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Convex Sets and Convex Combinations on Complex Linear Spaces

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Summary. In this article, convex sets, convex combinations and convex hulls on complex linear spaces are introduced.

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The articles [19], [18], [9], [23], [24], [6], [25], [7], [20], [3], [22], [17], [2], [11], [8], [1], [5], [10], [14], [15], [4], [16], [21], [12], and [13] provide the terminology and notation for this paper.

1. Complex Linear Combinations

Let V be a non empty zero structure. An element of $\mathbb{C}^{\text{the carrier of } V}$ is said to be a $\mathbb{C}$-linear combination of V if:

(Def. 1) There exists a finite subset T of V such that for every element v of V such that $v \notin T$ holds $\mathrm{it}(v) = 0$.

Let V be a non empty additive loop structure and let L be an element of $\mathbb{C}^{\text{the carrier of } V}$. The support of L yielding a subset of V is defined by:

(Def. 2) The support of $L = \{v \in V \colon L(v) \neq 0_{\mathbb{C}}\}$.

Let V be a non empty additive loop structure and let L be a $\mathbb{C}$-linear combination of V. One can check that the support of L is finite.

The following proposition is true

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(1) Let V be a non empty additive loop structure, L be a $\mathbb{C}$-linear combination of V, and v be an element of V. Then $L(v) = 0_{\mathbb{C}}$ if and only if $v \notin$ the support of L.

Let V be a non empty additive loop structure. The functor ZeroCLC V yields a $\mathbb{C}$-linear combination of V and is defined by:

(Def. 3) The support of ZeroCLC $V = \emptyset$.

Let V be a non empty additive loop structure. Note that the support of ZeroCLC V is empty.

We now state the proposition

(2) For every non empty additive loop structure V and for every element v of V holds $(\mathrm{ZeroCLC}\,V)(v) = 0_{\mathbb{C}}$.

Let V be a non empty additive loop structure and let A be a subset of V. A $\mathbb{C}$-linear combination of V is said to be a $\mathbb{C}$-linear combination of A if:

(Def. 4) The support of it $\subseteq A$.

Next we state three propositions:

(3) Let V be a non empty additive loop structure, A, B be subsets of V, and l be a $\mathbb{C}$-linear combination of A. If $A \subseteq B$, then l is a $\mathbb{C}$-linear combination of B.

(4) Let V be a non empty additive loop structure and A be a subset of V. Then ZeroCLC V is a $\mathbb{C}$-linear combination of A.

(5) Let V be a non empty additive loop structure and l be a $\mathbb{C}$-linear combination of $\emptyset_{\text{the carrier of } V}$. Then $l = \mathrm{ZeroCLC}\,V$.

In the sequel i is a natural number.

Let V be a non empty CLS structure, let F be a finite sequence of elements of the carrier of V, and let f be a function from the carrier of V into $\mathbb{C}$. The functor $f\,F$ yields a finite sequence of elements of the carrier of V and is defined as follows:

(Def. 5) $\mathrm{len}(f\,F) = \mathrm{len}\,F$ and for every i such that $i \in \mathrm{dom}(f\,F)$ holds $(f\,F)(i) = f(F_i) \cdot F_i$.

For simplicity, we follow the rules: V denotes a non empty CLS structure, v, v_1, v_2, v_3 denote vectors of V, A denotes a subset of V, l denotes a $\mathbb{C}$-linear combination of A, x denotes a set, a, b denote complex numbers, F denotes a finite sequence of elements of the carrier of V, and f denotes a function from the carrier of V into $\mathbb{C}$.

The following propositions are true:

(6) If $x \in \mathrm{dom}\,F$ and $v = F(x)$, then $(f\,F)(x) = f(v) \cdot v$.

(7) $f\,\varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.

(8) $f\,\langle v \rangle = \langle f(v) \cdot v \rangle$.

