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Several Integrability Formulas of Some Functions, Orthogonal Polynomials and Norm Functions

Bo Li
Qingdao University of Science and Technology
China

Yanping Zhuang
Qingdao University of Science and Technology
China

Bing Xie
Qingdao University of Science and Technology
China

Pan Wang
Qingdao University of Science and Technology
China

Summary. In this article, we give several integrability formulas of some functions including the trigonometric function and the index function [3]. We also give the definitions of the orthogonal polynomial and norm function, and some of their important properties [19].

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The terminology and notation used here are introduced in the following articles: [10], [21], [17], [6], [20], [1], [9], [13], [2], [4], [18], [15], [5], [8], [11], [14], [12], [16], and [7].

For simplicity, we use the following convention: r, p, x denote real numbers, n denotes an element of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, f, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

We now state a number of propositions:

(1) $-(\text{the function exp}) \cdot ((-1)\square{+}0)$ is differentiable on $\mathbb{R}$ and for every x holds $(-(\text{the function exp}) \cdot ((-1)\square{+}0))'_{\restriction \mathbb{R}}(x) = \exp(-x)$.

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(2) $\int\limits_{A}((\text{the function exp}) \cdot ((-1)\square{+}0))(x)dx = -\exp(-\sup A)+\exp(-\inf A)$.

(3) $\frac{1}{2}$ ((the function exp) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}\,((\text{the function exp}) \cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(2 \cdot x)$.

(4) $\int\limits_{A}((\text{the function exp}) \cdot (2\square{+}0))(x)dx = \frac{1}{2}\cdot\exp(2\cdot\sup A)-\frac{1}{2}\cdot\exp(2\cdot\inf A)$.

(5) Suppose $r \neq 0$. Then $\frac{1}{r}$ ((the function exp) $\cdot(r\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{r}\,((\text{the function exp}) \cdot(r\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(r\cdot x)$.

(6) If $r \neq 0$, then $\int\limits_{A}((\text{the function exp}) \cdot (r\square{+}0))(x)dx = \frac{1}{r}\cdot\exp(r\cdot\sup A)-\frac{1}{r}\cdot\exp(r\cdot\inf A)$.

(7) $\int\limits_{A}((\text{the function sin}) \cdot (2\square{+}0))(x)dx = (-\frac{1}{2})\cdot\cos(2\cdot\sup A)-(-\frac{1}{2})\cdot\cos(2\cdot\inf A)$.

(8) Suppose $n \neq 0$. Then $(-\frac{1}{n})$ ((the function cos) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $((-\frac{1}{n})\,((\text{the function cos}) \cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \sin(n \cdot x)$.

(9) If $n \neq 0$, then $\int\limits_{A}((\text{the function sin}) \cdot (n\square{+}0))(x)dx = (-\frac{1}{n}) \cdot \cos(n \cdot \sup A) - (-\frac{1}{n}) \cdot \cos(n \cdot \inf A)$.

(10) $\frac{1}{2}$ ((the function sin) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}\,((\text{the function sin}) \cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(2 \cdot x)$.

(11) $\int\limits_{A}((\text{the function cos}) \cdot (2\square{+}0))(x)dx = \frac{1}{2}\cdot\sin(2\cdot\sup A)-\frac{1}{2}\cdot\sin(2\cdot\inf A)$.

(12) Suppose $n \neq 0$. Then $\frac{1}{n}$ ((the function sin) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n}\,((\text{the function sin}) \cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(n\cdot x)$.

(13) If $n \neq 0$, then $\int\limits_{A}((\text{the function cos}) \cdot (n\square{+}0))(x)dx = \frac{1}{n}\cdot\sin(n\cdot\sup A)-\frac{1}{n}\cdot\sin(n\cdot\inf A)$.

(14) If $A \subseteq Z$, then $\int\limits_{A}(\mathrm{id}_Z\,(\text{the function sin}))(x)dx = ((-\sup A)\cdot\cos\sup A + \sin\sup A) - ((-\inf A)\cdot\cos\inf A + \sin\inf A)$.

(15) If $A \subseteq Z$, then $\int\limits_{A}(\mathrm{id}_Z\,(\text{the function cos}))(x)dx = (\sup A\cdot\sin\sup A + \cos\sup A) - (\inf A\cdot\sin\inf A + \cos\inf A)$.

(16) id_Z (the function cos) is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z$ (the function cos)$)'_{\restriction Z}(x) = \cos x - x \cdot \sin x$.

(17)(i) $-$the function $\sin + \mathrm{id}_Z$ (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function $\sin + \mathrm{id}_Z$ (the function cos$))'_{\restriction Z}(x) = -x \cdot \sin x$.

