FORMALIZED MATHEMATICS
Vol. **16**, No. **1**, Pages 81–90, **2008**

Solutions of Linear Equations

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Summary. In this paper I present the Kronecker-Capelli theorem which states that a system of linear equations has a solution if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix.

MML identifier: `MATRIX15`, version: `7.8.09 4.97.1001`

The terminology and notation used in this paper are introduced in the following papers: [9], [24], [1], [2], [10], [25], [6], [8], [7], [3], [23], [21], [13], [5], [11], [12], [26], [15], [27], [19], [16], [22], [20], [28], [4], [17], [14], and [18].

1. Preliminaries

For simplicity, we follow the rules: x denotes a set, i, j, k, l, m, n denote natural numbers, K denotes a field, N denotes a without zero finite subset of $\mathbb{N}$, a, b denote elements of K, A, B, B_1, B_2, X, X_1, X_2 denote matrices over K, A' denotes a matrix over K of dimension $m \times n$, B' denotes a matrix over K of dimension $m \times k$, and M denotes a square matrix over K of dimension n.

We now state a number of propositions:

(1) If $\mathrm{width}\, A = \mathrm{len}\, B$, then $(a \cdot A) \cdot B = a \cdot (A \cdot B)$.

(2) $\mathbf{1}_K \cdot A = A$ and $a \cdot (b \cdot A) = (a \cdot b) \cdot A$.

(3) Let K be a non empty additive loop structure and f, g, h, w be finite sequences of elements of K. If $\mathrm{len}\, f = \mathrm{len}\, g$ and $\mathrm{len}\, h = \mathrm{len}\, w$, then $f \frown h + g \frown w = (f + g) \frown (h + w)$.

(4) Let K be a non empty multiplicative magma, f, g be finite sequences of elements of K, and a be an element of K. Then $a \cdot (f \frown g) = (a \cdot f) \frown (a \cdot g)$.

ISSN 1426–2630(p), 1898-9934(e)

(5) Let f be a function and p_1, p_2, f_1, f_2 be finite sequences. If $\operatorname{rng} p_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} p_2 \subseteq \operatorname{dom} f$ and $f_1 = f \cdot p_1$ and $f_2 = f \cdot p_2$, then $f \cdot (p_1 \frown p_2) = f_1 \frown f_2$.

(6) Let f be a finite sequence of elements of $\mathbb{N}$ and given n. Suppose f is one-to-one and $\operatorname{rng} f \subseteq \operatorname{Seg} n$ and for all i, j such that i, $j \in \operatorname{dom} f$ and $i < j$ holds $f(i) < f(j)$. Then $\operatorname{Sgm} \operatorname{rng} f = f$.

(7) Let K be an Abelian add-associative right zeroed right complementable non empty additive loop structure, p be a finite sequence of elements of K, and given i, j. Suppose i, $j \in \operatorname{dom} p$ and $i \neq j$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq i$ and $k \neq j$ holds $p(k) = 0_K$. Then $\sum p = p_i + p_j$.

(8) If $i \in \operatorname{Seg} m$, then $(\operatorname{Sgm}(\operatorname{Seg}(n+m) \setminus \operatorname{Seg} n))(i) = n + i$.

(9) Let D be a non empty set, A be a matrix over D, and B_3, B_4, C_1, C_2 be without zero finite subsets of $\mathbb{N}$. Suppose $B_3 \times B_4 \subseteq$ the indices of A and $C_1 \times C_2 \subseteq$ the indices of A. Let B be a matrix over D of dimension $\operatorname{card} B_3 \times \operatorname{card} B_4$ and C be a matrix over D of dimension $\operatorname{card} C_1 \times \operatorname{card} C_2$. Suppose that for all natural numbers i, j, b_1, b_2, c_1, c_2 such that $\langle i, j \rangle \in (B_3 \times B_4) \cap (C_1 \times C_2)$ and $b_1 = (\operatorname{Sgm} B_3)^{-1}(i)$ and $b_2 = (\operatorname{Sgm} B_4)^{-1}(j)$ and $c_1 = (\operatorname{Sgm} C_1)^{-1}(i)$ and $c_2 = (\operatorname{Sgm} C_2)^{-1}(j)$ holds $B_{b_1,b_2} = C_{c_1,c_2}$. Then there exists a matrix M over D of dimension $\operatorname{len} A \times \operatorname{width} A$ such that $\operatorname{Segm}(M, B_3, B_4) = B$ and $\operatorname{Segm}(M, C_1, C_2) = C$ and for all i, j such that $\langle i, j \rangle \in$ (the indices of $M) \setminus (B_3 \times B_4 \cup C_1 \times C_2)$ holds $M_{i,j} = A_{i,j}$.

