be finite sequences of elements of D. Suppose $\mathrm{len}\, F = \mathrm{width}\, A'$ and $F_2 = F \cdot \mathrm{Sgm}\, Q$ and $[\!:P, Q\,:\!] \subseteq$ the indices of A'. Then $\mathrm{RLine}(\mathrm{Segm}(A', P, Q), i, F_2) = \mathrm{Segm}(\mathrm{RLine}(A', (\mathrm{Sgm}\, P)(i), F), P, Q)$.

(60) Let F be a finite sequence of elements of D and given i, P. If $i \notin P$ and $[\!:P, Q\,:\!] \subseteq$ the indices of A', then $\mathrm{Segm}(A', P, Q) = \mathrm{Segm}(\mathrm{RLine}(A', i, F), P, Q)$.

(61) If $[\!:P, Q\,:\!] \subseteq$ the indices of A and $\mathrm{card}\, P = 0$ iff $\mathrm{card}\, Q = 0$, then $(\mathrm{Segm}(A, P, Q))^{\mathrm{T}} = \mathrm{Segm}(A^{\mathrm{T}}, Q, P)$.

(62) If $[\!:P, Q\,:\!] \subseteq$ the indices of A and if $\mathrm{card}\, Q = 0$, then $\mathrm{card}\, P = 0$, then $\mathrm{Segm}(A, P, Q) = (\mathrm{Segm}(A^{\mathrm{T}}, Q, P))^{\mathrm{T}}$.

(63) If $[\!:P, Q\,:\!] \subseteq$ the indices of M, then $a \cdot \mathrm{Segm}(M, P, Q) = \mathrm{Segm}(a \cdot M, P, Q)$.

Let D be a non empty set, let A be a matrix over D, and let P, Q be without zero finite subsets of $\mathbb{N}$. Let us assume that $\mathrm{card}\, P = \mathrm{card}\, Q$. The functor $\mathrm{EqSegm}(A, P, Q)$ yields a matrix over D of dimension $\mathrm{card}\, P$ and is defined by:

(Def. 3) $\mathrm{EqSegm}(A, P, Q) = \mathrm{Segm}(A, P, Q)$.

Next we state several propositions:

(64) For all P, Q, i, j such that $i \in \mathrm{Seg}\,\mathrm{card}\, P$ and $j \in \mathrm{Seg}\,\mathrm{card}\, P$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ holds $\mathrm{Delete}(\mathrm{EqSegm}(M, P, Q), i, j) = \mathrm{EqSegm}(M, P \setminus \{(\mathrm{Sgm}\, P)(i)\}, Q \setminus \{(\mathrm{Sgm}\, Q)(j)\})$ and $\mathrm{card}(P \setminus \{(\mathrm{Sgm}\, P)(i)\}) = \mathrm{card}(Q \setminus \{(\mathrm{Sgm}\, Q)(j)\})$.

(65) For all M, P, P_1, Q_1 such that $\mathrm{card}\, P_1 = \mathrm{card}\, Q_1$ and $P \subseteq P_1$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists Q such that $Q \subseteq Q_1$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$.

(66) For all M, P_1, Q, Q_1 such that $\mathrm{card}\, P_1 = \mathrm{card}\, Q_1$ and $Q \subseteq Q_1$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists P such that $P \subseteq P_1$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$.

(67) If $\mathrm{card}\, P = \mathrm{card}\, Q$, then $[\!:P, Q\,:\!] \subseteq$ the indices of A iff $P \subseteq \mathrm{Seg}\,\mathrm{len}\, A$

and $Q \subseteq \mathrm{Seg}\,\mathrm{width}\,A$.

(68) Let given P, Q, i, j_0. Suppose $i \in \mathrm{Seg}\,n'$ and $j_0 \in \mathrm{Seg}\,n'$ and $i \in P$ and $j_0 \notin P$ and $\mathrm{card}\,P = \mathrm{card}\,Q$ and $[\![P, Q]\!] \subseteq$ the indices of M'. Then $\mathrm{card}\,P = \mathrm{card}((P \setminus \{i\}) \cup \{j_0\})$ but $[\![(P \setminus \{i\}) \cup \{j_0\}, Q]\!] \subseteq$ the indices of M' but $\mathrm{Det}\,\mathrm{EqSegm}(\mathrm{RLine}(M', i, \mathrm{Line}(M', j_0)), P, Q) = \mathrm{Det}\,\mathrm{EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$ or $\mathrm{Det}\,\mathrm{EqSegm}(\mathrm{RLine}(M', i, \mathrm{Line}(M', j_0)), P, Q) = -\mathrm{Det}\,\mathrm{EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$.

