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The Sum and Product of Finite Sequences of Complex Numbers

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Summary. This article extends the [10]. We define the sum and the product of the sequence of complex numbers, and formalize these theorems. Our method refers to the [11].

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The notation and terminology used in this paper have been introduced in the following papers: [5], [7], [6], [4], [8], [13], [9], [2], [3], [15], [10], [12], and [14].

1. Auxiliary Theorems

Let F be a complex-valued binary relation. Then $\mathrm{rng}\, F$ is a subset of $\mathbb{C}$.

Let D be a non empty set, let F be a function from $\mathbb{C}$ into D, and let F_1 be a complex-valued finite sequence. Note that $F \cdot F_1$ is finite sequence-like.

For simplicity, we adopt the following rules: i, j denote natural numbers, x, x_1 denote elements of $\mathbb{C}$, c denotes a complex number, F, F_1, F_2 denote complex-valued finite sequences, and R, R_1 denote i-element finite sequences of elements of $\mathbb{C}$.

The unary operation sqrcomplex on $\mathbb{C}$ is defined as follows:

(Def. 1) For every c holds $(\mathrm{sqrcomplex})(c) = c^{\mathbf{2}}$.

Next we state two propositions:

(1) sqrcomplex is distributive w.r.t. $\cdot_{\mathbb{C}}$.

(2) $\cdot_{\mathbb{C}}^{c}$ is distributive w.r.t. $+_{\mathbb{C}}$.

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2. Some Functors on the i-Tuples of Complex Numbers

Let us consider F_1, F_2. Then $F_1 + F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 2) $F_1 + F_2 = (+_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us observe that the functor $F_1 + F_2$ is commutative.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(3) $(R_1 + R_2)(s) = R_1(s) + R_2(s)$.

(4) $\varepsilon_{\mathbb{C}} + F = \varepsilon_{\mathbb{C}}$.

(5) $\langle c_1 \rangle + \langle c_2 \rangle = \langle c_1 + c_2 \rangle$.

(6) $i \mapsto c_1 + i \mapsto c_2 = i \mapsto (c_1 + c_2)$.

Let us consider F. Then $-F$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 3) $-F = -_{\mathbb{C}} \cdot F$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(7) $-\langle c \rangle = \langle -c \rangle$.

(8) $-i \mapsto c = i \mapsto (-c)$.

(9) If $R_1 + R = R_2 + R$, then $R_1 = R_2$.

(10) $-(F_1 + F_2) = -F_1 + -F_2$.

Let us consider F_1, F_2. Then $F_1 - F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 4) $F_1 - F_2 = (-_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(11) $(R_1 - R_2)(s) = R_1(s) - R_2(s)$.

(12) $\varepsilon_{\mathbb{C}} - F = \varepsilon_{\mathbb{C}}$ and $F - \varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{C}}$.

(13) $\langle c_1 \rangle - \langle c_2 \rangle = \langle c_1 - c_2 \rangle$.

(14) $i \mapsto c_1 - i \mapsto c_2 = i \mapsto (c_1 - c_2)$.

(15) $R - i \mapsto 0_{\mathbb{C}} = R$.

(16) $-(F_1 - F_2) = F_2 - F_1$.

(17) $-(F_1 - F_2) = -F_1 + F_2$.

(18) If $R_1 - R_2 = i \mapsto 0_{\mathbb{C}}$, then $R_1 = R_2$.

(19) $R_1 = (R_1 + R) - R$.

(20) $R_1 = (R_1 - R) + R$.

Let us consider F, c. We introduce $c \cdot F$ as a synonym of $c\,F$.

Let us consider F, c. Then $c \cdot F$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 5) $c \cdot F = \cdot^{c}_{\mathbb{C}} \cdot F$.

Let us consider i, R, c. Then $c \cdot R$ is an element of $\mathbb{C}^i$.

One can prove the following four propositions:

(21) $c \cdot \langle c_1 \rangle = \langle c \cdot c_1 \rangle$.

(22) $c_1 \cdot (i \mapsto c_2) = i \mapsto (c_1 \cdot c_2)$.

(23) $(c_1 + c_2) \cdot F = c_1 \cdot F + c_2 \cdot F$.

(24) $0_{\mathbb{C}} \cdot R = i \mapsto 0_{\mathbb{C}}$.

Let us consider F_1, F_2. We introduce $F_1 \bullet F_2$ as a synonym of $F_1\, F_2$.

Let us consider F_1, F_2. Then $F_1 \bullet F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 6) $F_1 \bullet F_2 = (\cdot_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us note that the functor $F_1 \bullet F_2$ is commutative.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{C}^i$.

Next we state four propositions:

(25) $\varepsilon_{\mathbb{C}} \bullet F = \varepsilon_{\mathbb{C}}$.

(26) $\langle c_1 \rangle \bullet \langle c_2 \rangle = \langle c_1 \cdot c_2 \rangle$.

(27) $i \mapsto c \bullet R = c \cdot R$.

(28) $i \mapsto c_1 \bullet i \mapsto c_2 = i \mapsto (c_1 \cdot c_2)$.

3. Finite Sum of Finite Sequence of Complex Numbers

One can prove the following propositions:

(29) $\sum(\varepsilon_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(30) $\sum\langle c \rangle = c$.

(31) $\sum(F \frown \langle c \rangle) = \sum F + c$.

(32) $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

(33) $\sum(\langle c \rangle \frown F) = c + \sum F$.

(34) $\sum\langle c_1, c_2 \rangle = c_1 + c_2$.

(35) $\sum\langle c_1, c_2, c_3 \rangle = c_1 + c_2 + c_3$.

(36) $\sum(i \mapsto c) = i \cdot c$.

(37) $\sum(i \mapsto 0_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(38) $\sum(c \cdot F) = c \cdot \sum F$.

(39) $\sum(-F) = -\sum F$.

(40) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(41) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

4. The Product of Finite Sequences of Complex Numbers

One can prove the following propositions:

(42) $\prod(\varepsilon_{\mathbb{C}}) = 1$.

(43) $\prod(\langle c \rangle \frown F) = c \cdot \prod F$.

(44) For every element R of $\mathbb{C}^0$ holds $\prod R = 1$.

(45) $\prod((i + j) \mapsto c) = \prod(i \mapsto c) \cdot \prod(j \mapsto c)$.

(46) $\prod((i \cdot j) \mapsto c) = \prod(j \mapsto \prod(i \mapsto c))$.

(47) $\prod(i \mapsto (c_1 \cdot c_2)) = \prod(i \mapsto c_1) \cdot \prod(i \mapsto c_2)$.

(48) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(49) $\prod(c \cdot R) = \prod(i \mapsto c) \cdot \prod R$.

5. Modified Part of [1]

We now state several propositions:

(50) For every complex-valued finite sequence x holds $\operatorname{len}(-x) = \operatorname{len} x$.

(51) For all complex-valued finite sequences x_1, x_2 such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 + x_2) = \operatorname{len} x_1$.

(52) For all complex-valued finite sequences x_1, x_2 such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 - x_2) = \operatorname{len} x_1$.

(53) For every real number a and for every complex-valued finite sequence x holds $\operatorname{len}(a \cdot x) = \operatorname{len} x$.

(54) For all complex-valued finite sequences x, y, z such that $\operatorname{len} x = \operatorname{len} y = \operatorname{len} z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

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Second-Order Partial Differentiation of Real Ternary Functions

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Summary. In this article, we shall extend the result of [17] to discuss second-order partial differentiation of real ternary functions (refer to [7] and [14] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [6], [11], [12], [1], [2], [3], [4], [5], [7], [16], [17], [13], [8], [15], [10], and [9].