(10) Let P, Q, Q' be without zero finite subsets of $\mathbb{N}$. Suppose $P \times Q' \subseteq$ the indices of A. Let given i, j. Suppose $i \in \operatorname{dom} A \setminus P$ and $j \in \operatorname{Seg} \operatorname{width} A \setminus Q$ and $A_{i,j} \neq 0_K$ and $Q \subseteq Q'$ and $\operatorname{Line}(A, i) \cdot \operatorname{Sgm} Q' = \operatorname{card} Q' \mapsto 0_K$. Then $\operatorname{rk}(A) > \operatorname{rk}(\operatorname{Segm}(A, P, Q))$.

(11) For every N such that $N \subseteq \operatorname{dom} A$ and for every i such that $i \in \operatorname{dom} A \setminus N$ holds $\operatorname{Line}(A, i) = \operatorname{width} A \mapsto 0_K$ holds $\operatorname{rk}(A) = \operatorname{rk}(\operatorname{Segm}(A, N, \operatorname{Seg} \operatorname{width} A))$.

(12) For every N such that $N \subseteq \operatorname{Seg} \operatorname{width} A$ and for every i such that $i \in \operatorname{Seg} \operatorname{width} A \setminus N$ holds $A_{\square,i} = \operatorname{len} A \mapsto 0_K$ holds $\operatorname{rk}(A) = \operatorname{rk}(\operatorname{Segm}(A, \operatorname{Seg} \operatorname{len} A, N))$.

(13) Let V be a vector space over K, U be a finite subset of V, u, v be vectors of V, and given a. If u, $v \in U$, then $\operatorname{Lin}((U \setminus \{u\}) \cup \{u + a \cdot v\})$ is a subspace of $\operatorname{Lin}(U)$.

(14) Let V be a vector space over K, U be a finite subset of V, u, v be vectors of V, and given a. Suppose u, $v \in U$ and if $u = v$, then $a \neq -\mathbf{1}_K$ or $u = 0_V$. Then $\operatorname{Lin}((U \setminus \{u\}) \cup \{u + a \cdot v\}) = \operatorname{Lin}(U)$.

2. Selected Properties of Joining Operation of two Matrices

Let D be a non empty set, let n, m, k be natural numbers, let A be a matrix over D of dimension $n \times m$, and let B be a matrix over D of dimension $n \times k$. Then $A \frown B$ is a matrix over D of dimension $n \times (\mathrm{width}\, A + \mathrm{width}\, B)$.

We now state a number of propositions:

(15) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i. If $i \in \mathrm{Seg}\, n$, then $\mathrm{Line}(A \frown B, i) = \mathrm{Line}(A, i) \frown \mathrm{Line}(B, i)$.

(16) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i. If $i \in \mathrm{Seg}\,\mathrm{width}\, A$, then $(A \frown B)_{\square,i} = A_{\square,i}$.

(17) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i. If $i \in \mathrm{Seg}\,\mathrm{width}\, B$, then $(A \frown B)_{\square,\mathrm{width}\, A+i} = B_{\square,i}$.

(18) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and p_3, p_4 be finite sequences of elements of D. If $\mathrm{len}\, p_3 = \mathrm{width}\, A$ and $\mathrm{len}\, p_4 = \mathrm{width}\, B$, then $\mathrm{ReplaceLine}(A \frown B, i, p_3 \frown p_4) = (\mathrm{ReplaceLine}(A, i, p_3)) \frown \mathrm{ReplaceLine}(B, i, p_4)$.

