FORMALIZED MATHEMATICS
Vol. **16**, No. **4**, Pages 319–324, **2008**

Integral of Complex-Valued Measurable Function

Keiko Narita
Hirosaki-city
Aomori, Japan

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalized the notion of the integral of a complex-valued function considered as a sum of its real and imaginary parts. Then we defined the measurability and integrability in this context, and proved the linearity and several other basic properties of complex-valued measurable functions. The set of properties showed in this paper is based on [15], where the case of real-valued measurable functions is considered.

MML identifier: `MESFUN6C`, version: `7.9.01 4.101.1015`

The notation and terminology used here are introduced in the following papers: [17], [1], [11], [18], [6], [19], [7], [2], [12], [14], [16], [5], [4], [3], [9], [10], [13], [8], and [15].

1. Definitions for Complex-valued Functions

One can prove the following proposition

(1) For all real numbers a, b holds $\overline{\mathbb{R}}(a)+\overline{\mathbb{R}}(b) = a+b$ and $-\overline{\mathbb{R}}(a) = -a$ and $\overline{\mathbb{R}}(a) - \overline{\mathbb{R}}(b) = a - b$ and $\overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b) = a \cdot b$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{C}$. The functor $\Re(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 1) $\operatorname{dom} \Re(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom} \Re(f)$ holds $\Re(f)(x) = \Re(f(x))$.

ISSN 1426–2630(p), 1898-9934(e)

The functor $\Im(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\operatorname{dom}\Im(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom}\Im(f)$ holds $\Im(f)(x) = \Im(f(x))$.

2. The Measurability of Complex-valued Functions

For simplicity, we use the following convention: X is a non empty set, Y is a set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\mathbb{C}$, r is a real number, c is a complex number, and E, A, B are elements of S.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\mathbb{C}$, and let E be an element of S. We say that f is measurable on E if and only if:

(Def. 3) $\Re(f)$ is measurable on E and $\Im(f)$ is measurable on E.

One can prove the following propositions:

(2) $r\,\Re(f) = \Re(r\,f)$ and $r\,\Im(f) = \Im(r\,f)$.

(3) $\Re(c\,f) = \Re(c)\,\Re(f) - \Im(c)\,\Im(f)$ and $\Im(c\,f) = \Im(c)\,\Re(f) + \Re(c)\,\Im(f)$.

(4) $-\Im(f) = \Re(i\,f)$ and $\Re(f) = \Im(i\,f)$.

(5) $\Re(f+g) = \Re(f) + \Re(g)$ and $\Im(f+g) = \Im(f) + \Im(g)$.

(6) $\Re(f-g) = \Re(f) - \Re(g)$ and $\Im(f-g) = \Im(f) - \Im(g)$.

(7) $\Re(f){\restriction}A = \Re(f{\restriction}A)$ and $\Im(f){\restriction}A = \Im(f{\restriction}A)$.

(8) $f = \Re(f) + i\,\Im(f)$.

(9) If $B \subseteq A$ and f is measurable on A, then f is measurable on B.

(10) If f is measurable on A and f is measurable on B, then f is measurable on $A \cup B$.

(11) If f is measurable on A and g is measurable on A, then $f+g$ is measurable on A.

(12) If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $f-g$ is measurable on A.

(13) If $Y \subseteq \operatorname{dom}(f+g)$, then $\operatorname{dom}(f{\restriction}Y + g{\restriction}Y) = Y$ and $(f+g){\restriction}Y = f{\restriction}Y + g{\restriction}Y$.

(14) If f is measurable on B and $A = \operatorname{dom} f \cap B$, then $f{\restriction}B$ is measurable on A.

(15) If $\operatorname{dom} f, \operatorname{dom} g \in S$, then $\operatorname{dom}(f+g) \in S$.

(16) If $\operatorname{dom} f = A$, then f is measurable on B iff f is measurable on $A \cap B$.

(17) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $c\,f$ is measurable on A.

(18) Given an element A of S such that $\operatorname{dom} f = A$. Let c be a complex number and B be an element of S. If f is measurable on B, then $c\,f$ is measurable on B.

3. The Integral of a Complex-valued Measurable Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. We say that f is integrable on M if and only if:

(Def. 4) $\Re(f)$ is integrable on M and $\Im(f)$ is integrable on M.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. Let us assume that f is integrable on M. The functor $\int f \, \mathrm{d}M$ yielding a complex number is defined by:

(Def. 5) There exist real numbers R, I such that $R = \int \Re(f) \, \mathrm{d}M$ and $I = \int \Im(f) \, \mathrm{d}M$ and $\int f \, \mathrm{d}M = R + I \cdot i$.

We now state several propositions:

(19) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M.

(20) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\mathbb{R}$, and E, A be elements of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M.

(21) Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M and $\int f{\restriction}A \, \mathrm{d}M = 0$.

(22) If $E = \mathrm{dom}\, f$ and f is integrable on M and $M(A) = 0$, then $\int f{\restriction}(E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

(23) If f is integrable on M, then $f{\restriction}A$ is integrable on M.