1. Second-order Partial Derivatives

For simplicity, we use the following convention: x, x_0, y, y_0, z, z_0, r denote real numbers, u, u_0 denote elements of $\mathcal{R}^3$, f, f_1, f_2 denote partial functions from $\mathcal{R}^3$ to $\mathbb{R}$, R denotes a rest, and L denotes a linear function.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. We say that f is partial differentiable on 1st-1st coordinate in u if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

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We say that f is partial differentiable on 1st-2nd coordinate in u if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 1st-3rd coordinate in u if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in u if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in u if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 2nd-3rd coordinate in u if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 3rd-1st coordinate in u if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exist real numbers x_0, y_0, z_0 such that
 (i) $u = \langle x_0, y_0, z_0 \rangle$, and
 (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 3rd-2nd coordinate in u if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exist real numbers x_0, y_0, z_0 such that
 (i) $u = \langle x_0, y_0, z_0 \rangle$, and
 (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 3rd-3rd coordinate in u if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exist real numbers x_0, y_0, z_0 such that
 (i) $u = \langle x_0, y_0, z_0 \rangle$, and
 (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-1st coordinate in u. The functor hpartdiff11(f, u) yielding a real number is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0, z_0 such that
 (i) $u = \langle x_0, y_0, z_0 \rangle$, and
 (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that $\mathrm{hpartdiff11}(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-2nd coordinate in u. The functor hpartdiff12(f, u) yielding a real number is defined by the condition (Def. 11).

(Def. 11) There exist real numbers x_0, y_0, z_0 such that
 (i) $u = \langle x_0, y_0, z_0 \rangle$, and
 (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that $\mathrm{hpartdiff12}(f, u) =$

$L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-3rd coordinate in u. The functor hpartdiff13(f, u) yielding a real number is defined by the condition (Def. 12).

(Def. 12) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that hpartdiff13$(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-1st coordinate in u. The functor hpartdiff21(f, u) yielding a real number is defined by the condition (Def. 13).

(Def. 13) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff21$(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-2nd coordinate in u. The functor hpartdiff22(f, u) yielding a real number is defined by the condition (Def. 14).

(Def. 14) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff22$(f, u) = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-3rd coordinate in u. The functor hpartdiff23(f, u) yielding a real number is defined by the condition (Def. 15).

(Def. 15) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff23$(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-1st coordinate in u. The functor hpartdiff31(f, u) yields a real number and is defined by the condition (Def. 16).

(Def. 16) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that hpartdiff31$(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-2nd coordinate in u. The functor hpartdiff32(f, u) yielding a real number is defined by the condition (Def. 17).

(Def. 17) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that hpartdiff32$(f, u) = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-3rd coordinate in u. The functor hpartdiff33(f, u) yielding a real number is defined by the condition (Def. 18).

(Def. 18) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that hpartdiff33$(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Next we state a number of propositions:

(1) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u, then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u)$ is differentiable in x_0.

(2) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u, then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u)$ is differentiable in y_0.

(3) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u, then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u)$ is differentiable in z_0.

(4) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u, then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u)$ is differentiable in x_0.

(5) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u, then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u)$ is differentiable in y_0.

(6) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u, then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u)$ is differentiable in z_0.

(7) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u, then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u)$ is differentiable in x_0.

(8) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u)$ is differentiable in y_0.

(9) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u)$ is differentiable in z_0.

(10) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u, then $\mathrm{hpartdiff11}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))'(x_0)$.

(11) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u, then $\mathrm{hpartdiff12}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))'(y_0)$.

(12) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u, then $\mathrm{hpartdiff13}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))'(z_0)$.

(13) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u, then $\mathrm{hpartdiff21}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))'(x_0)$.

(14) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u, then $\mathrm{hpartdiff22}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))'(y_0)$.

(15) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u, then $\mathrm{hpartdiff23}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))'(z_0)$.

(16) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u, then $\mathrm{hpartdiff31}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))'(x_0)$.

(17) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{hpartdiff32}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))'(y_0)$.

(18) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{hpartdiff33}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))'(z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. We say that f is partial differentiable on 1st-1st coordinate on D if and only if:

(Def. 19) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-1st coordinate in u.

We say that f is partial differentiable on 1st-2nd coordinate on D if and only if:

(Def. 20) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-2nd coordinate in u.

We say that f is partial differentiable on 1st-3rd coordinate on D if and only if:

(Def. 21) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-3rd coordinate in u.

We say that f is partial differentiable on 2nd-1st coordinate on D if and only if:

(Def. 22) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 2nd-1st coordinate in u.

We say that f is partial differentiable on 2nd-2nd coordinate on D if and only if:

(Def. 23) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 2nd-2nd coordinate in u.

We say that f is partial differentiable on 2nd-3rd coordinate on D if and only if:

(Def. 24) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f\restriction D$ is partial differentiable on 2nd-3rd coordinate in u.

We say that f is partial differentiable on 3rd-1st coordinate on D if and only if:

(Def. 25) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f\restriction D$ is partial differentiable on 3rd-1st coordinate in u.

We say that f is partial differentiable on 3rd-2nd coordinate on D if and only if:

(Def. 26) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f\restriction D$ is partial differentiable on 3rd-2nd coordinate in u.

We say that f is partial differentiable on 3rd-3rd coordinate on D if and only if:

(Def. 27) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f\restriction D$ is partial differentiable on 3rd-3rd coordinate in u.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on D. The functor $f{\restriction}_D^{\text{1st}-\text{1st}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 28) $\operatorname{dom}(f{\restriction}_D^{\text{1st}-\text{1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\text{1st}-\text{1st}}(u) = \text{hpartdiff11}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on D. The functor $f{\restriction}_D^{\text{1st}-\text{2nd}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined by:

(Def. 29) $\operatorname{dom}(f{\restriction}_D^{\text{1st}-\text{2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\text{1st}-\text{2nd}}(u) = \text{hpartdiff12}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-3rd coordinate on D. The functor $f{\restriction}_D^{\text{1st}-\text{3rd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 30) $\operatorname{dom}(f{\restriction}_D^{\text{1st}-\text{3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\text{1st}-\text{3rd}}(u) = \text{hpartdiff13}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on D. The functor $f{\restriction}_D^{\text{2nd}-\text{1st}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 31) $\operatorname{dom}(f{\restriction}_D^{\text{2nd}-\text{1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\text{2nd}-\text{1st}}(u) = \text{hpartdiff21}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on D. The functor $f{\restriction}_D^{\text{2nd}-\text{2nd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 32) $\operatorname{dom}(f{\restriction}_D^{\text{2nd}-\text{2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\text{2nd}-\text{2nd}}(u) = \text{hpartdiff22}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-3rd coordinate on D. The functor $f{\restriction}_D^{\text{2nd}-\text{3rd}}$

yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 33) $\mathrm{dom}(f{\restriction}_D^{\mathrm{2nd-3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{2nd-3rd}}(u) = \mathrm{hpartdiff23}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-1st coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-1st}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined as follows:

(Def. 34) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-1st}}(u) = \mathrm{hpartdiff31}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-2nd coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-2nd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 35) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-2nd}}(u) = \mathrm{hpartdiff32}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-3rd coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-3rd}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined by:

(Def. 36) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-3rd}}(u) = \mathrm{hpartdiff33}(f, u)$.

2. Main Properties of Second-order Partial Derivatives

Next we state a number of propositions:

(19) f is partial differentiable on 1st-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 1.

(20) f is partial differentiable on 1st-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 2.

(21) f is partial differentiable on 1st-3rd coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 3.

(22) f is partial differentiable on 2nd-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 1.

(23) f is partial differentiable on 2nd-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 2.

(24) f is partial differentiable on 2nd-3rd coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 3.

(25) f is partial differentiable on 3rd-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 1.

(26) f is partial differentiable on 3rd-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 2.

(27) f is partial differentiable on 3rd-3rd coordinate in u if and only if $\mathrm{pdiff1}(f,3)$ is partially differentiable in u w.r.t. 3.