(19) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, and B be a matrix over D of dimension $n \times k$. Then $\mathrm{Segm}(A \frown B, \mathrm{Seg}\, n, \mathrm{Seg}\,\mathrm{width}\, A) = A$ and $\mathrm{Segm}(A \frown B, \mathrm{Seg}\, n, \mathrm{Seg}(\mathrm{width}\, A + \mathrm{width}\, B) \setminus \mathrm{Seg}\,\mathrm{width}\, A) = B$.

(20) For all matrices A, B over K such that $\mathrm{len}\, A = \mathrm{len}\, B$ holds $\mathrm{rk}(A) \leq \mathrm{rk}(A \frown B)$ and $\mathrm{rk}(B) \leq \mathrm{rk}(A \frown B)$.

(21) For all matrices A, B over K such that $\mathrm{len}\, A = \mathrm{len}\, B$ and $\mathrm{len}\, A = \mathrm{rk}(A)$ holds $\mathrm{rk}(A) = \mathrm{rk}(A \frown B)$.

(22) For all matrices A, B over K such that $\mathrm{len}\, A = \mathrm{len}\, B$ and $\mathrm{width}\, A = 0$ holds $A \frown B = B$ and $B \frown A = B$.

(23) For all matrices A, B over K such that $B = 0_K^{(\mathrm{len}\, A) \times m}$ holds $\mathrm{rk}(A) = \mathrm{rk}(A \frown B)$.

(24) Let A, B be matrices over K. Suppose $\mathrm{rk}(A) = \mathrm{rk}(A \frown B)$ and $\mathrm{len}\, A = \mathrm{len}\, B$. Let given N. Suppose $N \subseteq \mathrm{dom}\, A$ and for every i such that $i \in N$ holds $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$. Let given i. If $i \in N$, then $\mathrm{Line}(B, i) = \mathrm{width}\, B \mapsto 0_K$.

3. Basic Properties of two Transformations which Transform Finite Sequences to Matrices

For simplicity, we follow the rules: D is a non empty set, b_3 is a finite sequence of elements of D, b, f, g are finite sequences of elements of K, and M_1 is a matrix over D.

Let D be a non empty set and let b be a finite sequence of elements of D. The functor LineVec2Mx b yielding a matrix over D of dimension $1 \times \operatorname{len} b$ is defined by:

(Def. 1) $\text{LineVec2Mx}\, b = \langle b \rangle$.

The functor ColVec2Mx b yielding a matrix over D of dimension $\operatorname{len} b \times 1$ is defined by:

(Def. 2) $\text{ColVec2Mx}\, b = \langle b \rangle^{\mathrm{T}}$.

One can prove the following propositions:

(25) $M_1 = \text{LineVec2Mx}\, b_3$ iff $\text{Line}(M_1, 1) = b_3$ and $\operatorname{len} M_1 = 1$.

(26) If $\operatorname{len} M_1 \neq 0$ or $\operatorname{len} b_3 \neq 0$, then $M_1 = \text{ColVec2Mx}\, b_3$ iff $(M_1)_{\square,1} = b_3$ and $\operatorname{width} M_1 = 1$.

(27) If $\operatorname{len} f = \operatorname{len} g$, then $\text{LineVec2Mx}\, f + \text{LineVec2Mx}\, g = \text{LineVec2Mx}(f + g)$.

(28) If $\operatorname{len} f = \operatorname{len} g$, then $\text{ColVec2Mx}\, f + \text{ColVec2Mx}\, g = \text{ColVec2Mx}(f + g)$.

(29) $a \cdot \text{LineVec2Mx}\, f = \text{LineVec2Mx}(a \cdot f)$.

(30) $a \cdot \text{ColVec2Mx}\, f = \text{ColVec2Mx}(a \cdot f)$.

(31) $\text{LineVec2Mx}(k \mapsto 0_K) = 0_K^{1 \times k}$.

(32) $\text{ColVec2Mx}(k \mapsto 0_K) = 0_K^{k \times 1}$.

