On $L^p$ Space Formed by Real-Valued Partial Functions

Yasushige Watase
Graduate School of Science and Technology
Shinshu University
Nagano, Japan

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. This article is the continuation of [31]. We define the set of $L^p$ integrable functions – the set of all partial functions whose absolute value raised to the $p$-th power is integrable. We show that $L^p$ integrable functions form the $L^p$ space. We also prove Minkowski’s inequality, H"older’s inequality and that $L^p$ space is Banach space ([15], [27]).

The notation and terminology used in this paper have been introduced in the following papers: [7], [8], [9], [10], [4], [1], [31], [6], [19], [20], [13], [28], [14], [2], [24], [3], [11], [25], [22], [21], [16], [32], [29], [23], [18], [17], [26], [30], [5], and [12].

1. Preliminaries on Powers of Numbers and Operations on Real Sequences

For simplicity, we follow the rules: $X$ denotes a non empty set, $x$ denotes an element of $X$, $S$ denotes a $\sigma$-field of subsets of $X$, $M$ denotes a $\sigma$-measure on $S$, $f$, $g$, $f_1$, $g_1$ denote partial functions from $X$ to $\mathbb{R}$, and $a$, $b$, $c$ denote real numbers.

The following propositions are true:

(1) For all positive real numbers $m$, $n$ such that $\frac{1}{m} + \frac{1}{n} = 1$ holds $m > 1$. 

MML identifier: LPSPACE2, version: 7.11.07.4.156.1112
(2) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, $M$ be a $\sigma$-measure on $S$, $A$ be an element of $S$, and $f$ be a partial function from $X$ to $\mathbb{R}$. Suppose $A = \text{dom } f$ and $f$ is measurable on $A$ and $f$ is non-negative. Then $\int f \, dM \in \mathbb{R}$ if and only if $f$ is integrable on $M$.

Let $r$ be a real number. We say that $r$ is great or equal to 1 if and only if:

(Def. 1) \[ 1 \leq r. \]

Let us note that every real number which is great or equal to 1 is also positive.

One can verify that there exists a real number which is great or equal to 1. In the sequel $k$ denotes a positive real number.

We now state several propositions:

(3) For all real numbers $a$, $b$, $p$ such that $0 < p$ and $0 \leq a < b$ holds $a^p < b^p$.

(4) If $a \geq 0$ and $b > 0$, then $a^b \geq 0$.

(5) If $a \geq 0$ and $b \geq 0$ and $c > 0$, then $(a \cdot b)^c = a^c \cdot b^c$.

(6) For all real numbers $a$, $b$ and for every $f$ such that $f$ is non-negative and $a > 0$ and $b > 0$ holds $(f^a)^b = f^{a\cdot b}$.

(7) For all real numbers $a$, $b$ and for every $f$ such that $f$ is non-negative and $a > 0$ and $b > 0$ holds $f^a \cdot f^b = f^{a+b}$.

(8) $f^1 = f$.

(9) Let $s_1$, $s_2$ be sequences of real numbers and $k$ be a positive real number.

Suppose that for every element $n$ of $\mathbb{N}$ holds $s_1(n) = s_2(n)^k$ and $s_2(n) \geq 0$. Then $s_1$ is convergent if and only if $s_2$ is convergent.

(10) Let $s_3$ be a sequence of real numbers and $n$, $m$ be elements of $\mathbb{N}$. If $m \leq n$, then $|\sum_{n=0}^{\infty} s_3(n) - \sum_{n=0}^{\infty} s_3(m)| \leq |\sum_{n=0}^{\infty} (s_3(n) - s_3(m))|$.

(11) Let $s_3$, $s_2$ be sequences of real numbers and $k$ be a positive real number.

Suppose $s_3$ is convergent and for every element $n$ of $\mathbb{N}$ holds $s_2(n) = \lim s_3 - s_3(n)^k$. Then $s_2$ is convergent and $\lim s_2 = 0$.

2. Real Linear Space of $L^p$ Integrable Functions

Next we state two propositions:

(12) For every positive real number $k$ and for every non empty set $X$ holds $(X \rightarrow 0)^k = X \rightarrow 0$.

(13) For every partial function $f$ from $X$ to $\mathbb{R}$ and for every set $D$ holds $|f|D = |f||D$.

Let us consider $X$ and let $f$ be a partial function from $X$ to $\mathbb{R}$. Observe that $|f|$ is non-negative.
One can prove the following two propositions:

(14) For every partial function $f$ from $X$ to $\mathbb{R}$ such that $f$ is non-negative holds $|f| = f$.

(15) If $X = \text{dom } f$ and for every $x$ such that $x \in \text{dom } f$ holds $0 = f(x)$, then $f$ is integrable on $M$ and $\int f \, dM = 0$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $L^p$ functions($M, k$) yielding a non empty subset of PFunct$_{RLS}$ $X$ is defined by the condition (Def. 2).

(Def. 2) $L^p$ functions($M, k$) = \{ $f$: $f$ ranges over partial functions from $X$ to $\mathbb{R}$: $
abla_{E_1}$: element of $S$ \quad ($M(E_1^\circ) = 0 \land \text{dom } f = E_1 \land f$ is measurable on $E_1 \land |f|^k$ is integrable on $M$) \}.

Next we state a number of propositions:

(16) For all real numbers $a, b, k$ such that $k > 0$ holds $|a + b|^k \leq (|a| + |b|)^k$ and $(|a| + |b|)^k \leq (2 \cdot \max(|a|, |b|))^k$ and $|a + b|^k \leq (2 \cdot \max(|a|, |b|))^k$.

(17) For all real numbers $a, b, k$ such that $a \geq 0$ and $b \geq 0$ and $k > 0$ holds $(a b)^k \leq a^k + b^k$.

(18) For every partial function $f$ from $X$ to $\mathbb{R}$ and for all real numbers $a, b$ such that $b > 0$ holds $|a|^b \cdot |f|^b = |a f|^b$.

(19) Let $f$ be a partial function from $X$ to $\mathbb{R}$ and $a, b$ be real numbers. If $a > 0$ and $b > 0$, then $a^b \cdot |f|^b = (a |f|)^b$.

(20) For every partial function $f$ from $X$ to $\mathbb{R}$ and for every real number $k$ and for every set $E$ holds $(f|E)^k = f^k|E$.

(21) For all real numbers $a, b, k$ such that $k > 0$ holds $|a + b|^k \leq 2^k \cdot (|a|^k + |b|^k)$.

(22) Let $k$ be a positive real number and $f, g$ be partial functions from $X$ to $\mathbb{R}$. Suppose $f, g \in L^p$ functions($M, k$). Then $|f|^k$ is integrable on $M$ and $|g|^k$ is integrable on $M$ and $|f|^k + |g|^k$ is integrable on $M$.

(23) $X \longrightarrow 0$ is a partial function from $X$ to $\mathbb{R}$ and $X \longrightarrow 0 \in L^p$ functions($M, k$).

(24) Let $k$ be a real number. Suppose $k > 0$. Let $f, g$ be partial functions from $X$ to $\mathbb{R}$ and $x$ be an element of $X$. If $x \in \text{dom } f \cap \text{dom } g$, then $|f + g|^k(x) \leq (2^k \cdot |f|^k + |g|^k)(x)$.

(25) If $f, g \in L^p$ functions($M, k$), then $f + g \in L^p$ functions($M, k$).

(26) If $f \in L^p$ functions($M, k$), then $a f \in L^p$ functions($M, k$).

(27) If $f, g \in L^p$ functions($M, k$), then $f - g \in L^p$ functions($M, k$).

(28) If $f \in L^p$ functions($M, k$), then $|f| \in L^p$ functions($M, k$).