(9) $f\,\langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(10) $f\,\langle v_1, v_2, v_3\rangle = \langle f(v_1)\cdot v_1, f(v_2)\cdot v_2, f(v_3)\cdot v_3\rangle$.

In the sequel L, L_1, L_2, L_3 are $\mathbb{C}$-linear combinations of V.

Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and let L be a $\mathbb{C}$-linear combination of V. The functor $\sum L$ yields an element of V and is defined by the condition (Def. 6).

(Def. 6) There exists a finite sequence F of elements of the carrier of V such that F is one-to-one and $\operatorname{rng} F =$ the support of L and $\sum L = \sum L\,F$.

One can prove the following propositions:

(11) For every Abelian add-associative right zeroed right complementable non empty CLS structure V holds $\sum \mathrm{ZeroCLC}\, V = 0_V$.

(12) Let V be a complex linear space and A be a subset of V. Suppose $A \neq \emptyset$. Then A is linearly closed if and only if for every $\mathbb{C}$-linear combination l of A holds $\sum l \in A$.

(13) Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and l be a $\mathbb{C}$-linear combination of $\emptyset_{\text{the carrier of } V}$. Then $\sum l = 0_V$.

(14) Let V be a complex linear space, v be a vector of V, and l be a $\mathbb{C}$-linear combination of $\{v\}$. Then $\sum l = l(v)\cdot v$.

(15) Let V be a complex linear space and v_1, v_2 be vectors of V. Suppose $v_1 \neq v_2$. Let l be a $\mathbb{C}$-linear combination of $\{v_1, v_2\}$. Then $\sum l = l(v_1)\cdot v_1 + l(v_2)\cdot v_2$.

(16) Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and L be a $\mathbb{C}$-linear combination of V. If the support of $L = \emptyset$, then $\sum L = 0_V$.

(17) Let V be a complex linear space, L be a $\mathbb{C}$-linear combination of V, and v be a vector of V. If the support of $L = \{v\}$, then $\sum L = L(v)\cdot v$.

(18) Let V be a complex linear space, L be a $\mathbb{C}$-linear combination of V, and v_1, v_2 be vectors of V. If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1)\cdot v_1 + L(v_2)\cdot v_2$.

Let V be a non empty additive loop structure and let L_1, L_2 be $\mathbb{C}$-linear combinations of V. Let us observe that $L_1 = L_2$ if and only if:

(Def. 7) For every element v of V holds $L_1(v) = L_2(v)$.

Let V be a non empty additive loop structure and let L_1, L_2 be $\mathbb{C}$-linear combinations of V. Then $L_1 + L_2$ is a $\mathbb{C}$-linear combination of V and it can be characterized by the condition:

(Def. 8) For every element v of V holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

One can prove the following propositions:

(19) The support of $L_1 + L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(20) Suppose L_1 is a $\mathbb{C}$-linear combination of A and L_2 is a $\mathbb{C}$-linear combination of A. Then $L_1 + L_2$ is a $\mathbb{C}$-linear combination of A.

Let us consider V, A and let L_1, L_2 be $\mathbb{C}$-linear combinations of A. Then $L_1 + L_2$ is a $\mathbb{C}$-linear combination of A.

The following three propositions are true:

(21) For every non empty additive loop structure V and for all $\mathbb{C}$-linear combinations L_1, L_2 of V holds $L_1 + L_2 = L_2 + L_1$.

(22) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(23) $L + \mathrm{ZeroCLC}\, V = L$.

Let us consider V, a and let us consider L. The functor $a \cdot L$ yielding a $\mathbb{C}$-linear combination of V is defined as follows:

(Def. 9) For every v holds $(a \cdot L)(v) = a \cdot L(v)$.

One can prove the following propositions:

(24) If $a \neq 0_{\mathbb{C}}$, then the support of $a \cdot L =$ the support of L.

(25) $0_{\mathbb{C}} \cdot L = \mathrm{ZeroCLC}\, V$.