(18) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)$ (the function sin)$)(x)dx = (-\sin \sup A + \sup A \cdot \cos \sup A) - (-\sin \inf A + \inf A \cdot \cos \inf A)$.

(19)(i) $-$the function $\cos - \mathrm{id}_Z$ (the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function $\cos - \mathrm{id}_Z$ (the function sin$))'_{\restriction Z}(x) = -x \cdot \cos x$.

(20) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)$ (the function cos)$)(x)dx = -\cos \sup A - \sup A \cdot \sin \sup A - (-\cos \inf A - \inf A \cdot \sin \inf A)$.

(21) If $A \subseteq Z$, then $\int\limits_A (($the function sin$) + \mathrm{id}_Z$ (the function cos)$)(x)dx = \sup A \cdot \sin \sup A - \inf A \cdot \sin \inf A$.

(22) If $A \subseteq Z$, then $\int\limits_A (-$the function $\cos + \mathrm{id}_Z$ (the function sin)$)(x)dx = (-\sup A) \cdot \cos \sup A - (-\inf A) \cdot \cos \inf A$.

(23) $\int\limits_A ((1\square + 0)$ (the function exp)$)(x)dx = \exp(\sup A - 1) - \exp(\inf A - 1)$.

(24) $\frac{1}{n+1}\,(\square^{n+1})$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}\,(\square^{n+1}))'_{\restriction \mathbb{R}}(x) = x^n$.

(25) $\int\limits_A (\square^n)(x)dx = \dfrac{1}{n+1} \cdot (\sup A)^{n+1} - \dfrac{1}{n+1} \cdot (\inf A)^{n+1}$.

(26) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $(f - g) \restriction C = f \restriction C - g \restriction C$.

(27) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 + f_2) \restriction C)\,(g \restriction C) = (f_1\, g + f_2\, g) \restriction C$.

(28) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 - f_2) \restriction C)\,(g \restriction C) = (f_1\, g - f_2\, g) \restriction C$.

(29) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1\, f_2) \restriction C)\,(g \restriction C) = (f_1 \restriction C)\,((f_2\, g) \restriction C)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle f, g \rangle_A$ yielding a real number is defined by:

(Def. 1) $\langle f, g \rangle_A = \int\limits_A (f\, g)(x)dx$.

The following propositions are true:

(30) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f, g\rangle_A = \langle g, f\rangle_A$.

(31) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \upharpoonright A$ is total,
(ii) $(f_2\, g) \upharpoonright A$ is total,
(iii) $(f_1\, g) \upharpoonright A$ is bounded,
(iv) $f_1\, g$ is integrable on A,
(v) $(f_2\, g) \upharpoonright A$ is bounded, and
(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 + f_2, g\rangle_A = \langle (f_1), g\rangle_A + \langle (f_2), g\rangle_A$.

(32) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \upharpoonright A$ is total,
(ii) $(f_2\, g) \upharpoonright A$ is total,
(iii) $(f_1\, g) \upharpoonright A$ is bounded,
(iv) $f_1\, g$ is integrable on A,
(v) $(f_2\, g) \upharpoonright A$ is bounded, and
(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 - f_2, g\rangle_A = \langle (f_1), g\rangle_A - \langle (f_2), g\rangle_A$.

(33) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle -f, g\rangle_A = -\langle f, g\rangle_A$.

(34) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, g\rangle_A = r \cdot \langle f, g\rangle_A$.

(35) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, p\, g\rangle_A = r \cdot p \cdot \langle f, g\rangle_A$.

(36) For all partial functions f, g, h from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f\, g, h\rangle_A = \langle f, g\, h\rangle_A$.

(37) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\, f) \upharpoonright A$ is total and $(f\, g) \upharpoonright A$ is total and $(g\, g) \upharpoonright A$ is total and $(f\, f) \upharpoonright A$ is bounded and $(f\, g) \upharpoonright A$ is bounded and $(g\, g) \upharpoonright A$ is bounded and $f\, f$ is integrable on A and $f\, g$ is integrable on A and $g\, g$ is integrable on A. Then $\langle f + g, f + g\rangle_A = \langle f, f\rangle_A + 2 \cdot \langle f, g\rangle_A + \langle g, g\rangle_A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that f is orthogonal with g in A if and only if:

(Def. 2) $\langle f, g\rangle_A = 0$.