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. Note that $L^p$ functions($M, k$) is multiplicatively-closed and add closed.
Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. One can check that $\langle L^p \text{functions}(M, k)^0, \text{PFunc}_{\text{RLS}} X \in L^p \text{functions}(M, k), \cdot L^p \text{functions}(M, k) \rangle$ is Abelian, add-associative, and real linear space-like.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{RLSp LpFunct}(M, k)$ yields a strict Abelian add-associative real linear space-like non empty RLS structure and is defined by:

$$(\text{Def. 3}) \quad \text{RLSp LpFunct}(M, k) = \langle L^p \text{functions}(M, k)^0, \text{PFunc}_{\text{RLS}} X \in L^p \text{functions}(M, k), \cdot L^p \text{functions}(M, k) \rangle.$$ 

3. Preliminaries on Real Normed Space of $L^p$ Integrable Functions

In the sequel $v, u$ are vectors of $\text{RLSp LpFunct}(M, k)$.

We now state three propositions:

(29) \( (v) + (u) = v + u. \)

(30) \( a (u) = a \cdot u. \)

(31) Suppose $f = u$. Then

(i) \( u + (-1) \cdot u = (X \mapsto 0) \upharpoonright \text{dom } f, \) and

(ii) there exist partial functions $v, g$ from $X$ to $\mathbb{R}$ such that $v, g \in L^p \text{functions}(M, k)$ and $v = u + (-1) \cdot u$ and $g = X \mapsto 0$ and $v =_{a.e.} g.$

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{AlmostZeroLpFunctions}(M, k)$ yielding a non empty subset of $\text{RLSp LpFunct}(M, k)$ is defined by:

$$(\text{Def. 4}) \quad \text{AlmostZeroLpFunctions}(M, k) = \{ f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{R}; f \in L^p \text{functions}(M, k) \land f =_{a.e.} X \mapsto 0 \}. \)

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. One can check that $\text{AlmostZeroLpFunctions}(M, k)$ is add closed and multiplicatively-closed.

Next we state the proposition

(32) \( 0_{\text{RLSp LpFunct}(M, k)} = X \mapsto 0 \) and

\( 0_{\text{RLSp LpFunct}(M, k)} \in \text{AlmostZeroLpFunctions}(M, k). \)

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{RLSpAlmostZeroLpFunctions}(M, k)$ yielding a non empty RLS structure is defined by:

$$(\text{Def. 5}) \quad \text{RLSpAlmostZeroLpFunctions}(M, k) = (\text{AlmostZeroLpFunctions}(M, k), 0_{\text{RLSp LpFunct}(M, k)}(\in \text{AlmostZeroLpFunctions}(M, k)), \cdot (\text{AlmostZeroLpFunct}(M, k))).$$
Functions($M, k$), RLSp LpFunct($M, k$), \text{AlmostZero}LpFunctions($M, k$).

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. Observe that RLSp LpFunct($M, k$) is strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel $v, u$ are vectors of RLSpAlmostZeroLpFunctions($M, k$).

One can prove the following two propositions:

(33) \((v) + (u) = v + u.\)

(34) \(a (u) = a \cdot u.\)

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, let $f$ be a partial function from $X$ to $\mathbb{R}$, and let $k$ be a positive real number. The functor a.e.eq-class $L^p(f, M, k)$ yields a subset of $L^p$ functions($M, k$) and is defined as follows:

(Def. 6) a.e.eq-class $L^p(f, M, k) = \{h; h \text{ ranges over partial functions from } X \text{ to } \mathbb{R}; h \in L^p \text{ functions}(M, k) \wedge f =_{a.e.}^M h\}.$

Next we state a number of propositions:

(35) If $f \in L^p$ functions($M, k$), then there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $\text{dom } f = E$ and $f$ is measurable on $E$.

(36) If $g \in L^p$ functions($M, k$) and $g =_{a.e.}^M f$, then $g \in a.e.eq-class L^p(f, M, k)$.

(37) Suppose there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } f$ and $f$ is measurable on $E$ and $g \in a.e.eq-class L^p(f, M, k)$. Then $g =_{a.e.}^M f$ and $f \in L^p$ functions($M, k$).

(38) If $f \in L^p$ functions($M, k$), then $f \in a.e.eq-class L^p(f, M, k)$.

(39) Suppose there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } g$ and $g$ is measurable on $E$ and a.e.eq-class $L^p(f, M, k) \neq \emptyset$ and a.e.eq-class $L^p(f, M, k) = a.e.eq-class L^p(g, M, k)$. Then $f =_{a.e.}^M a.e. g$.

(40) Suppose $f \in L^p$ functions($M, k$) and there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } g$ and $g$ is measurable on $E$ and a.e.eq-class $L^p(f, M, k) = a.e.eq-class L^p(g, M, k)$. Then $f =_{a.e.}^M a.e. g$.

(41) If $f =_{a.e.}^M g$, then a.e.eq-class $L^p(f, M, k) = a.e.eq-class L^p(g, M, k)$.

(42) If $f =_{a.e.}^M g$, then a.e.eq-class $L^p(f, M, k) = a.e.eq-class L^p(g, M, k)$.

(43) If $f \in L^p$ functions($M, k$) and $g \in a.e.eq-class L^p(f, M, k)$, then a.e.eq-class $L^p(f, M, k) = a.e.eq-class L^p(g, M, k)$.

(44) Suppose that there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } f$ and $f$ is measurable on $E$ and there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } f_1$ and $f_1$ is measurable on $E$ and there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } g$ and $g$ is measurable on $E$ and there exists an element $E$ of $S$ such that $M(E^c) = 0$ and $E = \text{dom } g_1$ and $g_1$ is measurable on
$E$ and a.e-eq-class $L^p(f, M, k)$ is non empty and a.e-eq-class $L^p(g, M, k)$ is non empty and a.e-eq-class $L^p(f, M, k) = a.e$-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) = a.e$-eq-class $L^p(g_1, M, k)$. Then a.e-eq-class $L^p(f + g, M, k) = a.e$-eq-class $L^p(f_1 + g_1, M, k)$.

(45) If $f, f_1, g, g_1 \in L^p$ functions$(M, k)$ and a.e-eq-class $L^p(f, M, k) = a.e$-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) = a.e$-eq-class $L^p(g_1, M, k)$, then a.e-eq-class $L^p(f + g, M, k) = a.e$-eq-class $L^p(f_1 + g_1, M, k)$.

(46) Suppose that

(i) there exists an element $E$ of $S$ such that $M(E^c) = 0$ and dom $f = E$ and $f$ is measurable on $E$,

(ii) there exists an element $E$ of $S$ such that $M(E^c) = 0$ and dom $g = E$ and $g$ is measurable on $E$,

(iii) a.e-eq-class $L^p(f, M, k)$ is non empty, and

(iv) a.e-eq-class $L^p(f, M, k) = a.e$-eq-class $L^p(g, M, k)$.

Then a.e-eq-class $L^p(a f, M, k) = a.e$-eq-class $L^p(a g, M, k)$.

(47) If $f, g \in L^p$ functions$(M, k)$ and a.e-eq-class $L^p(f, M, k) = a.e$-eq-class $L^p(g, M, k)$, then

$a.e$-eq-class $L^p(a f, M, k) = a.e$-eq-class $L^p(a g, M, k)$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{CosetSet}(M, k)$ yielding a non empty family of subsets of $L^p$ functions$(M, k)$ is defined by:

(Def. 7) $\text{CosetSet}(M, k) = \{ \text{a.e-eq-class } L^p(f, M, k); f \text{ ranges over partial functions from } X \text{ to } \mathbb{R}; f \in L^p \text{ functions}(M, k) \}.$

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{addCoset}(M, k)$ yields a binary operation on $\text{CosetSet}(M, k)$ and is defined by the condition

(Def. 8).