(24) If f is integrable on M and A misses B, then $\int f{\restriction}(A \cup B) \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

(25) If f is integrable on M and $B = \mathrm{dom}\, f \setminus A$, then $f{\restriction}A$ is integrable on M and $\int f \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

Let k be a real number, let X be a non empty set, and let f be a partial function from X to $\mathbb{R}$. The functor f^k yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 6) $\mathrm{dom}(f^k) = \mathrm{dom}\, f$ and for every element x of X such that $x \in \mathrm{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Let us consider X. Observe that there exists a partial function from X to $\mathbb{R}$ which is non-negative.

Let k be a non negative real number, let us consider X, and let f be a non-negative partial function from X to $\mathbb{R}$. Observe that f^k is non-negative.

We now state a number of propositions:

(26) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative and $0 \leq k$, then f^k is non-negative.

(27) Let x be a set, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative, then $f(x)^{\frac{1}{2}} = \sqrt{f(x)}$.

(28) For every partial function f from X to $\mathbb{R}$ and for every real number a such that $A \subseteq \mathrm{dom}\, f$ holds $A \cap \text{LE-dom}(f, a) = A \setminus A \cap \text{GTE-dom}(f, a)$.

(29) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. Suppose f is non-negative and $0 \leq k$ and $E \subseteq \mathrm{dom}\, f$ and f is measurable on E. Then f^k is measurable on E.

(30) If f is measurable on A and $A \subseteq \mathrm{dom}\, f$, then $|f|$ is measurable on A.

(31) Given an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(32) If f is integrable on M and g is integrable on M, then $\mathrm{dom}(f + g) \in S$.

(33) If f is integrable on M and g is integrable on M, then $f + g$ is integrable on M.

(34) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then $f - g$ is integrable on M.

(35) If f is integrable on M and g is integrable on M, then $f - g$ is integrable on M.

(36) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f + g \,\mathrm{d}M = \int f \restriction E \,\mathrm{d}M + \int g \restriction E \,\mathrm{d}M$.

(37) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f - g \,\mathrm{d}M = \int f \restriction E \,\mathrm{d}M + \int (-g) \restriction E \,\mathrm{d}M$.

(38) If f is integrable on M, then $r\, f$ is integrable on M and $\int r\, f \,\mathrm{d}M = r \cdot \int f \,\mathrm{d}M$.

(39) If f is integrable on M, then $i\, f$ is integrable on M and $\int i\, f \,\mathrm{d}M = i \cdot \int f \,\mathrm{d}M$.

(40) If f is integrable on M, then $c\, f$ is integrable on M and $\int c\, f \,\mathrm{d}M = c \cdot \int f \,\mathrm{d}M$.

(41) For every partial function f from X to $\mathbb{R}$ and for all Y, r holds $(r\, f) \restriction Y = r\, (f \restriction Y)$.

(42) Let f, g be partial functions from X to $\mathbb{R}$. Suppose that

(i) there exists an element A of S such that $A = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and f is measurable on A and g is measurable on A,

(ii) f is integrable on M,

(iii) g is integrable on M, and

(iv) $g - f$ is non-negative.

Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f{\restriction}E \,\mathrm{d}M \leq \int g{\restriction}E \,\mathrm{d}M$.

(43) Suppose there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A and f is integrable on M. Then $|\int f \,\mathrm{d}M| \leq \int |f| \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{C}$, and let B be an element of S. The functor $\int\limits_B f \,\mathrm{d}M$ yields a complex number and is defined by:

(Def. 7) $\int\limits_B f \,\mathrm{d}M = \int f{\restriction}B \,\mathrm{d}M$.

Next we state two propositions:

(44) Suppose f is integrable on M and g is integrable on M and $B \subseteq \mathrm{dom}(f+g)$. Then $f+g$ is integrable on M and $\int\limits_B f+g \,\mathrm{d}M = \int\limits_B f \,\mathrm{d}M + \int\limits_B g \,\mathrm{d}M$.

(45) If f is integrable on M and f is measurable on B, then $\int\limits_B c\, f \,\mathrm{d}M = c \cdot \int\limits_B f \,\mathrm{d}M$.

4. Several Properties of Real-valued Measurable Functions

In the sequel f denotes a partial function from X to $\mathbb{R}$ and a denotes a real number.

One can prove the following four propositions:

(46) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{GTE-dom}(f, a) = A \setminus A \cap \text{LE-dom}(f, a)$.

(47) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{GT-dom}(f, a) = A \setminus A \cap \text{LEQ-dom}(f, a)$.

(48) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{LEQ-dom}(f, a) = A \setminus A \cap \text{GT-dom}(f, a)$.

(49) $A \cap \text{EQ-dom}(f, a) = A \cap \text{GTE-dom}(f, a) \cap \text{LEQ-dom}(f, a)$.

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Received July 30, 2008