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General Theory of Quasi-Commutative BCI-algebras

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Summary. It is known that commutative BCK-algebras form a variety, but BCK-algebras do not [4]. Therefore H. Yutani introduced the notion of quasi-commutative BCK-algebras. In this article we first present the notion and general theory of quasi-commutative BCI-algebras. Then we discuss the reduction of the type of quasi-commutative BCK-algebras and some special classes of quasi-commutative BCI-algebras.

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The articles [7], [2], [3], [1], [5], and [6] provide the terminology and notation for this paper.

Let X be a BCI-algebra, let x, y be elements of X, and let m, n be elements of $\mathbb{N}$. The functor Polynom(m,n,x,y) yields an element of X and is defined as follows:

(Def. 1) Polynom$(m,n,x,y) = ((x \setminus (x \setminus y))^{m+1} \setminus (y \setminus x))^n$.

We adopt the following convention: X denotes a BCI-algebra, x, y, z denote elements of X, and i, j, k, l, m, n denote elements of $\mathbb{N}$.

One can prove the following propositions:

(1) If $x \leq y \leq z$, then $x \leq z$.

(2) If $x \leq y \leq x$, then $x = y$.

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(3) For every BCK-algebra X and for all elements x, y of X holds $x \setminus y \le x$ and $(x \setminus y)^{n+1} \le (x \setminus y)^n$.

(4) For every BCK-algebra X and for every element x of X holds $(0_X \setminus x)^n = 0_X$.

(5) For every BCK-algebra X and for all elements x, y of X such that $m \ge n$ holds $(x \setminus y)^m \le (x \setminus y)^n$.

(6) Let X be a BCK-algebra and x, y be elements of X. Suppose $m > n$ and $(x \setminus y)^n = (x \setminus y)^m$. Let k be an element of $\mathbb{N}$. If $k \ge n$, then $(x \setminus y)^n = (x \setminus y)^k$.

(7) $\mathrm{Polynom}(0, 0, x, y) = x \setminus (x \setminus y)$.

(8) $\mathrm{Polynom}(m, n, x, y) = ((\mathrm{Polynom}(0, 0, x, y) \setminus (x \setminus y))^m \setminus (y \setminus x))^n$.

(9) $\mathrm{Polynom}(m + 1, n, x, y) = \mathrm{Polynom}(m, n, x, y) \setminus (x \setminus y)$.

(10) $\mathrm{Polynom}(m, n + 1, x, y) = \mathrm{Polynom}(m, n, x, y) \setminus (y \setminus x)$.

(11) $\mathrm{Polynom}(n + 1, n + 1, y, x) \le \mathrm{Polynom}(n, n + 1, x, y)$.

(12) $\mathrm{Polynom}(n, n + 1, x, y) \le \mathrm{Polynom}(n, n, y, x)$.

Let X be a BCI-algebra. We say that X is quasi-commutative if and only if:

(Def. 2) There exist elements i, j, m, n of $\mathbb{N}$ such that for all elements x, y of X holds $\mathrm{Polynom}(i, j, x, y) = \mathrm{Polynom}(m, n, y, x)$.

Let us observe that BCI-EXAMPLE is quasi-commutative.

One can check that there exists a BCI-algebra which is quasi-commutative.

Let i, j, m, n be elements of $\mathbb{N}$. A BCI-algebra is called a BCI-algebra commutating with i, j and m, n if:

(Def. 3) For all elements x, y of it holds $\mathrm{Polynom}(i, j, x, y) = \mathrm{Polynom}(m, n, y, x)$.

One can prove the following propositions:

(13) X is a BCI-algebra commutating with i, j and m, n if and only if X is a BCI-algebra commutating with m, n and i, j.

(14) Let X be a BCI-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCI-algebra commutating with $i + k$, j and m, $n + k$.

(15) Let X be a BCI-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCI-algebra commutating with i, $j + k$ and $m + k$, n.

One can verify that there exists a BCK-algebra which is quasi-commutative.

Let i, j, m, n be elements of $\mathbb{N}$. One can check that there exists a BCI-algebra commutating with i, j and m, n which is BCK-5.

Let i, j, m, n be elements of $\mathbb{N}$. A BCK-algebra commutating with i, j and m, n is BCK-5 BCI-algebra commutating with i, j and m, n.

One can prove the following propositions:

(16) X is a BCK-algebra commutating with i, j and m, n if and only if X is a BCK-algebra commutating with m, n and i, j.

(17) Let X be a BCK-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCK-algebra commutating with $i + k$, j and m, $n + k$.

(18) Let X be a BCK-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCK-algebra commutating with i, $j + k$ and $m + k$, n.

(19) For every BCK-algebra X commutating with i, j and m, n and for all elements x, y of X holds $(x \setminus y)^{i+1} = (x \setminus y)^{n+1}$.

(20) For every BCK-algebra X commutating with i, j and m, n and for all elements x, y of X holds $(x \setminus y)^{j+1} = (x \setminus y)^{m+1}$.

(21) Every BCK-algebra commutating with i, j and m, n is a BCK-algebra commutating with i, j and j, n.

(22) Every BCK-algebra commutating with i, j and m, n is a BCK-algebra commutating with n, j and m, n.

Let us consider i, j, m, n. The functor $\min(i, j, m, n)$ yielding an extended real number is defined as follows:

(Def. 4) $\min(i, j, m, n) = \min(\min(i, j), \min(m, n))$.