(69) If $\mathrm{card}\,P = \mathrm{card}\,Q$, then $[\![P, Q]\!] \subseteq$ the indices of A iff $[\![Q, P]\!] \subseteq$ the indices of A^{T}.

(70) If $[\![P, Q]\!] \subseteq$ the indices of M and $\mathrm{card}\,P = \mathrm{card}\,Q$, then $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) = \mathrm{Det}\,\mathrm{EqSegm}(M^{\mathrm{T}}, Q, P)$.

(71) For every matrix M over K of dimension n holds $\mathrm{Det}(a \cdot M) = \mathrm{power}_K(a, n) \cdot \mathrm{Det}\,M$.

(72) If $[\![P, Q]\!] \subseteq$ the indices of M and $\mathrm{card}\,P = \mathrm{card}\,Q$, then $\mathrm{Det}\,\mathrm{EqSegm}(a \cdot M, P, Q) = \mathrm{power}_K(a, \mathrm{card}\,P) \cdot \mathrm{Det}\,\mathrm{EqSegm}(M, P, Q)$.

Let K be a field and let M be a matrix over K. The functor $\mathrm{rk}(M)$ yielding an element of $\mathbb{N}$ is defined by the conditions (Def. 4).

(Def. 4)(i) There exist P, Q such that $[\![P, Q]\!] \subseteq$ the indices of M and $\mathrm{card}\,P = \mathrm{card}\,Q$ and $\mathrm{card}\,P = \mathrm{rk}(M)$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$, and

(ii) for all P_1, Q_1 such that $[\![P_1, Q_1]\!] \subseteq$ the indices of M and $\mathrm{card}\,P_1 = \mathrm{card}\,Q_1$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, Q_1) \neq 0_K$ holds $\mathrm{card}\,P_1 \leq \mathrm{rk}(M)$.

The following propositions are true:

(73) For all P, Q such that $[\![P, Q]\!] \subseteq$ the indices of M and $\mathrm{card}\,P = \mathrm{card}\,Q$ holds $\mathrm{card}\,P \leq \mathrm{len}\,M$ and $\mathrm{card}\,Q \leq \mathrm{width}\,M$.

(74) $\mathrm{rk}(M) \leq \mathrm{len}\,M$ and $\mathrm{rk}(M) \leq \mathrm{width}\,M$.

(75) If $[\![\mathrm{rng}\,n_2, \mathrm{rng}\,n_3]\!] \subseteq$ the indices of M and $\mathrm{Det}\,\mathrm{Segm}(M, n_2, n_3) \neq 0_K$, then there exist P_1, P_2 such that $P_1 = \mathrm{rng}\,n_2$ and $P_2 = \mathrm{rng}\,n_3$ and $\mathrm{card}\,P_1 = \mathrm{card}\,P_2$ and $\mathrm{card}\,P_1 = n$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, P_2) \neq 0_K$.

(76) Let R_1 be an element of $\mathbb{N}$. Then $\mathrm{rk}(M) = R_1$ if and only if the following conditions are satisfied:

(i) there exist elements r_1, r_2 of $\mathbb{N}^{R_1}$ such that $[\![\mathrm{rng}\,r_1, \mathrm{rng}\,r_2]\!] \subseteq$ the indices of M and $\mathrm{Det}\,\mathrm{Segm}(M, r_1, r_2) \neq 0_K$, and

(ii) for all n, n_2, n_3 such that $[\![\mathrm{rng}\,n_2, \mathrm{rng}\,n_3]\!] \subseteq$ the indices of M and $\mathrm{Det}\,\mathrm{Segm}(M, n_2, n_3) \neq 0_K$ holds $n \leq R_1$.

(77) If $n = 0$ or $m = 0$, then $\mathrm{rk}(\mathrm{Segm}(M, n_1, m_1)) = 0$.