The following propositions are true:

(38) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \restriction A$ is total and $(f\,g) \restriction A$ is total and $(g\,g) \restriction A$ is total and $(f\,f) \restriction A$ is bounded and $(f\,g) \restriction A$ is bounded and $(g\,g) \restriction A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A. Then $\langle f+g, f+g\rangle_A = \langle f, f\rangle_A + \langle g, g\rangle_A$.

(39) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \restriction A$ is total and $(f\,f) \restriction A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \restriction A)(x) \geq 0$. Then $\langle f, f\rangle_A \geq 0$.

(40) The function sin is orthogonal with the function cos in $[0, \pi]$.

(41) The function sin is orthogonal with the function cos in $[0, \pi \cdot 2]$.

(42) The function sin is orthogonal with the function cos in $[2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$.

(43) The function sin is orthogonal with the function cos in $[x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$.

(44) The function sin is orthogonal with the function cos in $[-\pi, \pi]$.

(45) The function sin is orthogonal with the function cos in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(46) The function sin is orthogonal with the function cos in $[-2 \cdot \pi, 2 \cdot \pi]$.

(47) The function sin is orthogonal with the function cos in $[-2 \cdot n \cdot \pi, 2 \cdot n \cdot \pi]$.

(48) The function sin is orthogonal with the function cos in $[x - 2 \cdot n \cdot \pi, x + 2 \cdot n \cdot \pi]$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $||f||_A$ yields a real number and is defined by:

(Def. 3) $||f||_A = \sqrt{\langle f, f\rangle_A}$.

Next we state three propositions:

(49) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \restriction A$ is total and $(f\,f) \restriction A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \restriction A)(x) \geq 0$. Then $0 \leq ||f||_A$.

(50) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $||1\,f||_A = ||f||_A$.

(51) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \restriction A$ is total and $(f\,g) \restriction A$ is total and $(g\,g) \restriction A$ is total and $(f\,f) \restriction A$ is bounded and $(f\,g) \restriction A$ is bounded and $(g\,g) \restriction A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A and for every x such that $x \in A$ holds $((f\,f) \restriction A)(x) \geq 0$ and for every x such that $x \in A$ holds $((g\,g) \restriction A)(x) \geq 0$. Then $(||f+g||_A)^{\mathbf{2}} = (||f||_A)^{\mathbf{2}} + (||g||_A)^{\mathbf{2}}$.

For simplicity, we follow the rules: a, b, x are real numbers, n is an element of $\mathbb{N}$, A is a closed-interval subset of $\mathbb{R}$, f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z is an open subset of $\mathbb{R}$.

Next we state several propositions:

(52) If $-a \notin A$, then $\frac{1}{1\square+a} \restriction A$ is continuous.

(53) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $Z = \mathrm{dom}\, f$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(a+x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(54) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $\mathrm{dom}((-1)\,\frac{1}{f}) = Z$,

(iv) $\mathrm{dom}((-1)\,\frac{1}{f}) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a+x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -f(\sup A)^{-1} + f(\inf A)^{-1}$.

(55) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) \neq 0$,

(iii) $\mathrm{dom}\, f = Z$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a-x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,

(iii) $\mathrm{dom}((\text{the function } \ln) \cdot f) = Z$,

(iv) $\mathrm{dom}((\text{the function } \ln) \cdot f) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a+x}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln(a + \sup A) - \ln(a + \inf A)$.

Next we state a number of propositions:

(57) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = x - a$ and $f(x) > 0$,
(iii) $\mathrm{dom}((\text{the function ln}) \cdot f) = Z$,
(iv) $\mathrm{dom}((\text{the function ln}) \cdot f) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x-a}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln f(\sup A) - \ln f(\inf A)$.

(58) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
(iii) $\mathrm{dom}(-(\text{the function ln}) \cdot f) = Z$,
(iv) $\mathrm{dom}(-(\text{the function ln}) \cdot f) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a-x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln(a - \sup A) + \ln(a - \inf A)$.

(59) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - a\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x}{a+x}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - a \cdot f(\sup A) - (\inf A - a \cdot f(\inf A))$.

(60) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\mathrm{dom}((2 \cdot a)\, f - \mathrm{id}_Z) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{a-x}{a+x}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = 2 \cdot a \cdot f(\sup A) - \sup A - (2 \cdot a \cdot f(\inf A) - \inf A)$.

(61) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - (2 \cdot a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+a}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - 2 \cdot a \cdot f(\sup A) - (\inf A - 2 \cdot a \cdot f(\inf A))$.

(62) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (2 \cdot a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-a}$ and $f_2 \restriction A$

is continuous. Then $\int_A f_2(x)dx = (\sup A + 2 \cdot a \cdot f(\sup A)) - (\inf A + 2 \cdot a \cdot f(\inf A))$.

