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Small Inductive Dimension of Topological Spaces. Part II

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Summary. In this paper we present basic properties of n-dimensional topological spaces according to the book [10]. In the article the formalization of Section 1.5 is completed.

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The papers [15], [1], [3], [9], [5], [8], [16], [2], [4], [6], [13], [12], [17], [14], [18], [7], and [11] provide the terminology and notation for this paper.

1. Order of a Family of Subsets of a Set

In this paper n denotes a natural number, X denotes a set, and F_1, G_1 denote families of subsets of X.

Let us consider X, F_1. We say that F_1 is finite-order if and only if:

(Def. 1) There exists n such that for every G_1 such that $G_1 \subseteq F_1$ and $n \in \mathrm{Card}\, G_1$ holds $\bigcap G_1$ is empty.

Let us consider X. Observe that there exists a family of subsets of X which is finite-order and every family of subsets of X which is finite is also finite-order.

Let us consider X, F_1. The functor order F_1 yielding an extended real number is defined as follows:

(Def. 2)(i) For every G_1 such that $\mathrm{order}\, F_1+1 \in \mathrm{Card}\, G_1$ and $G_1 \subseteq F_1$ holds $\bigcap G_1$ is empty and there exists G_1 such that $G_1 \subseteq F_1$ but $\mathrm{Card}\, G_1 = \mathrm{order}\, F_1+1$ but $\bigcap G_1$ is non empty or G_1 is empty if F_1 is finite-order,

(ii) $\mathrm{order}\, F_1 = +\infty$, otherwise.

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Let us consider X and let F be a finite-order family of subsets of X. Observe that $\operatorname{order} F + 1$ is natural and $\operatorname{order} F$ is integer.

Next we state three propositions:

(1) If $\operatorname{order} F_1 \leq n$, then F_1 is finite-order.

(2) If $\operatorname{order} F_1 \leq n$, then for every G_1 such that $G_1 \subseteq F_1$ and $n+1 \in \operatorname{Card} G_1$ holds $\bigcap G_1$ is empty.

(3) If for every finite family G of subsets of X such that $G \subseteq F_1$ and $n+1 < \overline{\overline{G}}$ holds $\bigcap G$ is empty, then $\operatorname{order} F_1 \leq n$.

2. Basic Properties of n-dimensional Topological Spaces

One can verify that there exists a topological space which is finite-ind, second-countable, and metrizable.

For simplicity, we adopt the following convention: T_1 is a metrizable topological space, T_2, T_3 are finite-ind second-countable metrizable topological spaces, A, B, L, H are subsets of T_1, U, W are open subsets of T_1, p is a point of T_1, F, G are finite families of subsets of T_1, and I is an integer.

We now state several propositions:

(4) Let given T_1. Suppose that

(i) T_1 is second-countable, and

(ii) there exists F such that F is closed, a cover of T_1, countable, and finite-ind and $\operatorname{ind} F \leq n$.

Then T_1 is finite-ind and $\operatorname{ind} T_1 \leq n$.

(5) Let A, B be finite-ind subsets of T_1. Suppose A is closed and $T_1 \restriction (A \cup B)$ is second-countable and $\operatorname{ind} A \leq I$ and $\operatorname{ind} B \leq I$. Then $\operatorname{ind}(A \cup B) \leq I$ and $A \cup B$ is finite-ind.

(6) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq n$. Then there exist A, B such that $\Omega_{(T_1)} = A \cup B$ and A misses B and $\operatorname{ind} A \leq n - 1$ and $\operatorname{ind} B \leq 0$.

(7) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq I$. Then there exists F such that

(i) F is a cover of T_1 and finite-ind,

(ii) $\operatorname{ind} F \leq 0$,

(iii) $\overline{\overline{F}} \leq I + 1$, and

(iv) for all A, B such that $A, B \in F$ and A meets B holds $A = B$.

(8) Let given T_1. Suppose T_1 is second-countable and there exists F such that F is a cover of T_1 and finite-ind and $\operatorname{ind} F \leq 0$ and $\overline{\overline{F}} \leq I + 1$. Then T_1 is finite-ind and $\operatorname{ind} T_1 \leq I$.

Let T_1 be a second-countable metrizable topological space and let A, B be finite-ind subsets of T_1. One can check that $A \cup B$ is finite-ind.

Next we state two propositions:

(9) If A is finite-ind and B is finite-ind and $T_1 \restriction (A \cup B)$ is second-countable, then $A \cup B$ is finite-ind and $\operatorname{ind}(A \cup B) \leq \operatorname{ind} A + \operatorname{ind} B + 1$.

(10) For all topological spaces T_4, T_5 and for every subset A_1 of T_4 and for every subset A_2 of T_5 holds $\operatorname{Fr}(A_1 \times A_2) = \operatorname{Fr} A_1 \times \overline{A_2} \cup \overline{A_1} \times \operatorname{Fr} A_2$.