(78) If $[\![\mathrm{rng}\,n_1, \mathrm{rng}\,m_1]\!] \subseteq$ the indices of M, then $\mathrm{rk}(M) \geq \mathrm{rk}(\mathrm{Segm}(M, n_1, m_1))$.

(79) If $[\![P, Q]\!] \subseteq$ the indices of M, then $\mathrm{rk}(M) \geq \mathrm{rk}(\mathrm{Segm}(M, P, Q))$.

(80) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $\mathrm{rk}(\mathrm{Segm}(M, P, Q)) \leq \mathrm{rk}(\mathrm{Segm}(M, P_1, Q_1))$.

(81) For all functions f, g such that $\operatorname{rng} f \subseteq \operatorname{rng} g$ there exists a function h such that $\operatorname{dom} h = \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $f = g \cdot h$.

(82) If $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1 :\!] =$ the indices of M, then $\operatorname{rk}(M) = \operatorname{rk}(\operatorname{Segm}(M, n_1, m_1))$.

(83) For every matrix M over K of dimension n holds $\operatorname{rk}(M) = n$ iff $\operatorname{Det} M \neq 0_K$.

(84) $\operatorname{rk}(M) = \operatorname{rk}(M^{\mathrm{T}})$.

(85) For every matrix M over K of dimension $n \times m$ and for every permutation F of $\operatorname{Seg} n$ holds $\operatorname{rk}(M) = \operatorname{rk}(M \cdot F)$.

(86) If $a \neq 0_K$, then $\operatorname{rk}(M) = \operatorname{rk}(a \cdot M)$.

(87) Let p, p_2 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} p$, then $a \cdot p \cdot f = a \cdot p_2$.

(88) Let p, p_2, q, q_1 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} p$ and $q_1 = q \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} q$, then $(p + q) \cdot f = p_2 + q_1$.

(89) If $a \neq 0_K$, then $\operatorname{rk}(M') = \operatorname{rk}(\operatorname{RLine}(M', i, a \cdot \operatorname{Line}(M', i)))$.

(90) If $\operatorname{Line}(M, i) = \operatorname{width} M \mapsto 0_K$, then rk(the deleting of i-row in M) $= \operatorname{rk}(M)$.

(91) For every p such that $\operatorname{len} p = \operatorname{width} M'$ holds rk(the deleting of i-row in M') $= \operatorname{rk}(\operatorname{RLine}(M', i, 0_K \cdot p))$.

(92) If $j \in \operatorname{Seg} \operatorname{len} M'$ and if $i = j$, then $a \neq -\mathbf{1}_K$, then $\operatorname{rk}(M') = \operatorname{rk}(\operatorname{RLine}(M', i, \operatorname{Line}(M', i) + a \cdot \operatorname{Line}(M', j)))$.

(93) If $j \in \operatorname{Seg} \operatorname{len} M'$ and $j \neq i$, then rk(the deleting of i-row in M') $= \operatorname{rk}(\operatorname{RLine}(M', i, a \cdot \operatorname{Line}(M', j)))$.

(94) $\operatorname{rk}(M) > 0$ iff there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$.

(95) $$\operatorname{rk}(M) = 0 \text{ iff } M = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} M) \times (\operatorname{width} M)}.$$

(96) $\operatorname{rk}(M) = 1$ if and only if the following conditions are satisfied:
 (i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and
 (ii) for all i_0, j_0, n_0, m_0 such that $i_0 \neq j_0$ and $n_0 \neq m_0$ and $[\!: \{i_0, j_0\}, \{n_0, m_0\} :\!] \subseteq$ the indices of M holds $\operatorname{Det} \operatorname{EqSegm}(M, \{i_0, j_0\}, \{n_0, m_0\}) = 0_K$.

(97) $\operatorname{rk}(M) = 1$ if and only if the following conditions are satisfied:
 (i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and
 (ii) for all i, j, n, m such that $[\!: \{i, j\}, \{n, m\} :\!] \subseteq$ the indices of M holds $M_{i,n} \cdot M_{j,m} = M_{i,m} \cdot M_{j,n}$.

(98) $\mathrm{rk}(M) = 1$ if and only if there exists i such that $i \in \mathrm{Seg}\,\mathrm{len}\, M$ and there exists j such that $j \in \mathrm{Seg}\,\mathrm{width}\, M$ and $M_{i,j} \neq 0_K$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, M$ there exists a such that $\mathrm{Line}(M, k) = a \cdot \mathrm{Line}(M, i)$.