(26) If L is a $\mathbb{C}$-linear combination of A, then $a \cdot L$ is a $\mathbb{C}$-linear combination of A.

(27) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(28) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(29) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(30) $1_{\mathbb{C}} \cdot L = L$.

Let us consider V, L. The functor $-L$ yielding a $\mathbb{C}$-linear combination of V is defined as follows:

(Def. 10) $-L = (-1_{\mathbb{C}}) \cdot L$.

We now state three propositions:

(31) $(-L)(v) = -L(v)$.

(32) If $L_1 + L_2 = \mathrm{ZeroCLC}\, V$, then $L_2 = -L_1$.

(33) $--L = L$.

Let us consider V and let us consider L_1, L_2. The functor $L_1 - L_2$ yields a $\mathbb{C}$-linear combination of V and is defined by:

(Def. 11) $L_1 - L_2 = L_1 + -L_2$.

One can prove the following propositions:

(34) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(35) The support of $L_1 - L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(36) Suppose L_1 is a $\mathbb{C}$-linear combination of A and L_2 is a $\mathbb{C}$-linear combination of A. Then $L_1 - L_2$ is a $\mathbb{C}$-linear combination of A.

(37) $L - L = \mathrm{ZeroCLC}\, V$.

Let us consider V. The functor $\mathbb{C}$-LinComb V yields a set and is defined as follows:

(Def. 12) $x \in \mathbb{C}\text{-LinComb}\,V$ iff x is a $\mathbb{C}$-linear combination of V.

Let us consider V. One can verify that $\mathbb{C}$-LinComb V is non empty.

In the sequel e, e_1, e_2 denote elements of $\mathbb{C}$-LinComb V.

Let us consider V and let us consider e. The functor ${}^{@}e$ yields a $\mathbb{C}$-linear combination of V and is defined as follows:

(Def. 13) ${}^{@}e = e$.

Let us consider V and let us consider L. The functor ${}^{@}L$ yielding an element of $\mathbb{C}$-LinComb V is defined by:

(Def. 14) ${}^{@}L = L$.

Let us consider V. The functor $\mathbb{C}$-LCAdd V yields a binary operation on $\mathbb{C}$-LinComb V and is defined by:

(Def. 15) For all e_1, e_2 holds $(\mathbb{C}\text{-LCAdd}\,V)(e_1, e_2) = ({}^{@}e_1) + {}^{@}e_2$.

Let us consider V. The functor $\mathbb{C}$-LCMult V yields a function from $\mathbb{C} \times \mathbb{C}\text{-LinComb}\,V$ into $\mathbb{C}$-LinComb V and is defined as follows:

(Def. 16) For all a, e holds $(\mathbb{C}\text{-LCMult}\,V)(\langle a, e\rangle) = a \cdot ({}^{@}e)$.

Let us consider V. The functor L$\mathbb{C}$-CLSpace V yielding a complex linear space is defined by:

(Def. 17) $\mathrm{L}\mathbb{C}\text{-CLSpace}\,V = \langle \mathbb{C}\text{-LinComb}\,V, {}^{@}\mathrm{ZeroCLC}\,V, \mathbb{C}\text{-LCAdd}\,V, \mathbb{C}\text{-LCMult}\,V\rangle$.

Let us consider V. Note that L$\mathbb{C}$-CLSpace V is strict and non empty.

We now state four propositions:

(38) ${L_1}^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} + {L_2}^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} = L_1 + L_2$.

(39) $a \cdot L^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} = a \cdot L$.

(40) $-L^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} = -L$.

(41) ${L_1}^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} - {L_2}^{\mathrm{L}\mathbb{C}\text{-CLSpace}\,V} = L_1 - L_2$.

Let us consider V and let us consider A. The functor L$\mathbb{C}$-CLSpace A yielding a strict subspace of L$\mathbb{C}$-CLSpace V is defined as follows:

(Def. 18) The carrier of $\mathrm{L}\mathbb{C}\text{-CLSpace}\,A = \{l\}$.

