

Small Inductive Dimension of Topological Spaces

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Summary. We present the concept and basic properties of the Menger-Urysohn small inductive dimension of topological spaces according to the books [7]. Namely, the paper includes the formalization of main theorems from Sections 1.1 and 1.2.

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The terminology and notation used here are introduced in the following articles: [17], [8], [15], [5], [16], [6], [18], [14], [1], [2], [3], [13], [11], [9], [12], [19], [20], [10], and [4].

1. PRELIMINARIES

For simplicity, we adopt the following rules: T, T_1, T_2 denote topological spaces, A, B denote subsets of T , F denotes a subset of $T \setminus A$, G, G_1, G_2 denote families of subsets of T , U, W denote open subsets of $T \setminus A$, p denotes a point of $T \setminus A$, n denotes a natural number, and I denotes an integer.

One can prove the following propositions:

- (1) $\text{Fr}(B \cap A) \subseteq \text{Fr} B \cap A$.
- (2) T is a T_4 space if and only if for all closed subsets A, B of T such that A misses B there exist open subsets U, W of T such that $A \subseteq U$ and $B \subseteq W$ and \overline{U} misses \overline{W} .

Let us consider T . The sequence of ind of T yields a sequence of subsets of $2^{\text{the carrier of } T}$ and is defined by the conditions (Def. 1).

- (Def. 1)(i) (The sequence of ind of T)(0) = $\{\emptyset_T\}$, and
(ii) $A \in$ (the sequence of ind of T)($n + 1$) iff $A \in$ (the sequence of ind of T)(n) or for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $\text{Fr } W \in$ (the sequence of ind of T)(n).

Let us consider T . Note that the sequence of ind of T is ascending.

We now state the proposition

- (3) For every F such that $F = B$ holds $F \in$ (the sequence of ind of $T \setminus A$)(n) iff $B \in$ (the sequence of ind of T)(n).

Let us consider T, A . We say that A has finite small inductive dimension if and only if:

- (Def. 2) There exists n such that $A \in$ (the sequence of ind of T)(n).

Let us consider T, A . We introduce A is finite-ind as a synonym of A has finite small inductive dimension.

Let us consider T, G . We say that G has finite small inductive dimension if and only if:

- (Def. 3) There exists n such that $G \subseteq$ (the sequence of ind of T)(n).

Let us consider T, G . We introduce G is finite-ind as a synonym of G has finite small inductive dimension.

The following proposition is true

- (4) If $A \in G$ and G is finite-ind, then A is finite-ind.

Let us consider T . One can check the following observations:

- * every subset of T which is finite is also finite-ind,
- * there exists a subset of T which is finite-ind,
- * every family of subsets of T which is empty is also finite-ind, and
- * there exists a family of subsets of T which is non empty and finite-ind.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty and finite-ind.

Let us consider T . We say that T has finite small inductive dimension if and only if:

- (Def. 4) Ω_T has finite small inductive dimension.

Let us consider T . We introduce T is finite-ind as a synonym of T has finite small inductive dimension.

One can verify that every topological space which is empty is also finite-ind.

Let X be a set. Note that $\{X\}_{\text{top}}$ is finite-ind.

One can check that there exists a topological space which is non empty and finite-ind.

In the sequel A_1 is a finite-ind subset of T and T_3 is a finite-ind topological space.

2. SMALL INDUCTIVE DIMENSION

Let us consider T and let us consider A . Let us assume that A is finite-ind. The functor $\text{ind } A$ yields an integer and is defined as follows:

(Def. 5) $A \in$ (the sequence of ind of T)($\text{ind } A + 1$) and $A \notin$ (the sequence of ind of T)($\text{ind } A$).

We now state two propositions:

- (5) $-1 \leq \text{ind } A_1$.
- (6) $\text{ind } A_1 = -1$ iff A_1 is empty.

Let T be a non empty topological space and let A be a non empty finite-ind subset of T . Observe that $\text{ind } A$ is natural.

The following three propositions are true:

- (7) $\text{ind } A_1 \leq n - 1$ iff $A_1 \in$ (the sequence of ind of T)(n).
- (8) For every finite subset A of T holds $\text{ind } A < \overline{\overline{A}}$.
- (9) $\text{ind } A_1 \leq n$ if and only if for every point p of $T \setminus A_1$ and for every open subset U of $T \setminus A_1$ such that $p \in U$ there exists an open subset W of $T \setminus A_1$ such that $p \in W$ and $W \subseteq U$ and $\text{Fr } W$ is finite-ind and $\text{ind } \text{Fr } W \leq n - 1$.