Let us consider K. Observe that there exists a matrix over K which is diagonal.

One can prove the following propositions:

(99) Let M be a diagonal matrix over K and N_1 be a set. Suppose $N_1 = \{i : \langle i, i\rangle \in$ the indices of $M \ \wedge\ M_{i,i} \neq 0_K\}$. Let given P, Q. If $[: P, Q :] \subseteq$ the indices of M and $\mathrm{card}\, P = \mathrm{card}\, Q$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$, then $P \subseteq N_1$ and $Q \subseteq N_1$.

(100) For every diagonal matrix M over K and for every P such that $[: P, P :] \subseteq$ the indices of M holds $\mathrm{Segm}(M, P, P)$ is diagonal.

(101) Let M be a diagonal matrix over K and N_1 be a set. If $N_1 = \{i : \langle i, i\rangle \in$ the indices of $M \ \wedge\ M_{i,i} \neq 0_K\}$, then $\mathrm{rk}(M) = \overline{\overline{N_1}}$.

For simplicity, we adopt the following rules: v, v_1, v_2, u denote vectors of the n-dimension vector space over K, t, t_1, t_2 denote elements of (the carrier of $K)^n$, L denotes a linear combination of the n-dimension vector space over K, and M, M_1 denote matrices over K of dimension $m \times n$.

We now state the proposition

(102)(i) The carrier of the n-dimension vector space over $K =$ (the carrier of $K)^n$,

(ii) $0_{\text{the } n\text{-dimension vector space over } K} = n \mapsto 0_K$,

(iii) if $t_1 = v_1$ and $t_2 = v_2$, then $t_1 + t_2 = v_1 + v_2$, and

(iv) if $t = v$, then $a \cdot t = a \cdot v$.

Let us consider K, n. Then the n-dimension vector space over K is a strict vector space over K.

Let us consider K, n. One can verify that every vector of the n-dimension vector space over K is function-like and relation-like.

Let us consider K, m, n and let M be a matrix over K of dimension $m \times n$. We introduce $\mathrm{lines}(M)$ as a synonym of $\mathrm{rng}\, M$. We introduce M is without repeated line as a synonym of M is one-to-one.

Let K be a field, let us consider m, n, and let M be a matrix over K of dimension $m \times n$. Then $\mathrm{lines}(M)$ is a subset of the n-dimension vector space over K.

Next we state two propositions:

(103) $x \in \mathrm{lines}(M)$ iff there exists i such that $i \in \mathrm{Seg}\, m$ and $x = \mathrm{Line}(M, i)$.

(104) Let V be a finite subset of the n-dimension vector space over K. Then there exists a matrix M over K of dimension $\mathrm{card}\, V \times n$ such that M is without repeated line and $\mathrm{lines}(M) = V$.

Let us consider K, n and let F be a finite sequence of elements of the n-dimension vector space over K. The functor FinS2MX F yielding a matrix over K of dimension $\operatorname{len} F \times n$ is defined by:

(Def. 5) FinS2MX $F = F$.

Let us consider K, m, n and let M be a matrix over K of dimension $m \times n$. The functor MX2FinS M yielding a finite sequence of elements of the n-dimension vector space over K is defined as follows:

(Def. 6) MX2FinS $M = M$.

One can prove the following propositions:

(105) If $\operatorname{rk}(M) = m$, then M is without repeated line.

(106) If $i \in \operatorname{Seg} \operatorname{len} M$ and $a = L(M(i))$, then $\operatorname{Line}(\text{FinS2MX}(L \text{ MX2FinS } M), i) = a \cdot \operatorname{Line}(M, i)$.

(107) If M is without repeated line and the support of $L \subseteq \operatorname{lines}(M)$ and $i \in \operatorname{Seg} n$, then $(\sum L)(i) = \sum((\text{FinS2MX}(L \text{ MX2FinS } M))_{\square,i})$.

(108) Let given M, M_1. Suppose M is without repeated line and for every i such that $i \in \operatorname{Seg} m$ there exists a such that $\operatorname{Line}(M_1, i) = a \cdot \operatorname{Line}(M, i)$. Then there exists a linear combination L of $\operatorname{lines}(M)$ such that L MX2FinS $M = M_1$.