2. Preliminaries for Complex Convex Sets

Let V be a complex linear space and let W be a subspace of V. The functor $\mathrm{Up}(W)$ yields a subset of V and is defined by:

(Def. 19) $\mathrm{Up}(W) =$ the carrier of W.

Let V be a complex linear space and let W be a subspace of V. One can check that $\mathrm{Up}(W)$ is non empty.

Let V be a non empty CLS structure and let S be a subset of V. We say that S is affine if and only if the condition (Def. 20) is satisfied.

(Def. 20) Let x, y be vectors of V and z be a complex number. If there exists a real number a such that $a = z$ and $x, y \in S$, then $(1_{\mathbb{C}} - z) \cdot x + z \cdot y \in S$.

Let V be a complex linear space. The functor Ω_V yields a strict subspace of V and is defined as follows:

(Def. 21) $\Omega_V =$ the CLS structure of V.

Let V be a non empty CLS structure. Observe that Ω_V is affine and $\emptyset_V$ is affine.

Let V be a non empty CLS structure. One can check that there exists a subset of V which is non empty and affine and there exists a subset of V which is empty and affine.

We now state three propositions:

(42) For every real number a and for every complex number z holds $\Re(a \cdot z) = a \cdot \Re(z)$.

(43) For every real number a and for every complex number z holds $\Im(a \cdot z) = a \cdot \Im(z)$.

(44) For every real number a and for every complex number z such that $0 \leq a \leq 1$ holds $|a \cdot z| = a \cdot |z|$ and $|(1_{\mathbb{C}} - a) \cdot z| = (1_{\mathbb{C}} - a) \cdot |z|$.

3. Complex Convex Sets

Let V be a non empty CLS structure, let M be a subset of V, and let r be an element of $\mathbb{C}$. The functor $r \cdot M$ yielding a subset of V is defined by:

(Def. 22) $r \cdot M = \{r \cdot v; v$ ranges over elements of V: $v \in M\}$.

Let V be a non empty CLS structure and let M be a subset of V. We say that M is convex if and only if the condition (Def. 23) is satisfied.

(Def. 23) Let u, v be vectors of V and z be a complex number. Suppose there exists a real number r such that $z = r$ and $0 < r < 1$ and $u, v \in M$. Then $z \cdot u + (1_{\mathbb{C}} - z) \cdot v \in M$.

One can prove the following propositions:

(45) Let V be a complex linear space-like non empty CLS structure, M be a subset of V, and z be a complex number. If M is convex, then $z \cdot M$ is convex.

(46) Let V be an Abelian add-associative complex linear space-like non empty CLS structure and M, N be subsets of V. If M is convex and N is convex, then $M + N$ is convex.

(47) Let V be a complex linear space and M, N be subsets of V. If M is convex and N is convex, then $M - N$ is convex.

(48) Let V be a non empty CLS structure and M be a subset of V. Then M is convex if and only if for every complex number z such that there exists a real number r such that $z = r$ and $0 < r < 1$ holds $z \cdot M + (1_{\mathbb{C}} - z) \cdot M \subseteq M$.

(49) Let V be an Abelian non empty CLS structure and M be a subset of V. Suppose M is convex. Let z be a complex number. If there exists a real number r such that $z = r$ and $0 < r < 1$, then $(1_{\mathbb{C}} - z) \cdot M + z \cdot M \subseteq M$.

(50) Let V be an Abelian add-associative complex linear space-like non empty CLS structure and M, N be subsets of V. Suppose M is convex and N is convex. Let z be a complex number. If there exists a real number r such that $z = r$, then $z \cdot M + (1_{\mathbb{C}} - z) \cdot N$ is convex.

(51) For every complex linear space-like non empty CLS structure V and for every subset M of V holds $1_{\mathbb{C}} \cdot M = M$.