(Def. 8) Let $A, B$ be elements of $\text{CosetSet}(M, k)$ and $a, b$ be partial functions from $X$ to $\mathbb{R}$. If $a \in A$ and $b \in B$, then $(\text{addCoset}(M, k))(A, B) = a.e$-eq-class $L^p(a + b, M, k)$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{zeroCoset}(M, k)$ yields an element of $\text{CosetSet}(M, k)$ and is defined as follows:

(Def. 9) $\text{zeroCoset}(M, k) = a.e$-eq-class $L^p(X \rightarrow 0, M, k)$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $k$ be a positive real number. The functor $\text{lmultCoset}(M, k)$ yielding a function from $\mathbb{R} \times \text{CosetSet}(M, k)$ into $\text{CosetSet}(M, k)$ is defined by the condition (Def. 10).
(Def. 10) Let \( z \) be an element of \( \mathbb{R} \), \( A \) be an element of \( \text{CosetSet}(M, k) \), and \( f \) be a partial function from \( X \) to \( \mathbb{R} \). If \( f \in A \), then \( (\text{ImultCoset}(M, k))(z, A) = a.e.-\text{eq-class } L^p(z, f, M, k) \).

Let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), let \( M \) be a \( \sigma \)-measure on \( S \), and let \( k \) be a positive real number. The functor \( \text{Pre-} L^p\text{-Space}(M, k) \) yielding a strict RLS structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of \( \text{Pre-} L^p\text{-Space}(M, k) = \text{CosetSet}(M, k) \),
(ii) the addition of \( \text{Pre-} L^p\text{-Space}(M, k) = \text{addCoset}(M, k) \), and
(iii) \( 0_{\text{Pre-} L^p\text{-Space}(M, k)} = \text{zeroCoset}(M, k) \), and
(iv) the external multiplication of \( \text{Pre-} L^p\text{-Space}(M, k) = \text{lmultCoset}(M, k) \).

Let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), let \( M \) be a \( \sigma \)-measure on \( S \), and let \( k \) be a positive real number. Observe that \( \text{Pre-} L^p\text{-Space}(M, k) \) is non empty.

Let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), let \( M \) be a \( \sigma \)-measure on \( S \), and let \( k \) be a positive real number. Observe that \( \text{Pre-} L^p\text{-Space}(M, k) \) is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

4. Real Normed Space of \( L^p \) Integrable Functions

The following propositions are true:

(48) If \( f, g \in L^p\text{functions}(M, k) \) and \( f =_{a.e.} M \cdot g \), then \( \int |f|^k \, dM = \int |g|^k \, dM \).

(49) If \( f \in L^p\text{functions}(M, k) \), then \( \int |f|^k \, dM \in \mathbb{R} \) and \( 0 \leq \int |f|^k \, dM \).

(50) If there exists a vector \( x \) of \( \text{Pre-} L^p\text{-Space}(M, k) \) such that \( f, g \in x \), then \( f =_{a.e.} M \cdot g \) and \( f, g \in L^p\text{functions}(M, k) \).

(51) Let \( k \) be a positive real number. Then there exists a function \( N_1 \) from the carrier of \( \text{Pre-} L^p\text{-Space}(M, k) \) into \( \mathbb{R} \) such that for every point \( x \) of \( \text{Pre-} L^p\text{-Space}(M, k) \) holds there exists a partial function \( f \) from \( X \) to \( \mathbb{R} \) such that \( f \in x \) and there exists a real number \( r \) such that \( r = \int |f|^k \, dM \) and \( N_1(x) = \frac{r}{m} \).

In the sequel \( x \) denotes a point of \( \text{Pre-} L^p\text{-Space}(M, k) \).

We now state two propositions:

(52) If \( f \in x \), then \( |f|^k \) is integrable on \( M \) and \( f \in L^p\text{functions}(M, k) \).

(53) If \( f, g \in x \), then \( f =_{a.e.} M \cdot g \) and \( \int |f|^k \, dM = \int |g|^k \, dM \).

Let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), let \( M \) be a \( \sigma \)-measure on \( S \), and let \( k \) be a positive real number. The functor \( L^p\text{-Norm}(M, k) \) yielding a function from the carrier of \( \text{Pre-} L^p\text{-Space}(M, k) \) into \( \mathbb{R} \) is defined by the condition (Def. 12).
(Def. 12) Let \( x \) be a point of \( \text{Pre-} L^p \)-\( \text{Space}(M, k) \). Then there exists a partial function \( f \) from \( X \) to \( \mathbb{R} \) such that \( f \in x \) and there exists a real number \( r \) such that \( r = \int |f|^k \, dM \) and \( (L^p - \text{Norm}(M, k))(x) = r^{\frac{1}{k}} \).

Let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), let \( M \) be a \( \sigma \)-measure on \( S \), and let \( k \) be a positive real number. The functor \( L^p - \text{Space}(M, k) \) yields a non empty normed structure and is defined by:

(Def. 13) \( L^p - \text{Space}(M, k) = (\text{the carrier of Pre-} L^p - \text{Space}(M, k), \) the zero of \( \text{Pre-} L^p - \text{Space}(M, k), \) the addition of \( \text{Pre-} L^p - \text{Space}(M, k), \) the external multiplication of \( \text{Pre-} L^p - \text{Space}(M, k), \) \( \text{L}^p - \text{Norm}(M, k) \)).

In the sequel \( x, y \) denote points of \( L^p - \text{Space}(M, k) \).

One can prove the following propositions:

(54)(i) There exists a partial function \( f \) from \( X \) to \( \mathbb{R} \) such that \( f \in L^p \) functions\((M, k)\) and \( x = \text{a.e.-eq-class} L^p(f, M, k) \), and

(ii) for every partial function \( f \) from \( X \) to \( \mathbb{R} \) such that \( f \in x \) there exists a real number \( r \) such that \( 0 \leq r = \int |f|^k \, dM \) and \( \| x \| = r^{\frac{1}{k}} \).

(55) If \( f \in x \) and \( g \in y \), then \( f + g \in x + y \) and if \( f \in x \), then \( a \, f \in a \cdot x \).

(56) If \( f \in x \), then \( x = \text{a.e.-eq-class} L^p(f, M, k) \) and there exists a real number \( r \) such that \( 0 \leq r = \int |f|^k \, dM \) and \( \| x \| = r^{\frac{1}{k}} \).

(57) \( X \mapsto 0 \in \) the \( L^1 \) functions of \( M \).

(58) If \( f \in L^p \) functions\((M, k)\) and \( f |f|^k \, dM = 0, \) then \( f = \text{M.a.e.} \) \( X \mapsto 0 \).

(59) \( \int |X | \mapsto 0| |^{k} \, dM = 0 \).

(60) Let \( m, n \) be positive real numbers. Suppose \( \frac{1}{m} + \frac{1}{n} = 1 \) and \( f \in L^p \) functions\((M, m)\) and \( g \in L^p \) functions\((M, n)\). Then \( fg \in \) the \( L^1 \) functions of \( M \) and \( fg \) is integrable on \( M \).

(61) Let \( m, n \) be positive real numbers. Suppose \( \frac{1}{m} + \frac{1}{n} = 1 \) and \( f \in L^p \) functions\((M, m)\) and \( g \in L^p \) functions\((M, n)\). Then there exists a real number \( r_1 \) such that \( r_1 = \int |f|^m \, dM \) and there exists a real number \( r_2 \) such that \( r_2 = \int |g|^n \, dM \) and \( \int |f| \, g \, |^{m} \, dM \leq r_1^{\frac{1}{m}} \cdot r_2^{\frac{1}{n}} \).

(62) Let \( m \) be a positive real number and \( r_1, r_2, r_3 \) be elements of \( \mathbb{R} \). Suppose \( 1 \leq m \) and \( f, g \in L^p \) functions\((M, m)\) and \( r_1 = \int |f|^m \, dM \) and \( r_2 = \int |g|^m \, dM \) and \( r_3 = \int |f + g|^m \, dM \). Then \( r_3^{\frac{1}{m}} \leq r_1^{\frac{1}{m}} + r_2^{\frac{1}{m}} \).