Let us consider T_2, T_3. Observe that $T_2 \times T_3$ is finite-ind.

We now state several propositions:

(11) Let given A, B. Suppose A is closed and B is closed and A misses B. Let given H. Suppose $\operatorname{ind} H \leq n$ and $T_1 \restriction H$ is second-countable and finite-ind. Then there exists L such that L separates A, B and $\operatorname{ind}(L \cap H) \leq n - 1$.

(12) Let given T_1. Suppose T_1 is finite-ind and second-countable and $\operatorname{ind} T_1 \leq n$. Let given A, B. Suppose A is closed and B is closed and A misses B. Then there exists L such that L separates A, B and $\operatorname{ind} L \leq n - 1$.

(13) Let given H. Suppose $T_1 \restriction H$ is second-countable. Then H is finite-ind and $\operatorname{ind} H \leq n$ if and only if for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $H \cap \operatorname{Fr} W$ is finite-ind and $\operatorname{ind}(H \cap \operatorname{Fr} W) \leq n - 1$.

(14) Let given H. Suppose $T_1 \restriction H$ is second-countable. Then H is finite-ind and $\operatorname{ind} H \leq n$ if and only if there exists a basis B_1 of T_1 such that for every A such that $A \in B_1$ holds $H \cap \operatorname{Fr} A$ is finite-ind and $\operatorname{ind}(H \cap \operatorname{Fr} A) \leq n - 1$.

(15) If T_2 is non empty or T_3 is non empty, then $\operatorname{ind}(T_2 \times T_3) \leq \operatorname{ind} T_2 + \operatorname{ind} T_3$.

(16) If $\operatorname{ind} T_3 = 0$, then $\operatorname{ind}(T_2 \times T_3) = \operatorname{ind} T_2$.

3. Small Inductive Dimension of Euclidean Spaces

For simplicity, we follow the rules: u denotes a point of $\mathcal{E}^1$, U denotes a point of $\mathcal{E}^1_{\mathrm{T}}$, r, u_1 denote real numbers, and s denotes a real number.

Next we state three propositions:

(17) If $\langle u_1 \rangle = u$ and $r > 0$, then $\overline{\operatorname{Ball}}(u, r) = \{\langle s \rangle : u_1 - r \leq s \ \wedge \ s \leq u_1 + r\}$.

(18) If $\langle u_1 \rangle = U$ and $r > 0$, then $\operatorname{Fr} \operatorname{Ball}(U, r) = \{\langle u_1 - r \rangle, \langle u_1 + r \rangle\}$.

(19) Let T be a topological space and A be a countable subset of T. If $T \restriction A$ is a T_4 space, then A is finite-ind and $\operatorname{ind} A \leq 0$.

Let T_1 be a metrizable topological space. Observe that every subset of T_1 which is countable is also finite-ind.

Let n be a natural number. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is finite-ind.

One can prove the following propositions:

(20) If $n \leq 1$, then $\operatorname{ind}(\mathcal{E}^n_{\mathrm{T}}) = n$.

(21) $\operatorname{ind}(\mathcal{E}^n_{\mathrm{T}}) \leq n$.

(22) Let given A. Suppose $T_1 \restriction A$ is second-countable and finite-ind and $\operatorname{ind} A \leq 0$. Let given F. Suppose F is open and a cover of A. Then there exists a function g from F into $2^{\text{the carrier of } T_1}$ such that
  (i) $\operatorname{rng} g$ is open,
  (ii) $\operatorname{rng} g$ is a cover of A,
  (iii) for every set a such that $a \in F$ holds $g(a) \subseteq a$, and
  (iv) for all sets a, b such that $a, b \in F$ and $a \neq b$ holds $g(a)$ misses $g(b)$.

(23) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq n$. Let given F. Suppose F is open and a cover of T_1. Then there exists G such that G is open, a cover of T_1, and finer than F and $\overline{\overline{G}} \leq \overline{\overline{F}} \cdot (n+1)$ and $\operatorname{order} G \leq n$.

(24) Let given T_1. Suppose T_1 is finite-ind. Let given A. Suppose $\operatorname{ind}(A^{\mathrm{c}}) \leq n$ and $T_1 \restriction A^{\mathrm{c}}$ is second-countable. Let A_1, A_2 be closed subsets of T_1. Suppose $A = A_1 \cup A_2$. Then there exist closed subsets X_1, X_2 of T_1 such that $\Omega_{(T_1)} = X_1 \cup X_2$ and $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ and $A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$ and $\operatorname{ind}(X_1 \cap X_2 \setminus A_1 \cap A_2) \leq n - 1$.

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