(28) If f is partial differentiable on 1st-1st coordinate in u, then $\mathrm{hpartdiff11}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,1)$.

(29) If f is partial differentiable on 1st-2nd coordinate in u, then $\mathrm{hpartdiff12}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,2)$.

(30) If f is partial differentiable on 1st-3rd coordinate in u, then $\mathrm{hpartdiff13}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,3)$.

(31) If f is partial differentiable on 2nd-1st coordinate in u, then $\mathrm{hpartdiff21}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,1)$.

(32) If f is partial differentiable on 2nd-2nd coordinate in u, then $\mathrm{hpartdiff22}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,2)$.

(33) If f is partial differentiable on 2nd-3rd coordinate in u, then $\mathrm{hpartdiff23}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,3)$.

(34) If f is partial differentiable on 3rd-1st coordinate in u, then $\mathrm{hpartdiff31}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,1)$.

(35) If f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{hpartdiff32}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,2)$.

(36) If f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{hpartdiff33}(f,u)=\mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,3)$.

(37) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 1st-1st coordinate in u_0 and $N\subseteq\mathrm{dom}\,\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c=\{(\mathrm{proj}(1,3))(u_0)\}$ and $\mathrm{rng}(h+c)\subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*(h+c))-(\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*c))$ is convergent and $\mathrm{hpartdiff11}(f,u_0)=\lim(h^{-1}\,((\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*(h+c))-(\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*c)))$.

(38) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in u_0 and $N\subseteq\mathrm{dom}\,\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c=\{(\mathrm{proj}(2,3))(u_0)\}$ and $\mathrm{rng}(h+c)\subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*(h+c))-(\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*c))$ is convergent and $\mathrm{hpartdiff12}(f,u_0)=\lim(h^{-1}\,((\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*(h+c))-(\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*c)))$.

(39) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(3,3))(u_0)$. Suppose f is partial differentiable on 1st-3rd coordinate in u_0 and $N\subseteq\mathrm{dom}\,\mathrm{SVF1}(3,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real num-

bers. Suppose $\operatorname{rng} c = \{(\text{proj}(3,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(3, \text{pdiff1}(f,1), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f,1), u_0)_*c))$ is convergent and $\text{hpartdiff13}(f, u_0) = \lim(h^{-1}((\text{SVF1}(3, \text{pdiff1}(f,1), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f,1), u_0)_*c)))$.

(40) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\text{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in u_0 and $N \subseteq \operatorname{dom} \text{SVF1}(1, \text{pdiff1}(f,2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\text{proj}(1,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(1, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f,2), u_0)_*c))$ is convergent and $\text{hpartdiff21}(f, u_0) = \lim(h^{-1}((\text{SVF1}(1, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f,2), u_0)_*c)))$.

(41) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\text{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in u_0 and $N \subseteq \operatorname{dom} \text{SVF1}(2, \text{pdiff1}(f,2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\text{proj}(2,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(2, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f,2), u_0)_*c))$ is convergent and $\text{hpartdiff22}(f, u_0) = \lim(h^{-1}((\text{SVF1}(2, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f,2), u_0)_*c)))$.

(42) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\text{proj}(3,3))(u_0)$. Suppose f is partial differentiable on 2nd-3rd coordinate in u_0 and $N \subseteq \operatorname{dom} \text{SVF1}(3, \text{pdiff1}(f,2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\text{proj}(3,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(3, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f,2), u_0)_*c))$ is convergent and $\text{hpartdiff23}(f, u_0) = \lim(h^{-1}((\text{SVF1}(3, \text{pdiff1}(f,2), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f,2), u_0)_*c)))$.

(43) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\text{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 3rd-1st coordinate in u_0 and $N \subseteq \operatorname{dom} \text{SVF1}(1, \text{pdiff1}(f,3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\text{proj}(1,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(1, \text{pdiff1}(f,3), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f,3), u_0)_*c))$ is convergent and $\text{hpartdiff31}(f, u_0) = \lim(h^{-1}((\text{SVF1}(1, \text{pdiff1}(f,3), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f,3), u_0)_*c)))$.

(44) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\text{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 3rd-2nd coordinate in u_0 and $N \subseteq \operatorname{dom} \text{SVF1}(2, \text{pdiff1}(f,3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\text{proj}(2,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then

h^{-1} ((SVF1(2, pdiff1(f, 3), u_0)$_*$(h+c))−(SVF1(2, pdiff1(f, 3), u_0)$_*c$)) is convergent and hpartdiff32(f, u_0) = lim(h^{-1} ((SVF1(2, pdiff1(f, 3), u_0)$_*$(h + c)) − (SVF1(2, pdiff1(f, 3), u_0)$_*c$))).

(45) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of (proj(3, 3))(u_0). Suppose f is partial differentiable on 3rd-3rd coordinate in u_0 and $N \subseteq$ dom SVF1(3, pdiff1(f, 3), u_0). Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose rng c = {(proj(3, 3))(u_0)} and rng(h + c) $\subseteq N$. Then h^{-1} ((SVF1(3, pdiff1(f, 3), u_0)$_*$(h+c))−(SVF1(3, pdiff1(f, 3), u_0)$_*c$)) is convergent and hpartdiff33(f, u_0) = lim(h^{-1} ((SVF1(3, pdiff1(f, 3), u_0)$_*$(h + c)) − (SVF1(3, pdiff1(f, 3), u_0)$_*c$))).

(46) Suppose that
  (i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 1) = hpartdiff11(f_1, u_0) + hpartdiff11(f_2, u_0).

(47) Suppose that
  (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 2) = hpartdiff12(f_1, u_0) + hpartdiff12(f_2, u_0).

(48) Suppose that
  (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 3 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 3) = hpartdiff13(f_1, u_0) + hpartdiff13(f_2, u_0).

(49) Suppose that
  (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
  (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.

Then pdiff1(f_1, 2) + pdiff1(f_2, 2) is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1(f_1, 2) + pdiff1(f_2, 2), u_0, 1) = hpartdiff21(f_1, u_0) + hpartdiff21(f_2, u_0).

(50) Suppose that
  (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
  (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.

Then pdiff1(f_1, 2) + pdiff1(f_2, 2) is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1(f_1, 2) + pdiff1(f_2, 2), u_0, 2) = hpartdiff22(f_1, u_0) + hpartdiff22(f_2, u_0).

(51) Suppose that
(i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 2) + \mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 2) + \mathrm{pdiff1}(f_2, 2), u_0, 3) = \mathrm{hpartdiff23}(f_1, u_0) + \mathrm{hpartdiff23}(f_2, u_0)$.

(52) Suppose that
(i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 1 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1), u_0, 1) = \mathrm{hpartdiff11}(f_1, u_0) - \mathrm{hpartdiff11}(f_2, u_0)$.

(53) Suppose that
(i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 2 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1), u_0, 2) = \mathrm{hpartdiff12}(f_1, u_0) - \mathrm{hpartdiff12}(f_2, u_0)$.

(54) Suppose that
(i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 3 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 1) - \mathrm{pdiff1}(f_2, 1), u_0, 3) = \mathrm{hpartdiff13}(f_1, u_0) - \mathrm{hpartdiff13}(f_2, u_0)$.

(55) Suppose that
(i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2), u_0, 1) = \mathrm{hpartdiff21}(f_1, u_0) - \mathrm{hpartdiff21}(f_2, u_0)$.

(56) Suppose that
(i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2), u_0, 2) = \mathrm{hpartdiff22}(f_1, u_0) - \mathrm{hpartdiff22}(f_2, u_0)$.

(57) Suppose that
(i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
Then $\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and $\mathrm{partdiff}(\mathrm{pdiff1}(f_1, 2) - \mathrm{pdiff1}(f_2, 2), u_0, 3) = \mathrm{hpartdiff23}(f_1, u_0) -$

hpartdiff23(f_2, u_0).