(63) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a-b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (a-b) \cdot f(\sup A)) - (\inf A + (a-b) \cdot f(\inf A))$.

(64) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a+b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (a+b) \cdot f(\sup A)) - (\inf A + (a+b) \cdot f(\inf A))$.

(65) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - (a+b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int_A f_2(x)dx = \sup A - (a+b) \cdot f(\sup A) - (\inf A - (a+b) \cdot f(\inf A))$.

(66) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (b-a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (b-a) \cdot f(\sup A)) - (\inf A + (b-a) \cdot f(\inf A))$.

(67) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$,

(iii) dom((the function ln) $\cdot f) = Z$,

(iv) dom((the function ln) $\cdot f) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int_A f_2(x)dx = \ln \sup A - \ln \inf A$.

(68) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $x > 0$,

(iii) dom((the function ln) $\cdot(\square^n)) = Z$,

(iv) $\mathrm{dom}((\text{the function ln}) \cdot (\square^n)) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{n}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln((\sup A)^n) - \ln((\inf A)^n)$.

(69) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = x$,
(iii) $\mathrm{dom}((\text{the function ln}) \cdot \frac{1}{f}) = Z$,
(iv) $\mathrm{dom}((\text{the function ln}) \cdot \frac{1}{f}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln \sup A + \ln \inf A$.

(70) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) $\mathrm{dom}(\frac{2}{3}\, f^{\frac{3}{2}}) = Z$,
(iv) $\mathrm{dom}(\frac{2}{3}\, f^{\frac{3}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \frac{2}{3} \cdot (a + \sup A)^{\frac{3}{2}} - \frac{2}{3} \cdot (a + \inf A)^{\frac{3}{2}}$.

(71) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
(iii) $\mathrm{dom}((-\frac{2}{3})\, f^{\frac{3}{2}}) = Z$,
(iv) $\mathrm{dom}((-\frac{2}{3})\, f^{\frac{3}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\frac{2}{3} \cdot (a - \sup A)^{\frac{3}{2}} + \frac{2}{3} \cdot (a - \inf A)^{\frac{3}{2}}$.

(72) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) $\mathrm{dom}(2\, f^{\frac{1}{2}}) = Z$,
(iv) $\mathrm{dom}(2\, f^{\frac{1}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{-\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_{A} f_2(x)dx = 2 \cdot (a + \sup A)^{\frac{1}{2}} - 2 \cdot (a + \inf A)^{\frac{1}{2}}$.

(73) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,

(iii) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = Z$,

(iv) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{-\frac{1}{2}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_{A} f_2(x)dx = -2 \cdot (a - \sup A)^{\frac{1}{2}} + 2 \cdot (a - \inf A)^{\frac{1}{2}}$.

(74) Suppose that

(i) $A \subseteq Z$,

(ii) $\mathrm{dom}((-\mathrm{id}_Z)$ (the function cos)+the function sin$) = Z$,

(iii) for every x such that $x \in Z$ holds $f(x) = x \cdot \sin x$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\sup A \cdot \cos \sup A + \sin \sup A) - (-\inf A \cdot \cos \inf A + \sin \inf A)$.

(75) Suppose $A \subseteq Z$ and dom (the function sec) $= Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x}{(\cos x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f \restriction A$ is continuous. Then $\int\limits_{A} f(x)dx = \sec \sup A - \sec \inf A$.

(76) Suppose $Z \subseteq$ dom($-$the function cosec). Then $-$the function cosec is differentiable on Z and for every x such that $x \in Z$ holds ($-$the function cosec$)'_{\restriction Z}(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$.

(77) Suppose $A \subseteq Z$ and dom($-$the function cosec) $= Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f \restriction A$ is continuous. Then $\int\limits_{A} f(x)dx = -\mathrm{cosec} \sup A + \mathrm{cosec} \inf A$.

References

[1] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[2] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[3] Chuanzhang Chen. *Mathematical Analysis*. Higher Education Press, Beijing, 1978.

[4] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.

[5] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[8] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[9] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.
[12] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[14] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.
[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[16] Andrzej Trybulec and Yatsuka Nakamura. On the decomposition of a simple closed curve into two arcs. *Formalized Mathematics*, 10(**3**):163–167, 2002.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Peng Wang and Bo Li. Several differentiation formulas of special functions. Part V. *Formalized Mathematics*, 15(**3**):73–79, 2007, doi:10.2478/v10037-007-0009-4.
[19] Renhong Wang. *Numerical approximation*. Higher Education Press, Beijing, 1999.
[20] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[21] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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