4. Basis Properties of the Solution of Linear Equations

Let us consider K and let us consider A, B. The set of solutions of A and B is a set and is defined as follows:

(Def. 3) The set of solutions of A and $B = \{X : \operatorname{len} X = \operatorname{width} A \ \wedge \ \operatorname{width} X = \operatorname{width} B \ \wedge \ A \cdot X = B\}$.

We now state a number of propositions:

(33) If the set of solutions of A and B is non empty, then $\operatorname{len} A = \operatorname{len} B$.

(34) If $X \in$ the set of solutions of A and B and $i \in \operatorname{Seg} \operatorname{width} X$ and $X_{\square,i} = \operatorname{len} X \mapsto 0_K$, then $B_{\square,i} = \operatorname{len} B \mapsto 0_K$.

(35) Suppose $X \in$ the set of solutions of A and B. Then $a \cdot X \in$ the set of solutions of A and $a \cdot B$ and $X \in$ the set of solutions of $a \cdot A$ and $a \cdot B$.

(36) If $a \neq 0_K$, then the set of solutions of A and $B =$ the set of solutions of $a \cdot A$ and $a \cdot B$.

(37) Suppose $X_1 \in$ the set of solutions of A and B_1 and $X_2 \in$ the set of solutions of A and B_2 and width $B_1 =$ width B_2. Then $X_1 + X_2 \in$ the set of solutions of A and $B_1 + B_2$.

(38) If $X \in$ the set of solutions of A' and B', then $X \in$ the set of solutions of $\mathrm{RLine}(A', i, a \cdot \mathrm{Line}(A', i))$ and $\mathrm{RLine}(B', i, a \cdot \mathrm{Line}(B', i))$.

(39) Suppose $X \in$ the set of solutions of A' and B' and $j \in \mathrm{Seg}\, m$ and $i \neq j$. Then $X \in$ the set of solutions of $\mathrm{RLine}(A', i, \mathrm{Line}(A', i) + a \cdot \mathrm{Line}(A', j))$ and $\mathrm{RLine}(B', i, \mathrm{Line}(B', i) + a \cdot \mathrm{Line}(B', j))$.

(40) Suppose $j \in \mathrm{Seg}\, m$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then the set of solutions of A' and $B' =$ the set of solutions of $\mathrm{RLine}(A', i, \mathrm{Line}(A', i) + a \cdot \mathrm{Line}(A', j))$ and $\mathrm{RLine}(B', i, \mathrm{Line}(B', i) + a \cdot \mathrm{Line}(B', j))$.

(41) If $X \in$ the set of solutions of A and B and $i \in \mathrm{dom}\, A$ and $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$, then $\mathrm{Line}(B, i) = \mathrm{width}\, B \mapsto 0_K$.

(42) Let n_1 be an element of $\mathbb{N}^n$. Suppose $\mathrm{rng}\, n_1 \subseteq \mathrm{dom}\, A$ and $n > 0$. Then the set of solutions of A and $B \subseteq$ the set of solutions of $\mathrm{Segm}(A, n_1, \mathrm{Sgm}\,\mathrm{Seg}\,\mathrm{width}\, A)$ and $\mathrm{Segm}(B, n_1, \mathrm{Sgm}\,\mathrm{Seg}\,\mathrm{width}\, B)$.

(43) Let n_1 be an element of $\mathbb{N}^n$. Suppose $\mathrm{rng}\, n_1 \subseteq \mathrm{dom}\, A = \mathrm{dom}\, B$ and $n > 0$ and for every i such that $i \in \mathrm{dom}\, A \setminus \mathrm{rng}\, n_1$ holds $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$ and $\mathrm{Line}(B, i) = \mathrm{width}\, B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of $\mathrm{Segm}(A, n_1, \mathrm{Sgm}\,\mathrm{Seg}\,\mathrm{width}\, A)$ and $\mathrm{Segm}(B, n_1, \mathrm{Sgm}\,\mathrm{Seg}\,\mathrm{width}\, B)$.

(44) Let given N. Suppose $N \subseteq \mathrm{dom}\, A$ and N is non empty. Then the set of solutions of A and $B \subseteq$ the set of solutions of $\mathrm{Segm}(A, N, \mathrm{Seg}\,\mathrm{width}\, A)$ and $\mathrm{Segm}(B, N, \mathrm{Seg}\,\mathrm{width}\, B)$.