(52) For every complex linear space V and for every non empty subset M of V holds $0_{\mathbb{C}} \cdot M = \{0_V\}$.

(53) For every add-associative non empty additive loop structure V and for all subsets M_1, M_2, M_3 of V holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(54) Let V be a complex linear space-like non empty CLS structure, M be a subset of V, and z_1, z_2 be complex numbers. Then $z_1 \cdot (z_2 \cdot M) = (z_1 \cdot z_2) \cdot M$.

(55) Let V be a complex linear space-like non empty CLS structure, M_1, M_2 be subsets of V, and z be a complex number. Then $z \cdot (M_1 + M_2) = z \cdot M_1 + z \cdot M_2$.

(56) Let V be a complex linear space, M be a subset of V, and v be a vector of V. Then M is convex if and only if $v + M$ is convex.

(57) For every complex linear space V holds $\mathrm{Up}(\mathbf{0}_V)$ is convex.

(58) For every complex linear space V holds $\mathrm{Up}(\Omega_V)$ is convex.

(59) For every non empty CLS structure V and for every subset M of V such that $M = \emptyset$ holds M is convex.

(60) Let V be an Abelian add-associative complex linear space-like non empty CLS structure, M_1, M_2 be subsets of V, and z_1, z_2 be complex numbers. If M_1 is convex and M_2 is convex, then $z_1 \cdot M_1 + z_2 \cdot M_2$ is convex.

(61) Let V be a complex linear space-like non empty CLS structure, M be a subset of V, and z_1, z_2 be complex numbers. Then $(z_1 + z_2) \cdot M \subseteq z_1 \cdot M + z_2 \cdot M$.

(62) Let V be a non empty CLS structure, M, N be subsets of V, and z be a complex number. If $M \subseteq N$, then $z \cdot M \subseteq z \cdot N$.

(63) For every non empty CLS structure V and for every empty subset M of V and for every complex number z holds $z \cdot M = \emptyset$.

(64) Let V be a non empty additive loop structure, M be an empty subset of V, and N be a subset of V. Then $M + N = \emptyset$.

(65) For every right zeroed non empty additive loop structure V and for every subset M of V holds $M + \{0_V\} = M$.

(66) Let V be a complex linear space, M be a subset of V, and z_1, z_2 be complex numbers. Suppose there exist real numbers r_1, r_2 such that $z_1 = r_1$ and $z_2 = r_2$ and $r_1 \geq 0$ and $r_2 \geq 0$ and M is convex. Then $z_1 \cdot M + z_2 \cdot M = (z_1 + z_2) \cdot M$.

(67) Let V be an Abelian add-associative complex linear space-like non empty CLS structure, M_1, M_2, M_3 be subsets of V, and z_1, z_2, z_3 be complex numbers. If M_1 is convex and M_2 is convex and M_3 is convex, then $z_1 \cdot M_1 + z_2 \cdot M_2 + z_3 \cdot M_3$ is convex.

(68) Let V be a non empty CLS structure and F be a family of subsets of V. Suppose that for every subset M of V such that $M \in F$ holds M is convex. Then $\bigcap F$ is convex.

(69) For every non empty CLS structure V and for every subset M of V such that M is affine holds M is convex.

Let V be a non empty CLS structure. One can check that there exists a subset of V which is non empty and convex.

Let V be a non empty CLS structure. Observe that there exists a subset of V which is empty and convex.

One can prove the following propositions:

(70) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Re((u|v)) \geq r\}$, then M is convex.

(71) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Re((u|v)) > r\}$, then M is convex.

(72) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Re((u|v)) \leq r\}$, then M is convex.

(73) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Re((u|v)) < r\}$, then M is convex.

(74) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Im((u|v)) \geq r\}$, then M is convex.

(75) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Im((u|v)) > r\}$, then M is convex.

(76) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number.