The functor $\max(i, j, m, n)$ yielding an extended real number is defined by:

(Def. 5) $\max(i, j, m, n) = \max(\max(i, j), \max(m, n))$.

Next we state a number of propositions:

(23) $\min(i, j, m, n) = i$ or $\min(i, j, m, n) = j$ or $\min(i, j, m, n) = m$ or $\min(i, j, m, n) = n$.

(24) $\max(i, j, m, n) = i$ or $\max(i, j, m, n) = j$ or $\max(i, j, m, n) = m$ or $\max(i, j, m, n) = n$.

(25) If $i = \min(i, j, m, n)$, then $i \leq j$ and $i \leq m$ and $i \leq n$.

(26) $\max(i, j, m, n) \geq i$ and $\max(i, j, m, n) \geq j$ and $\max(i, j, m, n) \geq m$ and $\max(i, j, m, n) \geq n$.

(27) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. If $i = j$, then X is a BCK-algebra commutating with i, i and i, i.

(28) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i < n$. Then X is a BCK-algebra commutating with i, $i + 1$ and i, $i + 1$.

(29) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i = n$ and $i = m$. Then X is a BCK-algebra commutating with i, i and i, i.

(30) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i = n$ and $i < m < j$. Then X is a BCK-algebra commutating with i, $m + 1$ and m, i.

(31) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i = n$ and $j \leq m$. Then X is a BCK-algebra commutating with i, j and j, i.

(32) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $l \geq j$ and $k \geq n$. Then X is a BCK-algebra commutating with k, l and l, k.

(33) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $k \geq \max(i, j, m, n)$. Then X is a BCK-algebra commutating with k, k and k, k.

(34) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i \leq m$ and $j \leq n$. Then X is a BCK-algebra commutating with i, j and i, j.

(35) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i \leq m$ and $i < n$. Then X is a BCK-algebra commutating with i, j and i, $i + 1$.

(36) If X is a BCI-algebra commutating with i, j and $j + k$, $i + k$, then X is a BCK-algebra.

(37) X is a BCI-algebra commutating with 0, 0 and 0, 0 if and only if X is a BCK-algebra commutating with 0, 0 and 0, 0.

(38) X is a commutative BCK-algebra iff X is a BCI-algebra commutating with 0, 0 and 0, 0.

Let X be a BCI-algebra. We introduce p-Semisimple-part X as a synonym of AtomSet X.

In the sequel B, P are non empty subsets of X.

One can prove the following propositions:

(39) For every BCI-algebra X such that $B =$ BCK-part X and $P =$ p-Semisimple-part X holds $B \cap P = \{0_X\}$.

(40) For every BCI-algebra X such that $P = p$-Semisimple-part X holds X is a BCK-algebra iff $P = \{0_X\}$.

(41) For every BCI-algebra X such that $B =$ BCK-part X holds X is a p-semisimple BCI-algebra iff $B = \{0_X\}$.

(42) If X is a p-semisimple BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(43) Suppose X is a p-semisimple BCI-algebra. Then X is a BCI-algebra commutating with $n + j$, n and m, $m + j + 1$.

(44) Suppose X is an associative BCI-algebra. Then X is a BCI-algebra commutating with 0, 1 and 0, 0 and a BCI-algebra commutating with 1, 0 and 0, 0.

(45) Suppose X is a weakly-positive-implicative BCI-algebra. Then X is a BCI-algebra commutating with 0, 1 and 1, 1.

(46) If X is a positive-implicative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 1, 1.

(47) If X is an implicative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(48) If X is an alternative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(49) X is a BCK-positive-implicative BCK-algebra if and only if X is a BCK-algebra commutating with 0, 1 and 0, 1.

(50) X is a BCK-implicative BCK-algebra iff X is a BCK-algebra commutating with 1, 0 and 0, 0.

One can check that every BCK-algebra which is BCK-implicative is also commutative and every BCK-algebra which is BCK-implicative is also BCK-positive-implicative.

The following propositions are true:

(51) X is a BCK-algebra commutating with 1, 0 and 0, 0 if and only if X is a BCK-algebra commutating with 0, 0 and 0, 0 and a BCK-algebra commutating with 0, 1 and 0, 1.

(52) Let X be a quasi-commutative BCK-algebra. Then X is a BCK-algebra commutating with 0, 1 and 0, 1 if and only if for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y$.

(53) Let X be a quasi-commutative BCK-algebra. Then X is a BCK-algebra commutating with n, $n+1$ and n, $n+1$ if and only if for all elements x, y of X holds $(x \setminus y)^{n+1} = (x \setminus y)^{n+2}$.

(54) If X is a BCI-algebra commutating with 0, 1 and 0, 0, then X is a BCI-commutative BCI-algebra.

(55) If X is a BCI-algebra commutating with n, 0 and m, m, then X is a BCI-commutative BCI-algebra.

(56) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $j = 0$ and $m > 0$. Then X is a BCK-algebra commutating with 0, 0 and 0, 0.

(57) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $m = 0$ and $j > 0$. Then X is a BCK-algebra commutating with 0, 1 and 0, 1.

(58) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $n = 0$ and $i \neq 0$. Then X is a BCK-algebra commutating with 0, 0 and 0, 0.

(59) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = 0$ and $n \neq 0$. Then X is a BCK-algebra commutating with 0, 1 and 0, 1.

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