(58) Suppose f is partial differentiable on 1st-1st coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 1$), u_0, 1) $= r \cdot$ hpartdiff11(f, u_0).

(59) Suppose f is partial differentiable on 1st-2nd coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 1$), u_0, 2) $= r \cdot$ hpartdiff12(f, u_0).

(60) Suppose f is partial differentiable on 1st-3rd coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 1$), u_0, 3) $= r \cdot$ hpartdiff13(f, u_0).

(61) Suppose f is partial differentiable on 2nd-1st coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 2$), u_0, 1) $= r \cdot$ hpartdiff21(f, u_0).

(62) Suppose f is partial differentiable on 2nd-2nd coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 2$), u_0, 2) $= r \cdot$ hpartdiff22(f, u_0).

(63) Suppose f is partial differentiable on 2nd-3rd coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 2$), u_0, 3) $= r \cdot$ hpartdiff23(f, u_0).

(64) Suppose f is partial differentiable on 3rd-1st coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 3$), u_0, 1) $= r \cdot$ hpartdiff31(f, u_0).

(65) Suppose f is partial differentiable on 3rd-2nd coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 3$), u_0, 2) $= r \cdot$ hpartdiff32(f, u_0).

(66) Suppose f is partial differentiable on 3rd-3rd coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 3$), u_0, 3) $= r \cdot$ hpartdiff33(f, u_0).

(67) Suppose that
 (i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
 (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.
 Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 1.

(68) Suppose that
 (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.
 Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 2.

(69) Suppose that
 (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.
 Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 3.

(70) Suppose that
(i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.
Then $\text{pdiff1}(f_1, 2)\ \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1.

(71) Suppose that
(i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.
Then $\text{pdiff1}(f_1, 2)\ \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2.

(72) Suppose that
(i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
Then $\text{pdiff1}(f_1, 2)\ \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3.

(73) Suppose that
(i) f_1 is partial differentiable on 3rd-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 3rd-1st coordinate in u_0.
Then $\text{pdiff1}(f_1, 3)\ \text{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 1.

(74) Suppose that
(i) f_1 is partial differentiable on 3rd-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 3rd-2nd coordinate in u_0.
Then $\text{pdiff1}(f_1, 3)\ \text{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 2.

(75) Suppose that
(i) f_1 is partial differentiable on 3rd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 3rd-3rd coordinate in u_0.
Then $\text{pdiff1}(f_1, 3)\ \text{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 3.

(76) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-1st coordinate in u_0. Then $\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(1, 3))(u_0)$.

(77) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-2nd coordinate in u_0. Then $\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(2, 3))(u_0)$.

(78) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-3rd coordinate in u_0. Then $\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(3, 3))(u_0)$.

(79) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-1st coordinate in u_0. Then $\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)$ is continuous in $(\text{proj}(1, 3))(u_0)$.

(80) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-2nd coordinate in u_0. Then $\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)$ is continuous in $(\text{proj}(2, 3))(u_0)$.

(81) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-3rd coordinate in u_0. Then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)$ is continuous in $(\mathrm{proj}(3, 3))(u_0)$.

(82) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-1st coordinate in u_0. Then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(1, 3))(u_0)$.

(83) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-2nd coordinate in u_0. Then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(2, 3))(u_0)$.

(84) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-3rd coordinate in u_0. Then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(3, 3))(u_0)$.

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Integrability Formulas. Part II

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Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including trigonometric function, and polynomial function.

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The terminology and notation used here have been introduced in the following articles: [12], [13], [2], [3], [9], [1], [6], [11], [14], [4], [18], [7], [8], [5], [19], [10], [16], [17], and [15].

For simplicity, we use the following convention: a, x are real numbers, n is an element of $\mathbb{N}$, A is a closed-interval subset of $\mathbb{R}$, f, h, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z is an open subset of $\mathbb{R}$.

The following propositions are true:

(1) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{1}{(\text{the function sin})\,(\text{the function cos})}$,
(iii) $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function tan}))$,
(iv) $Z = \text{dom}\, f$, and
(v) f is continuous on A.

Then $\int_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function tan}))(\inf A)$.

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(2) Suppose that
(i) $A \subseteq Z$,
(ii) $f = -\frac{1}{(\text{the function sin})\,(\text{the function cos})}$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cot}))$,
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cot}))(\inf A)$.

(3) Suppose that
(i) $A \subseteq Z$,
(ii) $f = 2\,((\text{the function exp})\ (\text{the function sin}))$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function exp})\ ((\text{the function sin}) - (\text{the function cos})))$,
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ ((\text{the function sin}) - (\text{the function cos})))(\sup A) - ((\text{the function exp})\ ((\text{the function sin}) - (\text{the function cos})))(\inf A)$.

(4) Suppose that
(i) $A \subseteq Z$,
(ii) $f = 2\,((\text{the function exp})\ (\text{the function cos}))$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function exp})\ ((\text{the function sin}) + (\text{the function cos})))$,
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ ((\text{the function sin}) + (\text{the function cos})))(\sup A) - ((\text{the function exp})\ ((\text{the function sin}) + (\text{the function cos})))(\inf A)$.

(5) Suppose $A \subseteq Z = \operatorname{dom}((\text{the function cos}) - (\text{the function sin}))$ and (the function cos)$-$(the function sin) is continuous on A. Then $\int\limits_A ((\text{the function cos}) - (\text{the function sin}))(x)dx = ((\text{the function sin}) + (\text{the function cos}))(\sup A) - ((\text{the function sin}) + (\text{the function cos}))(\inf A)$.

(6) Suppose $A \subseteq Z = \operatorname{dom}((\text{the function cos}) + (\text{the function sin}))$ and (the function cos)$+$(the function sin) is continuous on A. Then $\int\limits_A ((\text{the function cos}) + (\text{the function sin}))(x)dx = ((\text{the function sin}) - (\text{the function cos}))(\sup A) - ((\text{the function sin}) - (\text{the function cos}))(\inf A)$.

(7) Suppose $Z \subseteq \mathrm{dom}((-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}})$. Then
(i) $(-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds
$((-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}})'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)}{(\text{the function exp})(x)}$.

(8) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\text{the function sin}}{\text{the function exp}}$,
(iii) $Z \subseteq \mathrm{dom}((-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int\limits_{A} f(x)dx = ((-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}})(\sup A) - ((-\frac{1}{2})\,\frac{(\text{the function sin})+(\text{the function cos})}{\text{the function exp}})(\inf A)$.

(9) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}})$. Then
(i) $\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds
$(\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}})'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)}{(\text{the function exp})(x)}$.

(10) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\text{the function cos}}{\text{the function exp}}$,
(iii) $Z \subseteq \mathrm{dom}(\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int\limits_{A} f(x)dx = (\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}})(\sup A) - (\frac{1}{2}\,\frac{(\text{the function sin})-(\text{the function cos})}{\text{the function exp}})(\inf A)$.

(11) Suppose that
(i) $A \subseteq Z$,
(ii) $f = (\text{the function exp})\ ((\text{the function sin})+(\text{the function cos}))$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function exp})\ (\text{the function sin}))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int\limits_{A} f(x)dx = ((\text{the function exp})\ (\text{the function sin}))(\sup A) - ((\text{the function exp})\ (\text{the function sin}))(\inf A)$.

(12) Suppose that
(i) $A \subseteq Z$,

(ii) $f =$ (the function exp) ((the function cos)$-$(the function sin)),
(iii) $Z \subseteq \operatorname{dom}$((the function exp) (the function cos)),
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx =$ ((the function exp) (the function cos))($\sup A$) $-$ ((the function exp) (the function cos))($\inf A$).