(45) Let given N. Suppose $N \subseteq \mathrm{dom}\, A$ and N is non empty and $\mathrm{dom}\, A = \mathrm{dom}\, B$ and for every i such that $i \in \mathrm{dom}\, A \setminus N$ holds $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$ and $\mathrm{Line}(B, i) = \mathrm{width}\, B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of $\mathrm{Segm}(A, N, \mathrm{Seg}\,\mathrm{width}\, A)$ and $\mathrm{Segm}(B, N, \mathrm{Seg}\,\mathrm{width}\, B)$.

(46) Suppose $i \in \mathrm{dom}\, A$ and $\mathrm{len}\, A > 1$. Then the set of solutions of A and $B \subseteq$ the set of solutions of the deleting of i-row in A and the deleting of i -row in B.

(47) Let given A, B, i. Suppose $i \in \mathrm{dom}\, A$ and $\mathrm{len}\, A > 1$ and $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$ and $i \in \mathrm{dom}\, B$ and $\mathrm{Line}(B, i) = \mathrm{width}\, B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of the deleting of i-row in A and the deleting of i -row in B.

(48) Let A be a matrix over K of dimension $n \times m$, B be a matrix over K of dimension $n \times k$, and P be a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$. Then

(i) the set of solutions of A and $B \subseteq$ the set of solutions of $A \cdot P$ and $B \cdot P$, and

(ii) if P is one-to-one, then the set of solutions of A and B = the set of solutions of $A \cdot P$ and $B \cdot P$.

(49) Let A be a matrix over K of dimension $n \times m$ and given N. Suppose $\operatorname{card} N = n$ and $N \subseteq \operatorname{Seg} m$ and $\operatorname{Segm}(A, \operatorname{Seg} n, N) = I_K^{n \times n}$ and $n > 0$. Then there exists a matrix M_2 over K of dimension $m -' n \times m$ such that

(i) $\operatorname{Segm}(M_2, \operatorname{Seg}(m -' n), \operatorname{Seg} m \setminus N) = I_K^{(m-'n)\times(m-'n)}$,

(ii) $\operatorname{Segm}(M_2, \operatorname{Seg}(m -' n), N) = -(\operatorname{Segm}(A, \operatorname{Seg} n, \operatorname{Seg} m \setminus N))^{\mathrm{T}}$, and

(iii) for every l and for every matrix M over K of dimension $m \times l$ such that for every i such that $i \in \operatorname{Seg} l$ holds there exists j such that $j \in \operatorname{Seg}(m-'n)$ and $M_{\square,i} = \operatorname{Line}(M_2, j)$ or $M_{\square,i} = m \mapsto 0_K$ holds $M \in$ the set of solutions of A and $0_K^{n \times l}$.

(50) Let A be a matrix over K of dimension $n \times m$, B be a matrix over K of dimension $n \times l$, and given N. Suppose $\operatorname{card} N = n$ and $N \subseteq \operatorname{Seg} m$ and $n > 0$ and $\operatorname{Segm}(A, \operatorname{Seg} n, N) = I_K^{n \times n}$. Then there exists a matrix X over K of dimension $m \times l$ such that $\operatorname{Segm}(X, \operatorname{Seg} m \setminus N, \operatorname{Seg} l) = 0_K^{(m-'n)\times l}$ and $\operatorname{Segm}(X, N, \operatorname{Seg} l) = B$ and $X \in$ the set of solutions of A and B.

(51) Let A be a matrix over K of dimension $0 \times n$ and B be a matrix over K of dimension $0 \times m$. Then the set of solutions of A and $B = \{\emptyset\}$.

(52) For every matrix B over K such that the set of solutions of $0_K^{n \times k}$ and B is non empty holds $B = 0_K^{n \times (\operatorname{width} B)}$.

(53) Let A be a matrix over K of dimension $n \times k$ and B be a matrix over K of dimension $n \times m$. Suppose $n > 0$. Suppose $x \in$ the set of solutions of A and B. Then x is a matrix over K of dimension $k \times m$.