Let \( k \) be a great or equal to 1 real number, let \( X \) be a non empty set, let \( S \) be a \( \sigma \)-field of subsets of \( X \), and let \( M \) be a \( \sigma \)-measure on \( S \). Note that \( L^p - \text{Space}(M, k) \) is reflexive, discernible, real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.
The following propositions are true:

(63) Let $S_1$ be a sequence of $L^p$-Space($M, k$). Then there exists a sequence $F_1$ of partial functions from $X$ into $\mathbb{R}$ such that for every element $n$ of $\mathbb{N}$ holds

$$F_1(n) \in L^p\text{-functions}(M, k) \text{ and } F_1(n) \in S_1(n) \text{ and } S_1(n) = a.e\text{-eq-class } L^p(F_1(n), M, k) \text{ and there exists a real number } r \text{ such that } r = \int |F_1(n)|^k \, dM \text{ and } \|S_1(n)\| = r^\frac{1}{k}.$$

(64) Let $S_1$ be a sequence of $L^p$-Space($M, k$). Then there exists a sequence $F_1$ of partial functions from $X$ into $\mathbb{R}$ with the same dom such that for every element $n$ of $\mathbb{N}$ holds

$$F_1(n) \in L^p\text{-functions}(M, k) \text{ and } F_1(n) \in S_1(n) \text{ and } S_1(n) = a.e\text{-eq-class } L^p(F_1(n), M, k) \text{ and there exists a real number } r \text{ such that } 0 \leq r = \int |F_1(n)|^k \, dM \text{ and } \|S_1(n)\| = r^\frac{1}{k}.$$

(65) Let $X$ be a real normed space, $S_1$ be a sequence of $X$, and $S_0$ be a point of $X$. If $\|S_1 - S_0\|$ is convergent and $\lim\|S_1 - S_0\| = 0$, then $S_1$ is convergent and $\lim S_1 = S_0$.

(66) Let $X$ be a real normed space and $S_1$ be a sequence of $X$. Suppose $S_1$ is Cauchy sequence by norm. Then there exists an increasing function $N$ from $\mathbb{N}$ into $\mathbb{N}$ such that for all elements $i, j$ of $\mathbb{N}$ if $j \geq N(i)$, then $\|S_1(j) - S_1(N(i))\| < 2^{-i}$.

(67) Let $F$ be a sequence of partial functions from $X$ into $\mathbb{R}$. Suppose that for every natural number $m$ holds $F(m) \in L^p\text{-functions}(M, k)$. Let $m$ be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \in L^p\text{-functions}(M, k)$.

(68) Let $F$ be a sequence of partial functions from $X$ into $\mathbb{R}$. Suppose that for every natural number $m$ holds $F(m)$ is non-negative. Let $m$ be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is non-negative.

(69) Let $F$ be a sequence of partial functions from $X$ into $\mathbb{R}$, $x$ be an element of $X$, and $n, m$ be natural numbers. Suppose $F$ has the same dom and $x \in \text{dom } F(0)$ and for every natural number $k$ holds $F(k)$ is non-negative and $n \leq m$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$.

(70) For every sequence $F$ of partial functions from $X$ into $\mathbb{R}$ such that $F$ has the same dom holds $|F|$ has the same dom.

(71) Let $k$ be a great or equal to 1 real number and $S_1$ be a sequence of $L^p$-Space($M, k$). If $S_1$ is Cauchy sequence by norm, then $S_1$ is convergent.

Let us consider $X$, $S$, $M$ and let $k$ be a great or equal to 1 real number. Observe that $L^p$-Space($M, k$) is complete.
6. Relations between $L^1$ Space and $L^p$ Space

One can prove the following propositions:

(72) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{CosetSet} M = \text{CosetSet}(M, 1)$.

(73) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{addCoset} M = \text{addCoset}(M, 1)$.

(74) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{zeroCoset} M = \text{zeroCoset}(M, 1)$.

(75) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{lmultCoset} M = \text{lmultCoset}(M, 1)$.

(76) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{pre-L}_1$-Space $M = \text{pre-$L^1$-Space}(M, 1)$.

(77) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{L}_1$-Norm($M$) = $\text{L}_p$-Norm($M, 1$).

(78) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, and $M$ be a $\sigma$-measure on $S$. Then $\text{L}_1$-Space($M$) = $\text{L}_p$-Space($M, 1$).

References


Received February 4, 2010
Miscellaneous Facts about Open Functions and Continuous Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. In this article we give definitions of open functions and continuous functions formulated in terms of “balls” of given topological spaces.

MML identifier: TOPS_4, version: 7.11.07.4.156.1112

The notation and terminology used here have been introduced in the following papers: [6], [4], [5], [8], [1], [2], [3], [10], [11], [12], [7], [9], and [13].

1. Open Functions

We adopt the following rules: \( n, m \) are elements of \( \mathbb{N} \), \( T \) is a non empty topological space, and \( M, M_1, M_2 \) are non empty metric spaces.

The following propositions are true:

(1) Let \( A, B, S, T \) be topological spaces, \( f \) be a function from \( A \) into \( S \), and \( g \) be a function from \( B \) into \( T \). Suppose that
   (i) the topological structure of \( A = \) the topological structure of \( B \),
   (ii) the topological structure of \( S = \) the topological structure of \( T \),
   (iii) \( f = g \), and
   (iv) \( f \) is open.

   Then \( g \) is open.

(2) Let \( P \) be a subset of \( \mathcal{E}_T^m \). Then \( P \) is open if and only if for every point \( p \) of \( \mathcal{E}_T^m \) such that \( p \in P \) there exists a positive real number \( r \) such that \( \text{Ball}(p, r) \subseteq P \).
(3) Let $X, Y$ be non-empty topological spaces and $f$ be a function from $X$ into $Y$. Then $f$ is open if and only if for every point $p$ of $X$ and for every open subset $V$ of $X$ such that $p \in V$ there exists an open subset $W$ of $Y$ such that $f(p) \in W$ and $W \subseteq f^o V$.

(4) Let $f$ be a function from $T$ into $M_{\text{top}}$. Then $f$ is open if and only if for every point $p$ of $T$ and for every open subset $V$ of $T$ and for every point $q$ of $M$ such that $q = f(p)$ and $p \in V$ there exists a positive real number $r$ such that $\text{Ball}(q, r) \subseteq f^o V$.

(5) Let $f$ be a function from $M_{\text{top}}$ into $T$. Then $f$ is open if and only if for every point $p$ of $M$ and for every positive real number $r$ there exists an open subset $W$ of $T$ such that $f(p) \in W$ and $W \subseteq f^o \text{Ball}(p, r)$.

(6) Let $f$ be a function from $(M_1)_{\text{top}}$ into $(M_2)_{\text{top}}$. Then $f$ is open if and only if for every point $p$ of $M_1$ and for every point $q$ of $M_2$ and for every positive real number $r$ such that $q = f(p)$ there exists a positive real number $s$ such that $\text{Ball}(q, s) \subseteq f^o \text{Ball}(p, r)$.

(7) Let $f$ be a function from $T$ into $E^m_T$. Then $f$ is open if and only if for every point $p$ of $T$ and for every open subset $V$ of $T$ such that $p \in V$ there exists a positive real number $r$ such that $\text{Ball}(f(p), r) \subseteq f^o V$.

(8) Let $f$ be a function from $E^m_T$ into $T$. Then $f$ is open if and only if for every point $p$ of $E^m_T$ and for every positive real number $r$ there exists an open subset $W$ of $T$ such that $f(p) \in W$ and $W \subseteq f^o \text{Ball}(p, r)$.