If $M = \{u; u$ ranges over vectors of V: $\Im((u|v)) \leq r\}$, then M is convex.

(77) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $\Im((u|v)) < r\}$, then M is convex.

(78) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $|(u|v)| \leq r\}$, then M is convex.

(79) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $|(u|v)| < r\}$, then M is convex.

4. Complex Convex Combinations

Let V be a complex linear space and let L be a $\mathbb{C}$-linear combination of V. We say that L is convex if and only if the condition (Def. 24) is satisfied.

(Def. 24) There exists a finite sequence F of elements of the carrier of V such that

(i) F is one-to-one,

(ii) $\operatorname{rng} F =$ the support of L, and

(iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = \operatorname{len} F$ and $\sum f = 1$ and for every natural number n such that $n \in \operatorname{dom} f$ holds $f(n) = L(F(n))$ and $f(n) \geq 0$.

We now state several propositions:

(80) Let V be a complex linear space and L be a $\mathbb{C}$-linear combination of V. If L is convex, then the support of $L \neq \emptyset$.

(81) Let V be a complex linear space, L be a $\mathbb{C}$-linear combination of V, and v be a vector of V. Suppose L is convex and there exists a real number r such that $r = L(v)$ and $r \leq 0$. Then $v \notin$ the support of L.

(82) For every complex linear space V and for every $\mathbb{C}$-linear combination L of V such that L is convex holds $L \neq \operatorname{ZeroCLC} V$.

(83) Let V be a complex linear space, v be a vector of V, and L be a $\mathbb{C}$-linear combination of V. Suppose L is convex and the support of $L = \{v\}$. Then there exists a real number r such that $r = L(v)$ and $r = 1$ and $\sum L = L(v) \cdot v$.

(84) Let V be a complex linear space, v_1, v_2 be vectors of V, and L be a $\mathbb{C}$-linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then there exist real numbers r_1, r_2 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_1 + r_2 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(85) Let V be a complex linear space, v_1, v_2, v_3 be vectors of V, and L be a $\mathbb{C}$-linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2, v_3\}$ and $v_1 \neq v_2 \neq v_3 \neq v_1$. Then

(i) there exist real numbers r_1, r_2, r_3 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_3 = L(v_3)$ and $r_1 + r_2 + r_3 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $r_3 \geq 0$, and

(ii) $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

(86) Let V be a complex linear space, v be a vector of V, and L be a $\mathbb{C}$-linear combination of $\{v\}$. Suppose L is convex. Then there exists a real number r such that $r = L(v)$ and $r = 1$ and $\sum L = L(v) \cdot v$.

(87) Let V be a complex linear space, v_1, v_2 be vectors of V, and L be a $\mathbb{C}$-linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is convex. Then there exist real numbers r_1, r_2 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(88) Let V be a complex linear space, v_1, v_2, v_3 be vectors of V, and L be a $\mathbb{C}$-linear combination of $\{v_1, v_2, v_3\}$. Suppose $v_1 \neq v_2 \neq v_3 \neq v_1$ and L is convex. Then

(i) there exist real numbers r_1, r_2, r_3 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_3 = L(v_3)$ and $r_1 + r_2 + r_3 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $r_3 \geq 0$, and

(ii) $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

5. Complex Convex Hull

Let V be a non empty CLS structure and let M be a subset of V. The functor Convex-Family M yielding a family of subsets of V is defined by:

(Def. 25) For every subset N of V holds $N \in$ Convex-Family M iff N is convex and $M \subseteq N$.

Let V be a non empty CLS structure and let M be a subset of V. The functor $\operatorname{conv} M$ yielding a convex subset of V is defined as follows:

(Def. 26) $\operatorname{conv} M = \bigcap \text{Convex-Family}\, M$.

The following proposition is true

(89) Let V be a non empty CLS structure, M be a subset of V, and N be a convex subset of V. If $M \subseteq N$, then $\operatorname{conv} M \subseteq N$.

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