(13) Suppose that
(i) $A \subseteq Z$,
(ii) $f_1 = \square^2$,
(iii) $f = -\frac{\frac{\text{the function sin}}{\text{the function cos}}}{f_1} + \frac{\frac{1}{\text{id}_Z}}{(\text{the function cos})^{\mathbf{2}}}$,
(iv) $Z \subseteq \operatorname{dom}(\frac{1}{\text{id}_Z}$ (the function tan)),
(v) $Z = \operatorname{dom} f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx = (\frac{1}{\text{id}_Z}$ (the function tan))($\sup A$) $-$ ($\frac{1}{\text{id}_Z}$ (the function tan))($\inf A$).

(14) Suppose that
(i) $A \subseteq Z$,
(ii) $f = -\frac{\frac{\text{the function cos}}{\text{the function sin}}}{f_1} - \frac{\frac{1}{\text{id}_Z}}{(\text{the function sin})^{\mathbf{2}}}$,
(iii) $f_1 = \square^2$,
(iv) $Z \subseteq \operatorname{dom}(\frac{1}{\text{id}_Z}$ (the function cot)),
(v) $Z = \operatorname{dom} f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx = (\frac{1}{\text{id}_Z}$ (the function cot))($\sup A$) $-$ ($\frac{1}{\text{id}_Z}$ (the function cot))($\inf A$).

(15) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\frac{\text{the function sin}}{\text{the function cos}}}{\text{id}_Z} + \frac{\text{the function ln}}{(\text{the function cos})^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}$((the function ln) (the function tan)),
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function tan))($\sup A$) $-$ ((the function ln) (the function tan))($\inf A$).

(16) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\frac{\text{the function cos}}{\text{the function sin}}}{\text{id}_Z} - \frac{\text{the function ln}}{(\text{the function sin})^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cot}))$,
(iv) $Z = \operatorname{dom} f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cot}))(\inf A)$.

(17) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\text{the function tan}}{\mathrm{id}_Z} + \frac{\text{the function ln}}{(\text{the function cos})^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function tan}))$,
(iv) $Z \subseteq \operatorname{dom}(\text{the function tan})$,
(v) $Z = \operatorname{dom} f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function tan}))(\inf A)$.

(18) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{\text{the function cot}}{\mathrm{id}_Z} - \frac{\text{the function ln}}{(\text{the function sin})^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cot}))$,
(iv) $Z \subseteq \operatorname{dom}(\text{the function cot})$,
(v) $Z = \operatorname{dom} f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cot}))(\inf A)$.

(19) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(iii) $f = \frac{\text{the function arctan}}{\mathrm{id}_Z} + \frac{\text{the function ln}}{f_1 + \square^2}$,
(iv) $Z \subseteq]-1, 1[$,
(v) $Z = \operatorname{dom} f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function arctan}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function arctan}))(\inf A)$.

(20) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(iii) $f = \frac{\text{the function arccot}}{\mathrm{id}_Z} - \frac{\text{the function ln}}{f_1 + \square^2}$,

(iv) $Z \subseteq]-1, 1[$,
(v) $Z = \mathrm{dom}\, f$, and
(vi) f is continuous on A.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function arccot))(sup A) − ((the function ln) (the function arccot))(inf A).

(21) Suppose $A \subseteq Z$ and $f = \frac{\text{(the function exp)}\cdot\text{(the function tan)}}{\text{(the function cos)}^2}$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx =$ ((the function exp) ·(the function tan))(sup A) − ((the function exp) ·(the function tan))(inf A).

(22) Suppose $A \subseteq Z$ and $f = -\frac{\text{(the function exp)}\cdot\text{(the function cot)}}{\text{(the function sin)}^2}$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx =$ ((the function exp) ·(the function cot))(sup A) − ((the function exp) ·(the function cot))(inf A).

(23) Suppose $Z \subseteq$ dom((the function exp) ·(the function cot)). Then
(i) −(the function exp) · (the function cot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds (−(the function exp) · (the function cot))$'_{\restriction Z}(x) = \frac{\text{(the function exp)((the function cot)}(x))}{\text{(the function sin)}(x)^2}$.

(24) Suppose $A \subseteq Z$ and $f = \frac{\text{(the function exp)}\cdot\text{(the function cot)}}{\text{(the function sin)}^2}$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx =$ (−(the function exp) · (the function cot))(sup A) − (−(the function exp) · (the function cot))(inf A).

(25) Suppose that
(i) $A \subseteq Z$,
(ii) $f = \frac{1}{\mathrm{id}_Z \text{ ((the function cos)}\cdot\text{(the function ln))}^2}$,
(iii) $Z \subseteq$ dom((the function tan) ·(the function ln)),
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx =$ ((the function tan) ·(the function ln))(sup A) − ((the function tan) ·(the function ln))(inf A).

(26) Suppose that
(i) $A \subseteq Z$,
(ii) $f = -\frac{1}{\mathrm{id}_Z \text{ ((the function sin)}\cdot\text{(the function ln))}^2}$,
(iii) $Z \subseteq$ dom((the function cot) ·(the function ln)),
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int_A f(x)dx = ((\text{the function cot}) \cdot (\text{the function ln}))(\sup A) - ((\text{the function cot}) \cdot (\text{the function ln}))(\inf A)$.

(27) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function ln}))$. Then

(i) $-(\text{the function cot}) \cdot (\text{the function ln})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot}) \cdot (\text{the function ln}))'_{\restriction Z}(x) = \frac{1}{x \cdot (\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$.

(28) Suppose that

(i) $A \subseteq Z$,

(ii) $f = \frac{1}{\text{id}_Z\ ((\text{the function sin}) \cdot (\text{the function ln}))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function ln}))$,

(iv) $Z = \text{dom} f$, and

(v) f is continuous on A.

Then $\int_A f(x)dx = (-(\text{the function cot}) \cdot (\text{the function ln}))(\sup A) - (-(\text{the function cot}) \cdot (\text{the function ln}))(\inf A)$.

(29) Suppose that

(i) $A \subseteq Z$,

(ii) $f = \frac{\text{the function exp}}{((\text{the function cos}) \cdot (\text{the function exp}))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan}) \cdot (\text{the function exp}))$,

(iv) $Z = \text{dom} f$, and

(v) f is continuous on A.

Then $\int_A f(x)dx = ((\text{the function tan}) \cdot (\text{the function exp}))(\sup A) - ((\text{the function tan}) \cdot (\text{the function exp}))(\inf A)$.

(30) Suppose that

(i) $A \subseteq Z$,

(ii) $f = -\frac{\text{the function exp}}{((\text{the function sin}) \cdot (\text{the function exp}))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function exp}))$,

(iv) $Z = \text{dom} f$, and

(v) f is continuous on A.

Then $\int_A f(x)dx = ((\text{the function cot}) \cdot (\text{the function exp}))(\sup A) - ((\text{the function cot}) \cdot (\text{the function exp}))(\inf A)$.

(31) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function exp}))$. Then

(i) $-(\text{the function cot}) \cdot (\text{the function exp})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$.

(32) Suppose that

(i) $A \subseteq Z$,

(ii) $f = \frac{\text{the function exp}}{((\text{the function sin})\cdot(\text{the function exp}))^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function cot}) \cdot(\text{the function exp}))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = (-(\text{the function cot}) \cdot (\text{the function exp}))(\sup A) - (-(\text{the function cot}) \cdot (\text{the function exp}))(\inf A)$.

(33) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{x^{\mathbf{2}}\cdot(\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot\frac{1}{\mathrm{id}_Z})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function tan}) \cdot\frac{1}{\mathrm{id}_Z})(\sup A) - ((\text{the function tan}) \cdot\frac{1}{\mathrm{id}_Z})(\inf A)$.

(34) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot\frac{1}{\mathrm{id}_Z})$. Then
(i) $-(\text{the function tan}) \cdot \frac{1}{\mathrm{id}_Z}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(-(\text{the function tan}) \cdot \frac{1}{\mathrm{id}_Z})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}\cdot(\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$.