(9) Let $f$ be a function from $E^m_T$ into $E^m_T$. Then $f$ is open if and only if for every point $p$ of $E^m_T$ and for every positive real number $r$ there exists a positive real number $s$ such that $\text{Ball}(f(p), s) \subseteq f^o \text{Ball}(p, r)$.

(10) Let $f$ be a function from $T$ into $\mathbb{R}^1$. Then $f$ is open if and only if for every point $p$ of $T$ and for every open subset $V$ of $T$ such that $p \in V$ there exists a positive real number $r$ such that $|f(p) - r, f(p) + r| \subseteq f^o V$.

(11) Let $f$ be a function from $\mathbb{R}^1$ into $\mathbb{R}^1$. Then $f$ is open if and only if for every point $p$ of $\mathbb{R}^1$ and for every positive real number $r$ there exists an open subset $V$ of $T$ such that $f(p) \in V$ and $V \subseteq f^o |p - r, p + r|.$

(12) Let $f$ be a function from $\mathbb{R}^1$ into $\mathbb{R}^1$. Then $f$ is open if and only if for every point $p$ of $\mathbb{R}^1$ and for every positive real number $r$ there exists a positive real number $s$ such that $|f(p) - s, f(p) + s| \subseteq f^o |p - r, p + r|.$

(13) Let $f$ be a function from $E^m_T$ into $\mathbb{R}^1$. Then $f$ is open if and only if for every point $p$ of $E^m_T$ and for every positive real number $r$ there exists a positive real number $s$ such that $|f(p) - s, f(p) + s| \subseteq f^o \text{Ball}(p, r)$.

(14) Let $f$ be a function from $\mathbb{R}^1$ into $E^m_T$. Then $f$ is open if and only if for every point $p$ of $\mathbb{R}^1$ and for every positive real number $r$ there exists a positive real number $s$ such that $\text{Ball}(f(p), s) \subseteq f^o |p - r, p + r|.$
Next we state a number of propositions:

(15) Let $f$ be a function from $T$ into $M_{\text{top}}$. Then $f$ is continuous if and only if for every point $p$ of $T$ and for every point $q$ of $M$ and for every positive real number $r$ such that $q = f(p)$ there exists an open subset $W$ of $T$ such that $p \in W$ and $f^\circ W \subseteq \text{Ball}(q, r)$.

(16) Let $f$ be a function from $M_{\text{top}}$ into $T$. Then $f$ is continuous if and only if for every point $p$ of $M$ and for every open subset $V$ of $T$ such that $f(p) \in V$ there exists a positive real number $s$ such that $f^\circ W \subseteq \text{Ball}(q, r)$.

(17) Let $f$ be a function from $(M_1)_{\text{top}}$ into $(M_2)_{\text{top}}$. Then $f$ is continuous if and only if for every point $p$ of $M_1$ and for every point $q$ of $M_2$ and for every positive real number $r$ such that $q = f(p)$ there exists a positive real number $s$ such that $f^\circ \text{Ball}(p, s) \subseteq \text{Ball}(q, r)$.

(18) Let $f$ be a function from $T$ into $E^n_T$. Then $f$ is continuous if and only if for every point $p$ of $T$ and for every positive real number $r$ there exists an open subset $W$ of $T$ such that $p \in W$ and $f^\circ W \subseteq \text{Ball}(f(p), r)$.

(19) Let $f$ be a function from $E^n_T$ into $T$. Then $f$ is continuous if and only if for every point $p$ of $E^n_T$ and for every open subset $V$ of $T$ such that $f(p) \in V$ there exists a positive real number $s$ such that $f^\circ \text{Ball}(p, s) \subseteq \text{Ball}(f(p), r)$.

(20) Let $f$ be a function from $E^n_T$ into $E^n_T$. Then $f$ is continuous if and only if for every point $p$ of $E^n_T$ and for every positive real number $r$ there exists a positive real number $s$ such that $f^\circ \text{Ball}(p, s) \subseteq \text{Ball}(f(p), r)$.

(21) Let $f$ be a function from $T$ into $\mathbb{R}^1$. Then $f$ is continuous if and only if for every point $p$ of $T$ and for every positive real number $r$ there exists an open subset $W$ of $T$ such that $p \in W$ and $f^\circ W \subseteq |f(p) - r, f(p) + r|$. 

(22) Let $f$ be a function from $\mathbb{R}^1$ into $T$. Then $f$ is continuous if and only if for every point $p$ of $\mathbb{R}^1$ and for every open subset $V$ of $T$ such that $f(p) \in V$ there exists a positive real number $s$ such that $f^\circ |p - s, p + s| \subseteq V$.

(23) Let $f$ be a function from $\mathbb{R}^1$ into $\mathbb{R}^1$. Then $f$ is continuous if and only if for every point $p$ of $\mathbb{R}^1$ and for every positive real number $r$ there exists a positive real number $s$ such that $f^\circ |p - s, p + s| \subseteq |f(p) - r, f(p) + r|$.

(24) Let $f$ be a function from $E^n_T$ into $\mathbb{R}^1$. Then $f$ is continuous if and only if for every point $p$ of $E^n_T$ and for every positive real number $r$ there exists a positive real number $s$ such that $f^\circ \text{Ball}(p, s) \subseteq |f(p) - r, f(p) + r|$.

(25) Let $f$ be a function from $\mathbb{R}^1$ into $E^n_T$. Then $f$ is continuous if and only if for every point $p$ of $\mathbb{R}^1$ and for every positive real number $r$ there exists a positive real number $s$ such that $f^\circ |p - s, p + s| \subseteq \text{Ball}(f(p), r)$.
REFERENCES


Received February 9, 2010
On the Continuity of Some Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. We prove that basic arithmetic operations preserve continuity of functions.

MML identifier: TOPREALC, version: 7.11.07 4.156.1112

The terminology and notation used here have been introduced in the following articles: [20], [1], [6], [13], [4], [7], [19], [8], [9], [5], [21], [2], [3], [10], [18], [25], [26], [23], [12], [22], [24], [14], [16], [17], [15], and [11].

1. Preliminaries

For simplicity, we adopt the following rules: $x$, $X$ are sets, $i$, $n$, $m$ are natural numbers, $r$, $s$ are real numbers, $c$, $c_1$, $c_2$, $d$ are complex numbers, $f$, $g$ are complex-valued functions, $g_1$ is an $n$-element complex-valued finite sequence, $f_1$ is an $n$-element real-valued finite sequence, $T$ is a non empty topological space, and $p$ is an element of $E^n$.

Let $R$ be a binary relation and let $X$ be an empty set. Observe that $R^2X$ is empty and $R^{-1}(X)$ is empty.

Let $A$ be an empty set. Observe that every element of $A$ is empty.

We now state the proposition

(1) For every trivial set $X$ and for every set $Y$ such that $X \approx Y$ holds $Y$ is trivial.

Let $r$ be a real number. Observe that $r^2$ is non negative.

Let $r$ be a positive real number. Note that $r^2$ is positive.

Let us note that $\sqrt{0}$ is zero.
Let $f$ be an empty set. Note that $2f$ is empty and $\|f\|$ is zero.

The following propositions are true:

1. $f(c_1 + c_2) = f c_1 + f c_2$.
2. $f(c_1 - c_2) = f c_1 - f c_2$.
3. $f/c + g/c = (f + g)/c$.
4. $f/c - g/c = (f - g)/c$.
5. If $c_1 \neq 0$ and $c_2 \neq 0$, then $f/c - g/c = (f c_2 - g c_1)/(c_1 \cdot c_2)$.

Let us consider $f$, $x$, $c$. Observe that $f + \cdot (x, c)$ is complex-valued.