(54) Suppose $n > 0$ and $k > 0$. Then the set of solutions of $0_K^{n \times k}$ and $0_K^{n \times m} = \{X : X$ ranges over matrices over K of dimension $k \times m\}$.

(55) If $n > 0$ and the set of solutions of $0_K^{n \times 0}$ and $0_K^{n \times m}$ is non empty, then $m = 0$.

(56) The set of solutions of $0_K^{n \times 0}$ and $0_K^{n \times 0} = \{\emptyset\}$.

5. Gaussian Eliminations

In this article we present several logical schemes. The scheme *GAUSS1* deals with a field $\mathcal{A}$, natural numbers $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a matrix $\mathcal{E}$ over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$, a matrix $\mathcal{F}$ over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$, a 4-ary functor $\mathcal{F}$ yielding a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a matrix A' over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ and there exists a matrix B' over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$ and there exists a without zero finite subset N of $\mathbb{N}$ such that

$N \subseteq \operatorname{Seg} \mathcal{C}$ and $\operatorname{rk}(\mathcal{E}) = \operatorname{rk}(A')$ and $\operatorname{rk}(\mathcal{E}) = \operatorname{card} N$ and $\mathcal{P}[A', B']$ and $\operatorname{Segm}(A', \operatorname{Seg} \operatorname{card} N, N)$ is diagonal and for every i

such that $i \in \operatorname{Seg} \operatorname{card} N$ holds $A'_{i,(\operatorname{Sgm} N)_i} \neq 0_{\mathcal{A}}$ and for every i such that $i \in \operatorname{dom} A'$ and $i > \operatorname{card} N$ holds $\operatorname{Line}(A', i) = \mathcal{C} \mapsto 0_{\mathcal{A}}$ and for all i, j such that $i \in \operatorname{Seg} \operatorname{card} N$ and $j \in \operatorname{Seg} \operatorname{width} A'$ and $j < (\operatorname{Sgm} N)(i)$ holds $A'_{i,j} = 0_{\mathcal{A}}$

provided the parameters meet the following requirements:

- $\mathcal{P}[\mathcal{E}, \mathcal{F}]$, and
- Let A' be a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ and B' be a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$. Suppose $\mathcal{P}[A', B']$. Let given i, j. Suppose $i \neq j$ and $j \in \operatorname{dom} A'$. Let a be an element of $\mathcal{A}$. Then $\mathcal{P}[\operatorname{RLine}(A', i, \operatorname{Line}(A', i) + a \cdot \operatorname{Line}(A', j)), \mathcal{F}(B', i, j, a)]$.

The scheme *GAUSS2* deals with a field $\mathcal{A}$, natural numbers $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a matrix $\mathcal{E}$ over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$, a matrix $\mathcal{F}$ over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$, a 4-ary functor $\mathcal{F}$ yielding a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a matrix A' over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ and there exists a matrix B' over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$ and there exists a without zero finite subset N of $\mathbb{N}$ such that

$N \subseteq \operatorname{Seg} \mathcal{C}$ and $\operatorname{rk}(\mathcal{E}) = \operatorname{rk}(A')$ and $\operatorname{rk}(\mathcal{E}) = \operatorname{card} N$ and $\mathcal{P}[A', B']$ and $\operatorname{Segm}(A', \operatorname{Seg} \operatorname{card} N, N) = I_{\mathcal{A}}^{\operatorname{card} N \times \operatorname{card} N}$ and for every i such that $i \in \operatorname{dom} A'$ and $i > \operatorname{card} N$ holds $\operatorname{Line}(A', i) = \mathcal{C} \mapsto 0_{\mathcal{A}}$ and for all i, j such that $i \in \operatorname{Seg} \operatorname{card} N$ and $j \in \operatorname{Seg} \operatorname{width} A'$ and $j < (\operatorname{Sgm} N)(i)$ holds $A'_{i,j} = 0_{\mathcal{A}}$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\mathcal{E}, \mathcal{F}]$, and
- Let A' be a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ and B' be a matrix over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{D}$. Suppose $\mathcal{P}[A', B']$. Let a be an element of $\mathcal{A}$ and given i, j. If $j \in \operatorname{dom} A'$ and if $i = j$, then $a \neq -\mathbf{1}_{\mathcal{A}}$, then $\mathcal{P}[\operatorname{RLine}(A', i, \operatorname{Line}(A', i) + a \cdot \operatorname{Line}(A', j)), \mathcal{F}(B', i, j, a)]$.