(35) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{x^{\mathbf{2}}\cdot(\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot\frac{1}{\mathrm{id}_Z})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = (-(\text{the function tan}) \cdot \frac{1}{\mathrm{id}_Z})(\sup A) - (-(\text{the function tan}) \cdot \frac{1}{\mathrm{id}_Z})(\inf A)$.

(36) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{x^{\mathbf{2}}\cdot(\text{the function sin})(\frac{1}{x})^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function cot}) \cdot\frac{1}{\mathrm{id}_Z})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function cot}) \cdot\frac{1}{\mathrm{id}_Z})(\sup A) - ((\text{the function cot}) \cdot\frac{1}{\mathrm{id}_Z})(\inf A)$.

(37) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and (the function arctan)$(x) > 0$ and $f = \frac{1}{(f_1+\square^2) \text{ the function arctan}}$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}$((the function ln) $\cdot$(the function arctan)) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx =$ ((the function ln) $\cdot$(the function arctan))$(\sup A)$ $-$ ((the function ln) $\cdot$(the function arctan))$(\inf A)$.

(38) Suppose that $A \subseteq Z$ and $f = n\,\frac{(\square^{n-1})\cdot \text{the function arctan}}{f_1+\square^2}$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}((\square^n) \cdot$ the function arctan) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = ((\square^n) \cdot$ the function arctan)$(\sup A) - ((\square^n) \cdot$ the function arctan)$(\inf A)$.

(39) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = -n\,\frac{(\square^{n-1})\cdot \text{the function arccot}}{f_1+\square^2}$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}((\square^n) \cdot$ the function arccot) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = ((\square^n) \cdot$ the function arccot)$(\sup A) - ((\square^n) \cdot$ the function arccot)$(\inf A)$.

(40) Suppose $Z \subseteq \mathrm{dom}((\square^n) \cdot$ the function arccot) and $Z \subseteq]-1, 1[$. Then

(i) $-(\square^n) \cdot$ the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\square^n) \cdot \text{the function arccot})'_{\restriction Z}(x) = \frac{n \cdot (\text{the function arccot})(x)^{n-1}}{1+x^{\mathbf{2}}}$.

(41) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = n\,\frac{(\square^{n-1})\cdot \text{the function arccot}}{f_1+\square^2}$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}((\square^n) \cdot$ the function arccot) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (-(\square^n) \cdot$ the function arccot)$(\sup A) - (-(\square^n) \cdot$ the function arccot)$(\inf A)$.

(42) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = \frac{\text{the function arctan}}{f_1+\square^2}$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}((\square^2) \cdot$ the function arctan) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (\frac{1}{2}\,((\square^2) \cdot$ the function arctan))$(\sup A) - (\frac{1}{2}\,((\square^2) \cdot$ the function arctan))$(\inf A)$.

(43) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = -\frac{\text{the function arccot}}{f_1+\square^2}$ and $Z \subseteq]-1, 1[$ and $Z \subseteq \mathrm{dom}((\square^2) \cdot$ the function arccot) and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (\frac{1}{2}\,((\square^2) \cdot$ the function arccot))$(\sup A) - (\frac{1}{2}\,((\square^2) \cdot$ the function arccot))$(\inf A)$.

(44) Suppose $Z \subseteq \mathrm{dom}((\square^2) \cdot$ the function arccot) and $Z \subseteq]-1, 1[$. Then

(i) $-\frac{1}{2}\,((\square^2) \cdot$ the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds

$(-\frac{1}{2}\,((\square^2)\cdot \text{the function arccot}))'_{\restriction Z}(x) = \frac{(\text{the function arccot})(x)}{1+x^2}$.

(45) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = \frac{\text{the function arccot}}{f_1+\square^2}$ and $Z \subseteq]-1,1[$ and $Z \subseteq \text{dom}((\square^2) \cdot \text{the function arccot})$ and $Z = \text{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (-\frac{1}{2}\,((\square^2) \cdot \text{the function arccot}))(\sup A) - (-\frac{1}{2}\,((\square^2) \cdot \text{the function arccot}))(\inf A)$.

(46) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
 (iii) $f = (\text{the function arctan}) + \frac{\text{id}_Z}{f_1+\square^2}$,
 (iv) $Z \subseteq]-1,1[$,
 (v) $Z = \text{dom}\, f$, and
 (vi) f is continuous on A.

Then $\int\limits_A f(x)dx = (\text{id}_Z\,\text{the function arctan})(\sup A) - (\text{id}_Z\,\text{the function arctan})(\inf A)$.

(47) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
 (iii) $f = (\text{the function arccot}) - \frac{\text{id}_Z}{f_1+\square^2}$,
 (iv) $Z \subseteq]-1,1[$,
 (v) $Z = \text{dom}\, f$, and
 (vi) f is continuous on A.

Then $\int\limits_A f(x)dx = (\text{id}_Z\,\text{the function arccot})(\sup A) - (\text{id}_Z\,\text{the function arccot})(\inf A)$.

(48) Suppose that
 (i) $A \subseteq Z$,
 (ii) $Z \subseteq]-1,1[$,
 (iii) $f = \frac{(\text{the function exp})\cdot(\text{the function arctan})}{f_1+\square^2}$,
 (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
 (v) $Z = \text{dom}\, f$, and
 (vi) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\cdot(\text{the function arctan}))(\sup A) - ((\text{the function exp})\cdot(\text{the function arctan}))(\inf A)$.

(49) Suppose that
 (i) $A \subseteq Z$,
 (ii) $Z \subseteq]-1,1[$,

(iii) $f = -\frac{(\text{the function exp})\cdot(\text{the function arccot})}{f_1+\square^2}$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(v) $Z = \operatorname{dom} f$, and

(vi) f is continuous on A.

Then $\int\limits_A f(x)dx$ = ((the function exp) $\cdot$(the function arccot))($\sup A$) $-$ ((the function exp) $\cdot$(the function arccot))($\inf A$).

(50) Suppose $Z \subseteq$ dom((the function exp) $\cdot$(the function arccot)) and $Z \subseteq]-1, 1[$. Then

(i) $-$(the function exp) $\cdot$ (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function exp}) \cdot (\text{the function arccot}))'_{\restriction Z}(x) = \frac{(\text{the function exp})((\text{the function arccot})(x))}{1+x^{2}}$.

(51) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f = \frac{(\text{the function exp})\cdot(\text{the function arccot})}{f_1+\square^2}$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(v) $Z = \operatorname{dom} f$, and

(vi) f is continuous on A.

Then $\int\limits_A f(x)dx$ = ($-$(the function exp) $\cdot$ (the function arccot))($\sup A$) $-$ ($-$(the function exp) $\cdot$ (the function arccot))($\inf A$).

(52) Suppose that $A \subseteq Z \subseteq$ dom((the function ln) $\cdot(f_1 + f_2)$) and $f = \frac{\mathrm{id}_Z}{f_1+f_2}$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $Z = \operatorname{dom} f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (\frac{1}{2}$ ((the function ln) $\cdot(f_1+f_2)$))($\sup A$)$-$ ($\frac{1}{2}$ ((the function ln) $\cdot(f_1 + f_2)$))($\inf A$).

(53) Suppose that $A \subseteq Z \subseteq$ dom((the function ln) $\cdot(f_1+f_2)$) and $f = \frac{\mathrm{id}_Z}{a\,(f_1+f_2)}$ and for every x such that $x \in Z$ holds $h(x) = \frac{x}{a}$ and $f_1(x) = 1$ and $a \neq 0$ and $f_2 = (\square^2) \cdot h$ and $Z = \operatorname{dom} f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (\frac{a}{2}$ ((the function ln) $\cdot(f_1 + f_2)$))($\sup A$) $-$ ($\frac{a}{2}$ ((the function ln) $\cdot(f_1 + f_2)$))($\inf A$).