Let us consider T and let us consider G . Let us assume that G is finite-ind. The functor $\text{ind } G$ yielding an integer is defined by the conditions (Def. 6).

(Def. 6)(i) $G \subseteq$ (the sequence of ind of T)($\text{ind } G + 1$),
(ii) $-1 \leq \text{ind } G$, and
(iii) for every integer i such that $-1 \leq i$ and $G \subseteq$ (the sequence of ind of T)($i + 1$) holds $\text{ind } G \leq i$.

The following propositions are true:

- (10) $\text{ind } G = -1$ and G is finite-ind iff $G \subseteq \{\emptyset_T\}$.
- (11) G is finite-ind and $\text{ind } G \leq I$ iff $-1 \leq I$ and for every A such that $A \in G$ holds A is finite-ind and $\text{ind } A \leq I$.
- (12) If G_1 is finite-ind and $G_2 \subseteq G_1$, then G_2 is finite-ind and $\text{ind } G_2 \leq \text{ind } G_1$.

Let us consider T and let G_1, G_2 be finite-ind families of subsets of T . Observe that $G_1 \cup G_2$ is finite-ind.

The following proposition is true

- (13) If G is finite-ind and G_1 is finite-ind and $\text{ind } G \leq I$ and $\text{ind } G_1 \leq I$, then $\text{ind}(G \cup G_1) \leq I$.

Let us consider T . The functor $\text{ind } T$ yields an integer and is defined as follows:

(Def. 7) $\text{ind } T = \text{ind}(\Omega_T)$.

Let T be a non empty finite-ind topological space. One can verify that $\text{ind } T$ is natural.

The following three propositions are true:

- (14) For every non empty set X holds $\text{ind}(\{X\}_{\text{top}}) = 0$.
- (15) Given n such that let p be a point of T and U be an open subset of T . Suppose $p \in U$. Then there exists an open subset W of T such that $p \in W$ and $W \subseteq U$ and $\text{Fr } W$ is finite-ind and $\text{ind } \text{Fr } W \leq n - 1$. Then T is finite-ind.
- (16) $\text{ind } T_3 \leq n$ if and only if for every point p of T_3 and for every open subset U of T_3 such that $p \in U$ there exists an open subset W of T_3 such that $p \in W$ and $W \subseteq U$ and $\text{Fr } W$ is finite-ind and $\text{ind } \text{Fr } W \leq n - 1$.

3. MONOTONICITY OF THE SMALL INDUCTIVE DIMENSION

Let us consider T_3 . Observe that every subset of T_3 is finite-ind.

Let us consider T , A_1 . Note that $T \setminus A_1$ is finite-ind.

One can prove the following propositions:

- (17) $\text{ind}(T \setminus A_1) = \text{ind } A_1$.
- (18) If $T \setminus A$ is finite-ind, then A is finite-ind.
- (19) If $A \subseteq A_1$, then A is finite-ind and $\text{ind } A \leq \text{ind } A_1$.
- (20) For every subset A of T_3 holds $\text{ind } A \leq \text{ind } T_3$.
- (21) If $F = B$ and B is finite-ind, then F is finite-ind and $\text{ind } F = \text{ind } B$.
- (22) If $F = B$ and F is finite-ind, then B is finite-ind and $\text{ind } F = \text{ind } B$.
- (23) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is finite-ind and $\text{ind } T \leq n$ if and only if for every closed subset A of T and for every point p of T such that $p \notin A$ there exists a subset L of T such that L separates $\{p\}$, A and L is finite-ind and $\text{ind } L \leq n - 1$.
- (24) If T_1 and T_2 are homeomorphic, then T_1 is finite-ind iff T_2 is finite-ind.
- (25) If T_1 and T_2 are homeomorphic and T_1 is finite-ind, then $\text{ind } T_1 = \text{ind } T_2$.
- (26) Let A_2 be a subset of T_1 and A_3 be a subset of T_2 . Suppose A_2 and A_3 are homeomorphic. Then A_2 is finite-ind if and only if A_3 is finite-ind.
- (27) Let A_2 be a subset of T_1 and A_3 be a subset of T_2 . If A_2 and A_3 are homeomorphic and A_2 is finite-ind, then $\text{ind } A_2 = \text{ind } A_3$.
- (28) If $T_1 \times T_2$ is finite-ind, then $T_2 \times T_1$ is finite-ind and $\text{ind}(T_1 \times T_2) = \text{ind}(T_2 \times T_1)$.
- (29) For every family G_3 of subsets of $T \setminus A$ such that G_3 is finite-ind and $G_3 = G$ holds G is finite-ind and $\text{ind } G = \text{ind } G_3$.
- (30) For every family G_3 of subsets of $T \setminus A$ such that G is finite-ind and $G_3 = G$ holds G_3 is finite-ind and $\text{ind } G = \text{ind } G_3$.

