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# Complex Integral[1]

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**Summary.** In this article, we defined complex curve and complex integral. Then we have proved the linearity for the complex integral. Furthermore, we have proved complex integral of complex curve's connection is the sum of each complex integral of individual complex curve.

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The terminology and notation used here are introduced in the following articles: [10], [2], [14], [11], [12], [3], [4], [1], [7], [15], [5], [13], [8], [17], [9], [16], and [6].

## 1. The Definition of Complex Curve and Complex Integral

In this paper $t$ is an element of $\mathbb{R}$.

The function $\mathbb{R}^2 \to \mathbb{C}$ from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{C}$ is defined as follows:

(Def. 1) For every element $p$ of $\mathbb{R} \times \mathbb{R}$ and for all elements $a$, $b$ of $\mathbb{R}$ such that $a = p_{\mathbf{1}}$ and $b = p_{\mathbf{2}}$ holds $(\mathbb{R}^2 \to \mathbb{C})(\langle a, b \rangle) = a + b \cdot i$.

Let $a$, $b$ be real numbers, let $x$, $y$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$, let $Z$ be a subset of $\mathbb{R}$, and let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\int(f, x, y, a, b, Z)$ yielding a complex number is defined by the condition (Def. 2).

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(Def. 2) There exist partial functions $u_0$, $v_0$ from $\mathbb{R}$ to $\mathbb{R}$ such that $u_0 = \Re(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $v_0 = \Im(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $\int(f, x, y, a, b, Z) = \int_a^b (u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z})(x)dx + \int_a^b (v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z})(x)dx \cdot i$.

Let $C$ be a partial function from $\mathbb{R}$ to $\mathbb{C}$. We say that $C$ is $C_1$-curve-like if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers $a$, $b$ and there exist partial functions $x$, $y$ from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset $Z$ of $\mathbb{R}$ such that
$a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and $Z$ is open and $[a, b] \subseteq Z$ and $x$ is differentiable on $Z$ and $y$ is differentiable on $Z$ and $x$ is continuous on $Z$ and $y$ is continuous on $Z$ and $C = (x + i\, y){\restriction}[a, b]$.

Let us observe that there exists a partial function from $\mathbb{R}$ to $\mathbb{C}$ which is $C_1$-curve-like.

A $C_1$-curve is a $C_1$-curve-like partial function from $\mathbb{R}$ to $\mathbb{C}$.

Let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let $C$ be a $C_1$-curve. Let us assume that $\operatorname{rng} C \subseteq \operatorname{dom} f$. The functor $\int_C f(x)dx$ yields a complex number and is defined by the condition (Def. 4).

(Def. 4) There exist real numbers $a$, $b$ and there exist partial functions $x$, $y$ from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset $Z$ of $\mathbb{R}$ such that
$a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and $Z$ is open and $[a, b] \subseteq Z$ and $x$ is differentiable on $Z$ and $y$ is differentiable on $Z$ and $x$ is continuous on $Z$ and $y$ is continuous on $Z$ and $C = (x + i\, y){\restriction}[a, b]$ and $\int_C f(x)dx = \int(f, x, y, a, b, Z)$.

Let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let $C$ be a $C_1$-curve. We say that $f$ is integrable on $C$ if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let $a$, $b$ be real numbers, $x$, $y$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$, $Z$ be a subset of $\mathbb{R}$, and $u_0$, $v_0$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and $Z$ is open and $[a, b] \subseteq Z$ and $x$ is differentiable on $Z$ and $y$ is differentiable on $Z$ and $x$ is continuous on $Z$ and $y$ is continuous on $Z$ and $C = (x + i\, y){\restriction}[a, b]$. Then $u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z}$ is integrable on $[a, b]$ and $v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z}$ is integrable on $[a, b]$.

Let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let $C$ be a $C_1$-curve. We say that $f$ is bounded on $C$ if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let $a$, $b$ be real numbers, $x$, $y$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$, $Z$ be a subset of $\mathbb{R}$, and $u_0$, $v_0$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and $Z$ is open and $[a, b] \subseteq Z$ and $x$ is differentiable on $Z$ and $y$ is differentiable on $Z$ and $x$ is continuous on $Z$ and $y$ is continuous on $Z$ and $C = (x + i\,y)\restriction[a, b]$. Then $(u_0\,x'_{\restriction Z} - v_0\,y'_{\restriction Z})\restriction[a, b]$ is bounded and $(v_0\,x'_{\restriction Z} + u_0\,y'_{\restriction Z})\restriction[a, b]$ is bounded.

## 2. Linearity of Complex Intergal

Next we state two propositions:

(1) Let $f$, $g$ be partial functions from $\mathbb{C}$ to $\mathbb{C}$ and $C$ be a $C_1$-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and $\operatorname{rng} C \subseteq \operatorname{dom} g$ and $f$ is integrable on $C$ and $g$ is integrable on $C$ and $f$ is bounded on $C$ and $g$ is bounded on $C$. Then $\int\limits_C (f + g)(x)dx = \int\limits_C f(x)dx + \int\limits_C g(x)dx$.

(2) Let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and $C$ be a $C_1$-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and $f$ is integrable on $C$ and $f$ is bounded on $C$. Let $r$ be a real number. Then $\int\limits_C (r\,f)(x)dx = r \cdot \int\limits_C f(x)dx$.

## 3. Complex Integral of Complex Curve's Connection

We now state the proposition

(3) Let $f$ be a partial function from $\mathbb{C}$ to $\mathbb{C}$, $C$, $C_1$, $C_2$ be $C_1$-curves, and $a$, $b$, $d$ be real numbers. Suppose that $\operatorname{rng} C \subseteq \operatorname{dom} f$ and $f$ is integrable on $C$ and $f$ is bounded on $C$ and $a \leq b$ and $\operatorname{dom} C = [a, b]$ and $d \in [a, b]$ and $\operatorname{dom} C_1 = [a, d]$ and $\operatorname{dom} C_2 = [d, b]$ and for every $t$ such that $t \in \operatorname{dom} C_1$ holds $C(t) = C_1(t)$ and for every $t$ such that $t \in \operatorname{dom} C_2$ holds $C(t) = C_2(t)$. Then $\int\limits_C f(x)dx = \int\limits_{C_1} f(x)dx + \int\limits_{C_2} f(x)dx$.

## References


[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[10] Takashi Mitsuishi, Katsumi Wasaki, and Yasunari Shidama. Property of complex functions. *Formalized Mathematics*, 9(**1**):179–184, 2001.

[11] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[12] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[14] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Formalized Mathematics*, 6(**3**):403–410, 1997.

[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[17] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.


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