(54) Suppose $Z \subseteq$ dom($\frac{1}{\mathrm{id}_Z}$ the function arctan) and $Z \subseteq]-1, 1[$. Then

(i) $-\frac{1}{\mathrm{id}_Z}$ the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\mathrm{id}_Z}\text{ the function arctan})'_{\restriction Z}(x) = \frac{(\text{the function arctan})(x)}{x^{2}} - \frac{1}{x\cdot(1+x^{2})}$.

(55) Suppose $Z \subseteq$ dom($\frac{1}{\mathrm{id}_Z}$ the function arccot) and $Z \subseteq]-1, 1[$. Then

(i) $-\frac{1}{\mathrm{id}_Z}$ the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\mathrm{id}_Z}$ the function arccot$)'_{\restriction Z}(x) = \frac{(\text{the function arccot})(x)}{x^{\mathbf{2}}} + \frac{1}{x \cdot (1+x^{\mathbf{2}})}$.

(56) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = \frac{\text{the function arctan}}{\square^2} - \frac{1}{\mathrm{id}_Z\ (f_1 + \square^2)}$ and $Z \subseteq \mathrm{dom}(\frac{1}{\mathrm{id}_Z}$ the function arctan) and $Z \subseteq \,]-1,1[$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function arctan$)(\sup A) - (-\frac{1}{\mathrm{id}_Z}$ the function arctan$)(\inf A)$.

(57) Suppose that $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f = \frac{\text{the function arccot}}{\square^2} + \frac{1}{\mathrm{id}_Z\ (f_1 + \square^2)}$ and $Z \subseteq \mathrm{dom}(\frac{1}{\mathrm{id}_Z}$ the function arccot) and $Z \subseteq \,]-1,1[$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int\limits_A f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function arccot$)(\sup A) - (-\frac{1}{\mathrm{id}_Z}$ the function arccot$)(\inf A)$.

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Integrability Formulas. Part III

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Summary. In this article, we give several differentiation and integrability formulas of composite trigonometric function.

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The papers [9], [10], [15], [2], [3], [1], [6], [11], [4], [16], [7], [8], [5], [17], [13], [14], and [12] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following convention: a, x denote real numbers, n denotes a natural number, A denotes a closed-interval subset of $\mathbb{R}$, f, f_1 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1) Suppose $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})$. Then

(i) $-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})'_{\restriction Z}(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^{\mathbf{2}} \cdot (\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function exp}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function exp})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$.

(3) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function ln}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function ln})$ is differentiable on Z, and

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(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function ln}))'_{\restriction Z}(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \operatorname{dom}((\text{the function exp}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function exp}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function exp}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = \frac{(\text{the function exp})((\text{the function cosec})(x)) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function ln}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function ln}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = (\text{the function cot})(x)$.

(6) Suppose $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function cosec})$ and $1 \leq n$. Then

(i) $-(\square^n) \cdot$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\square^n) \cdot \text{the function cosec})'_{\restriction Z}(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$.

(7) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function sec). Then

(i) $-\frac{1}{\mathrm{id}_Z}$ the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\mathrm{id}_Z}\text{the function sec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x^{\mathbf{2}}} - \frac{\frac{(\text{the function sin})(x)}{x}}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(8) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function cosec). Then

(i) $-\frac{1}{\mathrm{id}_Z}$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\mathrm{id}_Z}\text{the function cosec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x^{\mathbf{2}}} + \frac{\frac{(\text{the function cos})(x)}{x}}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(9) Suppose $Z \subseteq \operatorname{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function sin}))'_{\restriction Z}(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{\mathbf{2}}}$.

(10) Suppose $Z \subseteq \operatorname{dom}((\text{the function sec}) \cdot (\text{the function cot}))$. Then

(i) $-(\text{the function sec}) \cdot (\text{the function cot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}}$.

(11) Suppose $Z \subseteq \operatorname{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function tan})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}}$.

(12) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot})\ (\text{the function sec}))$. Then

(i) $-(\text{the function cot})\,(\text{the function sec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot})\,(\text{the function$

sec))$'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^{2}}}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x)\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^{2}}$.

(13) Suppose $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function cosec}))$. Then

(i) $-(\text{the function cot})\,(\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot})\,(\text{the function cosec}))'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^{2}}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{2}}$.

(14) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ (\text{the function cot}))$. Then

(i) $-(\text{the function cos})\,(\text{the function cot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cos})\,(\text{the function cot}))'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^{2}}$.

2. Integrability Formulas

We now state a number of propositions:

(15) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^{2}\cdot(\text{the function cos})(\frac{1}{x})^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})(\sup A) - (-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})(\inf A)$.

(16) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(\frac{1}{x})}{x^{2}\cdot(\text{the function sin})(\frac{1}{x})^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot \frac{1}{\mathrm{id}_Z})$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function cosec}) \cdot \frac{1}{\mathrm{id}_Z})(\sup A) - ((\text{the function cosec}) \cdot \frac{1}{\mathrm{id}_Z})(\inf A)$.

(17) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)\cdot(\text{the function sin})((\text{the function exp})(x))}{(\text{the function cos})((\text{the function exp})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function exp}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function sec})\cdot(\text{the function exp}))(\sup A) - ((\text{the function sec})\cdot(\text{the function exp}))(\inf A)$.

(18) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)\cdot(\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cosec})\cdot(\text{the function exp}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cosec})\cdot(\text{the function exp}))(\sup A) - (-(\text{the function cosec})\cdot(\text{the function exp}))(\inf A)$.

(19) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})((\text{the function ln})(x))}{x\cdot(\text{the function cos})((\text{the function ln})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function sec})\cdot(\text{the function ln}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function sec})\cdot(\text{the function ln}))(\sup A) - ((\text{the function sec})\cdot(\text{the function ln}))(\inf A)$.

(20) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x\cdot(\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cosec})\cdot(\text{the function ln}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cosec})\cdot(\text{the function ln}))(\sup A) - (-(\text{the function cosec})\cdot(\text{the function ln}))(\inf A)$.

(21) Suppose that

(i) $A \subseteq Z$,

(ii) $f = ((\text{the function exp})\cdot(\text{the function sec}))\ \frac{\text{the function sin}}{(\text{the function cos})^{\mathbf{2}}}$,

(iii) $Z = \text{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function exp}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function exp}) \cdot (\text{the function sec}))(\inf A)$.

(22) Suppose that

(i) $A \subseteq Z$,

(ii) $f = ((\text{the function exp}) \cdot (\text{the function cosec}))\ \frac{\text{the function cos}}{(\text{the function sin})^{\mathbf{2}}}$,

(iii) $Z = \operatorname{dom} f$, and

(iv) $f \restriction A$ is continuous.

Then $\int_A f(x)dx = (-(\text{the function exp}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function exp}) \cdot (\text{the function cosec}))(\inf A)$.

(23) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function sec}))$,

(iii) $Z = \operatorname{dom}(\text{the function tan})$, and

(iv) $(\text{the function tan}) \restriction A$ is continuous.

Then $\int_A (\text{the function tan})(x)dx = ((\text{the function ln}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function sec}))(\inf A)$.

(24) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$,

(iii) $Z = \operatorname{dom}(\text{the function cot})$, and

(iv) $(-\text{the function cot}) \restriction A$ is continuous.

Then $\int_A (-\text{the function cot})(x)dx = ((\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$.

(25) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$,

(iii) $Z = \operatorname{dom}(\text{the function cot})$, and

(iv) $(\text{the function cot}) \restriction A$ is continuous.

Then $\int_A (\text{the function cot})(x)dx = (-(\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$.

(26) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$,

(iii) $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function sec})$,

(iv) $1 \leq n$,

(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\square^n) \cdot \text{the function sec})(\sup A) - ((\square^n) \cdot \text{the function sec})(\inf A)$.