6. The Main Theorem

We now state the proposition

(57) Let A, B be matrices over K. Suppose $\operatorname{len} A = \operatorname{len} B$ and if $\operatorname{width} A = 0$, then $\operatorname{width} B = 0$. Then $\operatorname{rk}(A) = \operatorname{rk}(A \frown B)$ if and only if the set of solutions of A and B is non empty.

7. Space of Solutions of Linear Equations

Let us consider K, let A be a matrix over K, and let b be a finite sequence of elements of K. The set of solutions of A and b is defined by:

(Def. 4) The set of solutions of A and $b = \{f : \mathrm{ColVec2Mx}\, f \in$ the set of solutions of A and $\mathrm{ColVec2Mx}\, b\}$.

We now state two propositions:

(58) For every x such that $x \in$ the set of solutions of A and $\mathrm{ColVec2Mx}\, b$ there exists f such that $x = \mathrm{ColVec2Mx}\, f$ and $\mathrm{len}\, f = \mathrm{width}\, A$.

(59) For every f such that $\mathrm{ColVec2Mx}\, f \in$ the set of solutions of A and $\mathrm{ColVec2Mx}\, b$ holds $\mathrm{len}\, f = \mathrm{width}\, A$.

Let us consider K, let A be a matrix over K, and let b be a finite sequence of elements of K. Then the set of solutions of A and b is a subset of the width A-dimension vector space over K.

Let us consider K, let A be a matrix over K, and let k be an element of $\mathbb{N}$. Note that the set of solutions of A and $k \mapsto 0_K$ is linearly closed.

We now state two propositions:

(60) If the set of solutions of A and b is non empty and $\mathrm{width}\, A = 0$, then $\mathrm{len}\, A = 0$.

(61) If $\mathrm{width}\, A \neq 0$ or $\mathrm{len}\, A = 0$, then the set of solutions of A and $\mathrm{len}\, A \mapsto 0_K$ is non empty.

Let us consider K and let A be a matrix over K. Let us assume that if $\mathrm{width}\, A = 0$, then $\mathrm{len}\, A = 0$. The space of solutions of A is a strict subspace of the width A-dimension vector space over K and is defined by:

(Def. 5) The carrier of the space of solutions of A = the set of solutions of A and $\mathrm{len}\, A \mapsto 0_K$.

The following propositions are true:

(62) Let A be a matrix over K and b be a finite sequence of elements of K. Suppose the set of solutions of A and b is non empty. Then the set of solutions of A and b is a coset of the space of solutions of A.

(63) Let given A. Suppose if $\mathrm{width}\, A = 0$, then $\mathrm{len}\, A = 0$ and $\mathrm{rk}(A) = 0$. Then the space of solutions of A = the width A-dimension vector space over K.

(64) For every A such that the space of solutions of A = the width A-dimension vector space over K holds $\mathrm{rk}(A) = 0$.

(65) Let given i, j. Suppose $j \in \mathrm{Seg}\, m$ and $n > 0$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then the space of solutions of A' = the space of solutions of $\mathrm{RLine}(A', i, \mathrm{Line}(A', i) + a \cdot \mathrm{Line}(A', j))$.

(66) Let given N. Suppose $N \subseteq \mathrm{dom}\, A$ and N is non empty and $\mathrm{width}\, A > 0$ and for every i such that $i \in \mathrm{dom}\, A \setminus N$ holds $\mathrm{Line}(A, i) = \mathrm{width}\, A \mapsto 0_K$. Then the space of solutions of A = the space of solutions of $\mathrm{Segm}(A, N, \mathrm{Seg}\,\mathrm{width}\, A)$.