We now state a number of propositions:

1. $(0, \ldots, 0) + \cdot (x, c) + (0, \ldots, 0) = (x, c^2)$.
2. $(x, r) + (0, \ldots, 0) = |x, r| = |r|$.
3. $0_{\mathbb{E}_T^n} + \cdot (x, 0) = 0_{\mathbb{E}_T^n}$.
4. $f_1 \cdot (0_{\mathbb{E}_T^n} + \cdot (x, r)) = 0_{\mathbb{E}_T^n} + \cdot (x, f_1(x) \cdot r)$.
5. $\|f_1 + 0_{\mathbb{E}_T^n} + \cdot (x, r)\| = f_1(x) \cdot r$.
6. $(g_1 + \cdot (i, c)) - g_1 = (0, \ldots, 0) + \cdot (i, c - g_1(i))$.

One can prove the following proposition

1. For all points $p$, $q$ of $\mathbb{E}_T^n$ holds $p \in \text{Ball}(q, r)$ iff $-p \in \text{Ball}(-q, r)$. 

Let $S$ be a 1-sorted structure. We say that $S$ is complex-functions-membered if and only if:

(Def. 1) The carrier of $S$ is complex-functions-membered.

We say that $S$ is real-functions-membered if and only if:

(Def. 2) The carrier of $S$ is real-functions-membered.

Let us consider $n$. One can verify that $E^n_T$ is real-functions-membered.

Let us observe that $E^0_T$ is real-membered.

One can check that $E^0_T$ is trivial.

Let us observe that every 1-sorted structure which is real-functions-membered is also complex-functions-membered.

Let us mention that there exists a 1-sorted structure which is strict, non empty, and real-functions-membered.

Let $S$ be a complex-functions-membered 1-sorted structure. One can check that the carrier of $S$ is complex-functions-membered.

Let $S$ be a real-functions-membered 1-sorted structure. Note that the carrier of $S$ is real-functions-membered.

Let us observe that there exists a topological space which is strict, non empty, and real-functions-membered.

Let $S$ be a complex-functions-membered topological space. Observe that every subspace of $S$ is complex-functions-membered.

Let $S$ be a real-functions-membered topological space. One can verify that every subspace of $S$ is real-functions-membered.

Let $X$ be a complex-functions-membered set. The functor $(-)X$ yields a complex-functions-membered set and is defined as follows:

(Def. 3) For every complex-valued function $f$ holds $-f \in (-)X$ iff $f \in X$.

Let us observe that the functor $(-)X$ is involutive.

Let $X$ be an empty set. One can verify that $(-)X$ is empty.

Let $X$ be a non empty complex-functions-membered set. Observe that $(-)X$ is non empty.

The following proposition is true

(24) Let $X$ be a complex-functions-membered set and $f$ be a complex-valued function. Then $-f \in X$ if and only if $f \in (-)X$.

Let $X$ be a real-functions-membered set. One can verify that $(-)X$ is real-functions-membered.

Next we state the proposition

(25) For every subset $X$ of $E^n_T$ holds $-X = (-)X$.

Let us consider $n$ and let $X$ be a subset of $E^n_T$. Then $(-)X$ is a subset of $E^n_T$.

Let us consider $n$ and let $X$ be an open subset of $E^n_T$. Observe that $(-)X$ is open.

Let us consider $n$, $p$, $x$. Then $p(x)$ is an element of $\mathbb{R}$.
Let $R$, $S$, $T$ be non-empty topological spaces, let $f$ be a function from $R \times S$ into $T$, and let $x$ be a point of $R \times S$. Then $f(x)$ is a point of $T$.

Let $R$, $S$, $T$ be non-empty topological spaces, let $f$ be a function from $R \times S$ into $T$, let $r$ be a point of $R$, and let $s$ be a point of $S$. Then $f(r, s)$ is a point of $T$.

Let us consider $n$, $p$, $r$. Then $p + r$ is a point of $E^n_T$.

Let us consider $n$, $p$, $r$. Then $p - r$ is a point of $E^n_T$.

Let us consider $n$, $p$, $r$. Then $pr$ is a point of $E^n_T$.

Let us consider $n$, $p$, $r$. Then $p/r$ is a point of $E^n_T$.

Let us consider $n$ and let $p_1$, $p_2$ be points of $E^n_T$. Then $p_1 p_2$ is a point of $E^n_T$.

Let us note that the functor $p_1 p_2$ is commutative.

Let us consider $n$ and let $p$ be a point of $E^n_T$. Then $2p$ is a point of $E^n_T$.

Let us consider $n$ and let $p_1$, $p_2$ be points of $E^n_T$. Then $p_1/p_2$ is a point of $E^n_T$.

Next we state the proposition (26) For all points $a$, $o$ of $E^n_T$ such that $n \neq 0$ and $a \in \text{Ball}(o, r)$ holds $|\sum (a - o)| < n \cdot r$.

Let us consider $n$. Note that $E^n$ is real-functions-membered.

One can prove the following propositions:

(27) Let $V$ be an add-associative right zeroed right complementable non empty additive loop structure and $v$, $u$ be elements of $V$. Then $(v + u) - u = v$.

(28) Let $V$ be an Abelian add-associative right zeroed right complementable non empty additive loop structure and $v$, $u$ be elements of $V$. Then $(v - u) + u = v$.

(29) For every complex-functions-membered set $Y$ and for every partial function $f$ from $X$ to $Y$ holds $f + c = f + (\text{dom } f \mapsto c)$.

(30) For every complex-functions-membered set $Y$ and for every partial function $f$ from $X$ to $Y$ holds $f - c = f - (\text{dom } f \mapsto c)$.

(31) For every complex-functions-membered set $Y$ and for every partial function $f$ from $X$ to $Y$ holds $f \cdot c = f \cdot (\text{dom } f \mapsto c)$.

(32) For every complex-functions-membered set $Y$ and for every partial function $f$ from $X$ to $Y$ holds $f/c = f/(\text{dom } f \mapsto c)$.

Let $D$ be a complex-functions-membered set and let $f$, $g$ be finite sequences of elements of $D$. One can verify the following observations:

* $f + g$ is finite sequence-like,
* $f - g$ is finite sequence-like,
* $f \cdot g$ is finite sequence-like, and
* $f/g$ is finite sequence-like.

Next we state a number of propositions:
(33) For every function $f$ from $X$ into $\mathcal{E}_T^n$ holds $-f$ is a function from $X$ into $\mathcal{E}_T^n$.

(34) For every function $f$ from $\mathcal{E}_T^n$ into $\mathcal{E}_T^n$ holds $f \circ -$ is a function from $\mathcal{E}_T^n$ into $\mathcal{E}_T^n$.

(35) For every function $f$ from $X$ into $\mathcal{E}_T^n$ holds $f + r$ is a function from $X$ into $\mathcal{E}_T^n$.

(36) For every function $f$ from $X$ into $\mathcal{E}_T^n$ holds $f - r$ is a function from $X$ into $\mathcal{E}_T^n$.

(37) For every function $f$ from $X$ into $\mathcal{E}_T^n$ holds $f \cdot r$ is a function from $X$ into $\mathcal{E}_T^n$.

(38) For every function $f$ from $X$ into $\mathcal{E}_T^n$ holds $f/r$ is a function from $X$ into $\mathcal{E}_T^n$.

(39) For all functions $f$, $g$ from $X$ into $\mathcal{E}_T^n$ holds $f + g$ is a function from $X$ into $\mathcal{E}_T^n$.

(40) For all functions $f$, $g$ from $X$ into $\mathcal{E}_T^n$ holds $f - g$ is a function from $X$ into $\mathcal{E}_T^n$.

(41) For all functions $f$, $g$ from $X$ into $\mathcal{E}_T^n$ holds $f \cdot g$ is a function from $X$ into $\mathcal{E}_T^n$.

(42) For all functions $f$, $g$ from $X$ into $\mathcal{E}_T^n$ holds $f/g$ is a function from $X$ into $\mathcal{E}_T^n$.