(27) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$,
(iii) $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function cosec})$,
(iv) $1 \leq n$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\square^n) \cdot \text{the function cosec})(\sup A) - (-(\square^n) \cdot \text{the function cosec})(\inf A)$.

(28) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function exp})\ (\text{the function sec}))$,
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ (\text{the function sec}))(\sup A) - ((\text{the function exp})\ (\text{the function sec}))(\inf A)$.

(29) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}((\text{the function exp})\ (\text{the function cosec}))$,
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ (\text{the function cosec}))(\sup A) - ((\text{the function exp})\ (\text{the function cosec}))(\inf A)$.

(30) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function sin})(a \cdot x) - (\text{the function cos})(a \cdot x)^{\mathbf{2}}}{(\text{the function cos})(a \cdot x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}(\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \mathrm{dom}\, f$, and
(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(31) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function cos)}(a \cdot x) - \text{(the function sin)}(a \cdot x)^2}{\text{(the function sin)}(a \cdot x)^2}$,
(iii) $Z \subseteq \mathrm{dom}((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \mathrm{dom}\, f$, and
(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(32) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{\text{(the function cos)}(x)}}{x} + \frac{\text{(the function ln)}(x) \cdot \text{(the function sin)}(x)}{\text{(the function cos)}(x)^2}$,
(iii) $Z \subseteq \mathrm{dom}$((the function ln) (the function sec)),
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function sec))(sup A) − ((the function ln) (the function sec))(inf A).

(33) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{\text{(the function sin)}(x)}}{x} - \frac{\text{(the function ln)}(x) \cdot \text{(the function cos)}(x)}{\text{(the function sin)}(x)^2}$,
(iii) $Z \subseteq \mathrm{dom}$((the function ln) (the function cosec)),
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function cosec))(sup A) − ((the function ln) (the function cosec))(inf A).

(34) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x^2} - \frac{\frac{(\text{the function sin})(x)}{x}}{(\text{the function cos})(x)^2}$,

(iii) $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z}$ the function sec),

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = (-\frac{1}{\text{id}_Z}$ the function sec)(sup A) $-$ $(-\frac{1}{\text{id}_Z}$ the function sec)(inf A).

(35) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x^2} + \frac{\frac{(\text{the function cos})(x)}{x}}{(\text{the function sin})(x)^2}$,

(iii) $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z}$ the function cosec),

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = (-\frac{1}{\text{id}_Z}$ the function cosec)(sup A) $-$ $(-\frac{1}{\text{id}_Z}$ the function cosec)(inf A).

(36) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function sin})((\text{the function sin})(x))}{(\text{the function cos})((\text{the function sin})(x))^2}$,

(iii) $Z \subseteq \text{dom}$((the function sec) $\cdot$(the function sin)),

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx =$ ((the function sec) $\cdot$(the function sin))(sup A)$-$((the function sec) $\cdot$(the function sin))(inf A).

(37) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(x) \cdot (\text{the function sin})((\text{the function cos})(x))}{(\text{the function cos})((\text{the function cos})(x))^2}$,

(iii) $Z \subseteq \text{dom}$((the function sec) $\cdot$(the function cos)),

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = (-$(the function sec) $\cdot$ (the function cos))(sup A) $-$ $(-$(the function sec) $\cdot$ (the function cos))(inf A).

(38) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function sin}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function sin}))(\inf A)$.

(39) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function sin})(x) \cdot (\text{the function cos})((\text{the function cos})(x))}{(\text{the function sin})((\text{the function cos})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function cos}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cos}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cos}))(\inf A)$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{\frac{(\text{the function sin})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function cos})((\text{the function tan})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function tan}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function sec}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function tan}))(\inf A)$.

(41) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function cot}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function sec}) \cdot (\text{the function cot}))(\inf A)$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function } \cos)((\text{the function } \tan)(x))}{(\text{the function } \cos)(x)^2}}{(\text{the function } \sin)((\text{the function } \tan)(x))^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f \restriction A$ is continuous.

Then $\int_A f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function tan}))(\inf A)$.

(43) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function } \cos)((\text{the function } \cot)(x))}{(\text{the function } \sin)(x)^2}}{(\text{the function } \sin)((\text{the function } \cot)(x))^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function cosec}) \cdot (\text{the function cot}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f \restriction A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cot}))(\inf A)$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function } \cos)(x)^2}}{(\text{the function } \cos)(x)} + \frac{(\text{the function } \tan)(x) \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function tan}) \ (\text{the function sec}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f \restriction A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function tan}) \ (\text{the function sec}))(\sup A) - ((\text{the function tan}) \ (\text{the function sec}))(\inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function } \sin)(x)^2}}{(\text{the function } \cos)(x)} - \frac{(\text{the function } \cot)(x) \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function cot}) \ (\text{the function sec}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f \restriction A$ is continuous.

Then $\int_A f(x)dx = (-(\text{the function cot}) \ (\text{the function sec}))(\sup A) - (-(\text{the}$

function cot) (the function sec))(inf A).

(46) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)^2}}{(\text{the function sin})(x)} - \frac{(\text{the function tan})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan})\ (\text{the function cosec}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function tan})\ (\text{the function cosec}))(\sup A) - ((\text{the function tan})\ (\text{the function cosec}))(\inf A)$.

(47) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)^2}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$,

(iii) $Z \subseteq \text{dom}((\text{the function cot})\ (\text{the function cosec}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cot})\ (\text{the function cosec}))(\sup A) - (-(\text{the function cot})\ (\text{the function cosec}))(\inf A)$.

(48) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})((\text{the function cot})(x))^2} \cdot \frac{1}{(\text{the function sin})(x)^2}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan}) \cdot (\text{the function cot}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function tan}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function tan}) \cdot (\text{the function cot}))(\inf A)$.

(49) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})((\text{the function tan})(x))^2} \cdot \frac{1}{(\text{the function cos})(x)^2}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan}) \cdot (\text{the function tan}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (($the function tan$) \cdot($the function tan$))(\sup A)-(($the function tan$) \cdot($the function tan$))(\inf A)$.

(50) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}(($the function cot$) \cdot($the function cot$))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (($the function cot$) \cdot($the function cot$))(\sup A)-(($the function cot$) \cdot($the function cot$))(\inf A)$.

(51) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}(($the function cot$) \cdot($the function tan$))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-($the function cot$) \cdot ($the function tan$))(\sup A) - (-($the function cot$) \cdot ($the function tan$))(\inf A)$.

(52) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} + \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}(($the function tan$)-($the function cot$))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (($the function tan$)-($the function cot$))(\sup A) - (($the function tan$)-($the function cot$))(\inf A)$.

(53) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} - \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}(($the function tan$)+($the function cot$))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function tan)+(the function cot))(sup A) − ((the function tan)+(the function cot))(inf A).

(54) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function sin)(x)) · (the function cos)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sin) ·(the function sin))(sup A)−((the function sin) ·(the function sin))(inf A).

(55) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function cos)(x)) · (the function sin)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ (−(the function sin) · (the function cos))(sup A) − (−(the function sin) · (the function cos))(inf A).

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function sin)(x)) · (the function cos)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ (−(the function cos) · (the function sin))(sup A) − (−(the function cos) · (the function sin))(inf A).

(57) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function cos)(x)) · (the function sin)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function cos) ·(the function cos))(sup A)−((the function cos) ·(the function cos))(inf A).

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)(x) + $\frac{\text{(the function cos)}(x)}{\text{(the function sin)}(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq$ dom((the function cos) (the function cot)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx =$ (−(the function cos) (the function cot))(sup A) − (−(the function cos) (the function cot))(inf A).

(59) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)(x) + $\frac{\text{(the function sin)}(x)}{\text{(the function cos)}(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq$ dom((the function sin) (the function tan)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx =$ ((the function sin) (the function tan))(sup A)−((the function sin) (the function tan))(inf A).

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