(67) Let A be a matrix over K of dimension $n \times m$ and given N. Suppose $\mathrm{card}\, N = n$ and $N \subseteq \mathrm{Seg}\, m$ and $\mathrm{Segm}(A, \mathrm{Seg}\, n, N) = I_K^{n \times n}$ and $n > 0$

and $m -' n > 0$. Then there exists a matrix M_2 over K of dimension $m -' n \times m$ such that $\mathrm{Segm}(M_2, \mathrm{Seg}(m -' n), \mathrm{Seg}\, m \setminus N) = I_K^{(m-'n)\times(m-'n)}$ and $\mathrm{Segm}(M_2, \mathrm{Seg}(m -' n), N) = -(\mathrm{Segm}(A, \mathrm{Seg}\, n, \mathrm{Seg}\, m \setminus N))^{\mathrm{T}}$ and $\mathrm{Lin}(\mathrm{lines}(M_2))$ = the space of solutions of A.

(68) For every A such that if $\mathrm{width}\, A = 0$, then $\mathrm{len}\, A = 0$ holds dim(the space of solutions of A) $= \mathrm{width}\, A - \mathrm{rk}(A)$.

(69) Let M be a matrix over K of dimension $n \times m$ and given i, j, a. Suppose M is without repeated line and $j \in \mathrm{dom}\, M$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then $\mathrm{Lin}(\mathrm{lines}(M)) = \mathrm{Lin}(\mathrm{lines}(\mathrm{RLine}(M, i, \mathrm{Line}(M, i) + a \cdot \mathrm{Line}(M, j))))$.

(70) Let W be a subspace of the m-dimension vector space over K. Then there exists a matrix A over K of dimension $\dim(W) \times m$ and there exists a without zero finite subset N of $\mathbb{N}$ such that $N \subseteq \mathrm{Seg}\, m$ and $\dim(W) = \mathrm{card}\, N$ and $\mathrm{Segm}(A, \mathrm{Seg}\dim(W), N) = I_K^{\dim(W)\times\dim(W)}$ and $\mathrm{rk}(A) = \dim(W)$ and $\mathrm{lines}(A)$ is a basis of W.

(71) Let W be a strict subspace of the m-dimension vector space over K. Suppose $\dim(W) < m$. Then there exists a matrix A over K of dimension $m -' \dim(W) \times m$ and there exists a without zero finite subset N of $\mathbb{N}$ such that $\mathrm{card}\, N = m -' \dim(W)$ and $N \subseteq \mathrm{Seg}\, m$ and $\mathrm{Segm}(A, \mathrm{Seg}(m -' \dim(W)), N) = I_K^{(m-'\dim(W))\times(m-'\dim(W))}$ and W = the space of solutions of A.

(72) Let A, B be matrices over K. Suppose $\mathrm{width}\, A = \mathrm{len}\, B$ and if $\mathrm{width}\, A = 0$, then $\mathrm{len}\, A = 0$ and if $\mathrm{width}\, B = 0$, then $\mathrm{len}\, B = 0$. Then the space of solutions of B is a subspace of the space of solutions of $A \cdot B$.

(73) For all matrices A, B over K such that $\mathrm{width}\, A = \mathrm{len}\, B$ holds $\mathrm{rk}(A \cdot B) \leq \mathrm{rk}(A)$ and $\mathrm{rk}(A \cdot B) \leq \mathrm{rk}(B)$.

(74) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K. Suppose $\mathrm{Det}\, A \neq 0_K$ and $\mathrm{width}\, A = \mathrm{len}\, B$ and if $\mathrm{width}\, B = 0$, then $\mathrm{len}\, B = 0$. Then the space of solutions of B = the space of solutions of $A \cdot B$.

(75) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K. If $\mathrm{width}\, A = \mathrm{len}\, B$ and $\mathrm{Det}\, A \neq 0_K$, then $\mathrm{rk}(A \cdot B) = \mathrm{rk}(B)$.

(76) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K. If $\mathrm{len}\, A = \mathrm{width}\, B$ and $\mathrm{Det}\, A \neq 0_K$, then $\mathrm{rk}(B \cdot A) = \mathrm{rk}(B)$.

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Received December 18, 2007