(43) Let $f$ be a function from $X$ into $\mathcal{E}_T^n$ and $g$ be a function from $X$ into $\mathbb{R}^1$.
Then $f + g$ is a function from $X$ into $\mathcal{E}_T^n$.

(44) Let $f$ be a function from $X$ into $\mathcal{E}_T^n$ and $g$ be a function from $X$ into $\mathbb{R}^1$.
Then $f - g$ is a function from $X$ into $\mathcal{E}_T^n$.

(45) Let $f$ be a function from $X$ into $\mathcal{E}_T^n$ and $g$ be a function from $X$ into $\mathbb{R}^1$.
Then $f \cdot g$ is a function from $X$ into $\mathcal{E}_T^n$.

(46) Let $f$ be a function from $X$ into $\mathcal{E}_T^n$ and $g$ be a function from $X$ into $\mathbb{R}^1$.
Then $f/g$ is a function from $X$ into $\mathcal{E}_T^n$.

Let $n$ be a natural number, let $T$ be a non empty set, let $R$ be a real-membered set, and let $f$ be a function from $T$ into $R$. The functor incl$(f, n)$ yields a function from $T$ into $\mathcal{E}_T^n$ and is defined by:

(Def. 4) For every element $t$ of $T$ holds $(\text{incl}(f, n))(t) = n \mapsto f(t)$.

We now state several propositions:

(47) Let $R$ be a real-membered set, $f$ be a function from $T$ into $R$, and $t$ be a point of $T$. If $x \in \text{Seg} n$, then $(\text{incl}(f, n))(t)(x) = f(t)$.

(48) For every non empty set $T$ and for every real-membered set $R$ and for every function $f$ from $T$ into $R$ holds incl$(f, 0) = T \mapsto 0$.

(49) For every function $f$ from $T$ into $\mathcal{E}_T^n$ and for every function $g$ from $T$ into $\mathbb{R}^1$ holds $f + g = f + \text{incl}(g, n)$. 
180

ARTUR KORNIŁOWICZ

(50) For every function $f$ from $T$ into $E^m_T$ and for every function $g$ from $T$ into $\mathbb{R}^1$ holds $f - g = f - \text{incl}(g, n)$.

(51) For every function $f$ from $T$ into $E^m_T$ and for every function $g$ from $T$ into $\mathbb{R}^1$ holds $f \cdot g = f \cdot \text{incl}(g, n)$.

(52) For every function $f$ from $T$ into $E^m_T$ and for every function $g$ from $T$ into $\mathbb{R}^1$ holds $f / g = f / \text{incl}(g, n)$.

Let us consider $n$. The functor $\otimes_n$ yields a function from $E^m_T \times E^m_T$ into $E^m_T$ and is defined by:

(Def. 5) For all points $x, y$ of $E^m_T$ holds $\otimes_n(x, y) = x \cdot y$.

Next we state two propositions:

(53) $\otimes_0 = E^m_T \times E^m_T \longrightarrow 0_{C^1_T}$.

(54) For all functions $f, g$ from $T$ into $E^m_T$ holds $f \cdot g = (\otimes_n)^\circ(f, g)$.

Let us consider $m, n$. The functor $\text{PROJ}(m, n)$ yields a function from $E^m_T$ into $\mathbb{R}^1$ and is defined as follows:

(Def. 6) For every element $p$ of $E^m_T$ holds $(\text{PROJ}(m, n))(p) = p_n$.

One can prove the following propositions:

(55) For every point $p$ of $E^m_T$ such that $n \in \text{dom } p$ holds $(\text{PROJ}(m, n)) \circ \text{Ball}(p, r) = \left[p_n - r, p_n + r\right]$.

(56) For every non zero natural number $m$ and for every function $f$ from $T$ into $\mathbb{R}^1$ holds $f = \text{PROJ}(m, m) \cdot \text{incl}(f, m)$.

2. Contiunity

Let us consider $T$. One can check that there exists a function from $T$ into $\mathbb{R}^1$ which is non-empty and continuous.

Next we state two propositions:

(57) If $n \in \text{Seg } m$, then $\text{PROJ}(m, n)$ is continuous.

(58) If $n \in \text{Seg } m$, then $\text{PROJ}(m, n)$ is open.

Let us consider $n, T$ and let $f$ be a continuous function from $T$ into $\mathbb{R}^1$. Observe that $\text{incl}(f, n)$ is continuous.

Let us consider $n$. One can verify that $\otimes_n$ is continuous.

One can prove the following proposition

(59) Let $f$ be a function from $E^m_T$ into $E^m_T$. Suppose $f$ is continuous. Then $f \circ -$ is a continuous function from $E^m_T$ into $E^m_T$.

Let us consider $T$ and let $f$ be a continuous function from $T$ into $\mathbb{R}^1$. Observe that $-f$ is continuous.

Let us consider $T$ and let $f$ be a non-empty continuous function from $T$ into $\mathbb{R}^1$. One can verify that $f^{-1}$ is continuous.
Let us consider $T$, let $f$ be a continuous function from $T$ into $\mathbb{R}^1$, and let us consider $r$. One can check the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous,
* $fr$ is continuous, and
* $f/r$ is continuous.

Let us consider $T$ and let $f, g$ be continuous functions from $T$ into $\mathbb{R}^1$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $fg$ is continuous.

Let us consider $T$, let $f$ be a continuous function from $T$ into $\mathbb{R}^1$, and let $g$ be a non-empty continuous function from $T$ into $\mathbb{R}^1$. Observe that $f/g$ is continuous.

Let us consider $n, T$ and let $f, g$ be continuous functions from $T$ into $E^n_T$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider $n, T$, let $f$ be a continuous function from $T$ into $E^n_T$, and let $g$ be a continuous function from $T$ into $\mathbb{R}^1$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider $n, T$, let $f$ be a continuous function from $T$ into $E^n_T$, and let $g$ be a non-empty continuous function from $T$ into $\mathbb{R}^1$. Observe that $f/g$ is continuous.

Let us consider $n, T, r$ and let $f$ be a continuous function from $T$ into $E^n_T$. One can verify the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous, and
* $f \cdot r$ is continuous, and
* $f/r$ is continuous.

We now state two propositions:

(60) Let $r$ be a non negative real number, $n$ be a non zero natural number, and $p$ be a point of $T_{\text{circle}}(0_{E^n_T}, r)$. Then $-p$ is a point of $T_{\text{circle}}(0_{E^n_T}, r)$. 


Let $r$ be a non-negative real number and $f$ be a function from $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$ into $E_{\mathbb{T}^m}^n$. Then $f \circ -$ is a function from $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$ into $E_{\mathbb{T}^m}^n$.

Let $n$ be a natural number, let $r$ be a non-negative real number, and let $X$ be a subset of $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$. Then $(-)X$ is a subset of $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$.

Let us consider $m$, let $r$ be a non-negative real number, and let $X$ be an open subset of $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$. One can verify that $(-)X$ is open.

The following proposition is true

Let $r$ be a non-negative real number and $f$ be a continuous function from $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$ into $E_{\mathbb{T}^m}^n$. Then $f \circ -$ is a continuous function from $T_{\mathbb{R}^n}^0(0_{\mathbb{R}^n}^m + 1, r)$ into $E_{\mathbb{T}^m}^n$.

REFERENCES


Received February 9, 2010
The Geometric Interior in Real Linear Spaces

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. We introduce the notions of the geometric interior and the centre of mass for subsets of real linear spaces. We prove a number of theorems concerning these notions which are used in the theory of abstract simplicial complexes.