(109) Let given M. Suppose M is without repeated line. Then for every i such that $i \in \operatorname{Seg} m$ holds $\operatorname{Line}(M, i) \neq n \mapsto 0_K$ and for every M_1 such that for every i such that $i \in \operatorname{Seg} m$ there exists a such that $\operatorname{Line}(M_1, i) = a \cdot \operatorname{Line}(M, i)$ and for every j such that $j \in \operatorname{Seg} n$ holds $\sum((M_1)_{\square,j}) = 0_K$ holds $M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{m \times n}$ if and only if $\operatorname{lines}(M)$ is linearly independent.

(110) If $\operatorname{rk}(M) = m$, then $\operatorname{lines}(M)$ is linearly independent.

(111) Let M be a diagonal n-dimensional matrix over K. Suppose $\operatorname{rk}(M) = n$. Then $\operatorname{lines}(M)$ is a basis of the n-dimension vector space over K.

Let us consider K, n. Then the n-dimension vector space over K is a strict finite dimensional vector space over K.

The following propositions are true:

(112) dim(the n-dimension vector space over K) $= n$.

(113) Let given M, i, a. Suppose that for every j such that $j \in \operatorname{Seg} m$ holds $M_{j,i} = a$. Then M is without repeated line if and only if $\operatorname{Segm}(M, \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M \setminus \{i\})$ is without repeated line.

(114) Let given M, i. Suppose M is without repeated line and $\operatorname{lines}(M)$ is linearly independent and for every j such that $j \in \operatorname{Seg} m$ holds $M_{j,i} = 0_K$. Then $\operatorname{lines}(\operatorname{Segm}(M, \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M \setminus \{i\}))$ is linearly independent.

(115) Let V be a vector space over K and U be a finite subset of V. Suppose U is linearly independent. Let u, v be vectors of V. If $u \in U$ and $v \in U$ and $u \neq v$, then $(U \setminus \{u\}) \cup \{u + a \cdot v\}$ is linearly independent.

(116) Let V be a vector space over K and u, v be vectors of V. Then $x \in \mathrm{Lin}(\{u, v\})$ if and only if there exist a, b such that $x = a \cdot u + b \cdot v$.

(117) Let given M. Suppose $\mathrm{lines}(M)$ is linearly independent and M is without repeated line. Let given i, j. Suppose $j \in \mathrm{Seg}\,\mathrm{len}\,M$ and $i \neq j$. Then $\mathrm{RLine}(M, i, \mathrm{Line}(M, i) + a \cdot \mathrm{Line}(M, j))$ is without repeated line and $\mathrm{lines}(\mathrm{RLine}(M, i, \mathrm{Line}(M, i) + a \cdot \mathrm{Line}(M, j)))$ is linearly independent.

(118) If $P \subseteq \mathrm{Seg}\,m$, then $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n)) \subseteq \mathrm{lines}(M)$.

(119) If $P \subseteq \mathrm{Seg}\,m$ and $\mathrm{lines}(M)$ is linearly independent, then $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n))$ is linearly independent.

(120) If $P \subseteq \mathrm{Seg}\,m$ and M is without repeated line, then $\mathrm{Segm}(M, P, \mathrm{Seg}\,n)$ is without repeated line.

(121) Let M be a matrix over K of dimension $m \times n$. Then $\mathrm{lines}(M)$ is linearly independent and M is without repeated line if and only if $\mathrm{rk}(M) = m$.

(122) Let U be a subset of the n-dimension vector space over K. Suppose $U \subseteq \mathrm{lines}(M)$. Then there exists P such that $P \subseteq \mathrm{Seg}\,m$ and $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n)) = U$ and $\mathrm{Segm}(M, P, \mathrm{Seg}\,n)$ is without repeated line.

(123) Let R_1 be an element of $\mathbb{N}$. Then $\mathrm{rk}(M) = R_1$ if and only if the following conditions are satisfied:

(i) there exists a finite subset U of the n-dimension vector space over K such that U is linearly independent and $U \subseteq \mathrm{lines}(M)$ and $\mathrm{card}\,U = R_1$, and

(ii) for every finite subset W of the n-dimension vector space over K such that W is linearly independent and $W \subseteq \mathrm{lines}(M)$ holds $\mathrm{card}\,W \leq R_1$.

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