4. BASIC PROPERTIES 0-DIMENSIONAL TOPOLOGICAL SPACES

Next we state several propositions:

- (31) T is finite-ind and $\text{ind } T \leq n$ if and only if there exists a basis B_1 of T such that for every A such that $A \in B_1$ holds $\text{Fr } A$ is finite-ind and $\text{ind } \text{Fr } A \leq n - 1$.
- (32) Let given T . Suppose that
- (i) T is a T_1 space, and
 - (ii) for all closed subsets A, B of T such that A misses B there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.
- Then T is finite-ind and $\text{ind } T \leq 0$.
- (33) Let X be a set and f be a sequence of subsets of X . Then there exists a sequence g of subsets of X such that
- (i) $\bigcup \text{rng } f = \bigcup \text{rng } g$,
 - (ii) for all natural numbers i, j such that $i \neq j$ holds $g(i)$ misses $g(j)$, and
 - (iii) for every n there exists a finite family f_1 of subsets of X such that $f_1 = \{f(i); i \text{ ranges over elements of } \mathbb{N}: i < n\}$ and $g(n) = f(n) \setminus \bigcup f_1$.
- (34) Let given T . Suppose T is finite-ind and $\text{ind } T \leq 0$ and T is Lindelöf. Let A, B be closed subsets of T . Suppose A misses B . Then there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.
- (35) Let given T . Suppose T is a T_1 space and Lindelöf. Then T is finite-ind and $\text{ind } T \leq 0$ if and only if for all closed subsets A, B of T such that A misses B holds \emptyset_T separates A, B .
- (36) Let given T . Suppose that
- (i) T is a T_4 space, a T_1 space, and Lindelöf, and
 - (ii) there exists a family F of subsets of T such that F is closed, a cover of T , countable, and finite-ind and $\text{ind } F \leq 0$.
- Then T is finite-ind and $\text{ind } T \leq 0$.

In the sequel T_4 is a metrizable topological space.

We now state four propositions:

- (37) Let A, B be closed subsets of T_4 . Suppose A misses B . Let N_1 be a finite-ind subset of T_4 . Suppose $\text{ind } N_1 \leq 0$ and $T_4 \upharpoonright N_1$ is second-countable. Then there exists a subset L of T_4 such that L separates A, B and L misses N_1 .
- (38) Let N_1 be a subset of T_4 . Suppose $T_4 \upharpoonright N_1$ is second-countable. Then N_1 is finite-ind and $\text{ind } N_1 \leq 0$ if and only if for every point p of T_4 and for every open subset U of T_4 such that $p \in U$ there exists an open subset W of T_4 such that $p \in W$ and $W \subseteq U$ and N_1 misses $\text{Fr } W$.

- (39) Let N_1 be a subset of T_4 . Suppose $T_4 \upharpoonright N_1$ is second-countable. Then N_1 is finite-ind and $\text{ind } N_1 \leq 0$ if and only if there exists a basis B of T_4 such that for every subset A of T_4 such that $A \in B$ holds N_1 misses $\text{Fr } A$.
- (40) Let N_1, A be subsets of T_4 . Suppose $T_4 \upharpoonright N_1$ is second-countable and N_1 is finite-ind and A is finite-ind and $\text{ind } N_1 \leq 0$. Then $A \cup N_1$ is finite-ind and $\text{ind}(A \cup N_1) \leq \text{ind } A + 1$.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(4):481–485, 1991.
- [4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [5] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(2):383–386, 1990.
- [6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [7] Roman Duda. *Wprowadzenie do topologii*. PWN, 1986.
- [8] Adam Grabowski. Properties of the product of compact topological spaces. *Formalized Mathematics*, 8(1):55–59, 1999.
- [9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [10] Robert Milewski. Bases of continuous lattices. *Formalized Mathematics*, 7(2):285–294, 1998.
- [11] Andrzej Nędzusiak. σ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [14] Karol Pał. Basic properties of metrizable topological spaces. *Formalized Mathematics*, 17(3):201–205, 2009, doi: 10.2478/v10037-009-0024-8.
- [15] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(2):233–236, 1996.
- [16] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [17] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(4):535–545, 1991.
- [18] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [20] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(1):231–237, 1990.

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