The papers [1], [6], [11], [2], [5], [3], [4], [13], [7], [16], [10], [14], [12], [8], [9], and [15] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: $x$ denotes a set, $r$, $s$ denote real numbers, $n$ denotes a natural number, $V$ denotes a real linear space, $v$, $u$, $w$, $p$ denote vectors of $V$, $A$, $B$ denote subsets of $V$, $A_1$ denotes a finite subset of $V$, $I$ denotes an affinely independent subset of $V$, $I_1$ denotes a finite affinely independent subset of $V$, $F$ denotes a family of subsets of $V$, and $L_1$, $L_2$ denote linear combinations of $V$.

Next we state four propositions:

1. Let $L$ be a linear combination of $A$. Suppose $L$ is convex and $v \neq \sum L$ and $L(v) \neq 0$. Then there exists $p \in \text{conv } A \setminus \{v\}$ and $\sum L = L(v) \cdot v + (1 - L(v)) \cdot p$ and $\frac{1}{L(v)} \cdot \sum L + (1 - \frac{1}{L(v)}) \cdot p = v$.

2. Let $p_1$, $p_2$, $w_1$, $w_2$ be elements of $V$. Suppose that $v, u \in \text{conv } I$ and $u \notin \text{conv } I \setminus \{p_1\}$ and $u \notin \text{conv } I \setminus \{p_2\}$ and $w_1 \in \text{conv } I \setminus \{p_1\}$ and
2. The Geometric Interior

Let $V$ be a non empty RLS structure and let $A$ be a subset of $V$. The functor $\text{Int} A$ yields a subset of $V$ and is defined by:

(Def. 1) \( x \in \text{Int} A \) iff \( x \in \text{conv} A \) and it is not true that there exists a subset \( B \) of $V$ such that \( B \subset A \) and \( x \in \text{conv} B \).

Let $V$ be a non empty RLS structure and let $A$ be an empty subset of $V$. Observe that $\text{Int} A$ is empty.

We now state a number of propositions:

(5) For every non empty RLS structure $V$ and for every subset $A$ of $V$ holds $\text{Int} A \subseteq \text{conv} A$.

(6) Let $V$ be a real linear space-like non empty RLS structure and $A$ be a subset of $V$. Then $\text{Int} A = A$ if and only if $A$ is trivial.

(7) If $A \subseteq B$, then $\text{conv} A$ misses $\text{Int} B$.

(8) $\text{conv} A = \bigcup \{ \text{Int} B : B \subseteq A \}$.

(9) $\text{conv} A = \text{Int} A \cup \bigcup \{ \text{conv} A \setminus \{v\} : v \in A \}$.

(10) If $x \in \text{Int} A$, then there exists a linear combination $L$ of $A$ such that $L$ is convex and $x = \sum L$.

(11) For every linear combination $L$ of $A$ such that $L$ is convex and $\sum L \in \text{Int} A$ holds the support of $L = A$.

(12) For every linear combination $L$ of $I$ such that $L$ is convex and the support of $L = I$ holds $\sum L \in \text{Int} I$.

(13) If $\text{Int} A$ is non empty, then $A$ is finite.

(14) If $v \in I$ and $u \in \text{Int} I$ and $p \in \text{conv} I \setminus \{v\}$ and $r \cdot v + (1 - r) \cdot p = u$, then $p \in \text{Int}(I \setminus \{v\})$. 

\[ w_2 \in \text{conv} I \setminus \{p_2\} \text{ and } r \cdot u + (1 - r) \cdot w_1 = v \text{ and } s \cdot u + (1 - s) \cdot w_2 = v \text{ and } r < 1 \text{ and } s < 1. \text{ Then } w_1 = w_2 \text{ and } r = s. \]
3. The Center of Mass

Let us consider $V$. The center of mass of $V$ yielding a function from the carrier of $V$ into the carrier of $V$ is defined by the conditions (Def. 2).

(Def. 2) (i) For every non empty finite subset $A$ of $V$ holds (the center of mass of $V)(A) = \frac{1}{A} \cdot \sum A$, and

(ii) for every $A$ such that $A$ is infinite holds (the center of mass of $V)(A) = 0_V$.

One can prove the following propositions:

(15) There exists a linear combination $L$ of $A_1$ such that $\sum L = r \cdot \sum A_1$ and

$$\text{sum } L = r \cdot \frac{1}{A_1} \text{ and } L = 0_{L_{C_V}} + \cdots (A_1 \rightarrow r).$$

(16) If $A_1$ is non empty, then (the center of mass of $V)(A_1) \in \text{conv } A_1$.

(17) If $\bigcup F$ is finite, then (the center of mass of $V)^{\circ} F \subseteq \text{conv } \bigcup F$.

(18) If $v \in I_1$, then ((the center of mass of $V)(I_1) \rightarrow I_1)(v) = \frac{1}{I_1}$.

(19) (The center of mass of $V)(I_1) \in I_1$ iff $\overline{I_1} = 1$.

(20) If $I_1$ is non empty, then (the center of mass of $V)(I_1) \in \text{Int } I_1$.

(21) If $A \subseteq I_1$ and (the center of mass of $V)(I_1) \in \text{Affin } A$, then $I_1 = A$.

(22) If $v \in A_1$ and $A_1 \setminus \{v\}$ is non empty, then (the center of mass of $V)(A_1) = (1 - \frac{1}{A_1}) \cdot (\text{the center of mass of } V)_{A_1 \setminus \{v\}} + \frac{1}{A_1} \cdot v$.

(23) If $\text{conv } A \subseteq \text{conv } I_1$ and $I_1$ is non empty and $\text{conv } A$ misses $\text{Int } I_1$, then there exists a subset $B$ of $V$ such that $B \subseteq I_1$ and $\text{conv } A \subseteq \text{conv } B$.

(24) If $\sum L_1 \neq \sum L_2$ and sum $L_1 = \text{sum } L_2$, then there exists $v$ such that

$L_1(v) > L_2(v)$.

(25) If $p$ be a real number. Suppose $(r \cdot L_1 + (1 - r) \cdot L_2)(v) \leq p \leq (s \cdot L_1 + (1 - s) \cdot L_2)(v)$. Then there exists a real number $r_1$ such that $(r_1 \cdot L_1 + (1 - r_1) \cdot L_2)(v) = p$ and if $r \leq s$, then $r \leq r_1 \leq s$ and if $s \leq r$, then $s \leq r_1 \leq r$.

(26) If $v, u \in \text{conv } A$ and $v \neq u$, then there exist $p, w, r$ such that $p \in A$ and $w \in \text{conv } A \setminus \{p\}$ and $0 \leq r < 1$ and $r \cdot u + (1 - r) \cdot w = v$.

(27) $A \cup \{v\}$ is affinely independent iff $A$ is affinely independent but $v \in A$ or $v \notin \text{Affin } A$.

(28) If $A_1 \subseteq I$ and $v \in A_1$, then $(I \setminus \{v\}) \cup \{(\text{the center of mass of } V)(A_1)\}$ is an affinely independent subset of $V$.

(29) Let $F$ be a $\subseteq$-linear family of subsets of $V$. Suppose $\bigcup F$ is finite and affinely independent. Then (the center of mass of $V)^{\circ} F$ is an affinely independent subset of $V$.

(30) Let $F$ be a $\subseteq$-linear family of subsets of $V$. Suppose $\bigcup F$ is affinely independent and finite. Then $\text{Int } ((\text{the center of mass of } V)^{\circ} F) \subseteq \text{Int } \bigcup F$. 

(31) The geometric interior in real linear . . .
REFERENCES


Received February 9, 2010
Contents

On $L^p$ Space Formed by Real-Valued Partial Functions
By Yasushige Watase et al. ............................. 159

Miscellaneous Facts about Open Functions and Continuous Functions
By Artur Korniłowicz ................................. 171

On the Continuity of Some Functions
By Artur Korniłowicz ................................. 175

The Geometric Interior in Real Linear Spaces
By Karol Pąk ................................. 185