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Ramsey's Theorem

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Summary. The goal of this article is to formalize two versions of Ramsey's theorem. The theorems are not phrased in the usually pictorial representation of a coloured graph but use a set-theoretic terminology. After some useful lemma, the second section presents a generalization of Ramsey's theorem on infinite set closely following the book [9]. The last section includes the formalization of the theorem in a more known version (see [1]).

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The notation and terminology used here are introduced in the following papers: [15], [16], [17], [4], [3], [6], [12], [7], [2], [5], [8], [14], [13], [10], and [11].

1. Preliminaries

For simplicity, we adopt the following convention: n, m, k are natural numbers, X, Y, Z are sets, f is a function from X into Y, and H is a subset of X.

Let us consider X, Y, H and let P be a partition of $[X]^Y$. We say that H is homogeneous for P if and only if:

(Def. 1) There exists an element p of P such that $[H]^Y \subseteq p$.

Let us consider n and let X be an infinite set. One can check that $[X]^n$ is non empty.

Let us consider n, X, Y, f. Let us assume that f is one-to-one and $\overline{\overline{n}} \subseteq \overline{\overline{X}}$ and X is non empty and Y is non empty. The functor $f||^n$ yields a function from $[X]^n$ into $[Y]^n$ and is defined by:

(Def. 2) For every element x of $[X]^n$ holds $(f||^n)(x) = f^\circ x$.

Next we state four propositions:

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(1) If f is one-to-one and $\overline{\overline{n}} \subseteq \overline{\overline{X}}$ and X is non empty and Y is non empty, then $[f^\circ H]^n = (f||^n)^\circ([H]^n)$.

(2) If X is infinite and $X \subseteq \omega$, then $\overline{\overline{X}} = \omega$.

(3) If X is infinite, then $X \cup Y$ is infinite.

(4) If X is infinite and Y is finite, then $X \setminus Y$ is infinite.

Let X be an infinite set and let Y be a set. Note that $X \cup Y$ is infinite.

Let X be an infinite set and let Y be a finite set. One can verify that $X \setminus Y$ is infinite.

The following propositions are true:

(5) $[X]^0 = \{0\}$.

(6) For every finite set X such that $\operatorname{card} X < n$ holds $[X]^n$ is empty.

(7) If $X \subseteq Y$, then $[X]^Z \subseteq [Y]^Z$.

(8) If X is finite and Y is finite and $\overline{\overline{Y}} = X$, then $[Y]^X = \{Y\}$.

(9) If X is non empty and Y is non empty, then f is constant iff there exists an element y of Y such that $\operatorname{rng} f = \{y\}$.

(10) For every finite set X such that $k \leq \operatorname{card} X$ there exists a subset Y of X such that $\operatorname{card} Y = k$.

(11) If $m \geq 1$, then $n + 1 \leq \binom{n+m}{m}$.

(12) If $m \geq 1$ and $n \geq 1$, then $m + 1 \leq \binom{n+m}{m}$.

(13) Let X be a non empty set, p_1, p_2 be elements of X, P be a partition of X, and A be an element of P. Suppose $p_1 \in A$ and (the projection onto P)(p_1) = (the projection onto P)(p_2). Then $p_2 \in A$.

2. Infinite Ramsey Theorem

We now state two propositions:

(14) Let F be a function from $[X]^n$ into k. Suppose $k \neq 0$ and X is infinite. Then there exists H such that H is infinite and $F{\restriction}[H]^n$ is constant.

(15) Let X be an infinite set and P be a partition of $[X]^n$. If $\overline{\overline{P}} = k$, then there exists a subset of X which is infinite and homogeneous for P.

3. Ramsey's Theorem

The scheme *BinInd2* concerns a binary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[m, n]$

provided the following conditions are satisfied:

- $\mathcal{P}[0, n]$ and $\mathcal{P}[n, 0]$, and
- If $\mathcal{P}[m + 1, n]$ and $\mathcal{P}[m, n + 1]$, then $\mathcal{P}[m + 1, n + 1]$.

We now state two propositions:

(16) Suppose $m \geq 2$ and $n \geq 2$. Then there exists a natural number r such that

(i) $r \leq \binom{(m+n)-'2}{m-'1}$,

(ii) $r \geq 2$, and

(iii) for every finite set X and for every function F from $[X]^2$ into $\operatorname{Seg} 2$ such that $\operatorname{card} X \geq r$ there exists a subset S of X such that $\operatorname{card} S \geq m$ and $\operatorname{rng}(F \restriction [S]^2) = \{1\}$ or $\operatorname{card} S \geq n$ and $\operatorname{rng}(F \restriction [S]^2) = \{2\}$.

(17) Let m be a natural number. Then there exists a natural number r such that for every finite set X and for every partition P of $[X]^2$ if $\operatorname{card} X \geq r$ and $\overline{\overline{P}} = 2$, then there exists a subset S of X such that $\operatorname{card} S \geq m$ and S is homogeneous for P.

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