FORMALIZED MATHEMATICS
Vol. 18, No. 3, Pages 159–169, 2010

On L^p Space Formed by Real-Valued Partial Functions

Yasushige Watase
Graduate School of Science and Technology
Shinshu University
Nagano, Japan

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. This article is the continuation of [31]. We define the set of L^p integrable functions – the set of all partial functions whose absolute value raised to the p-th power is integrable. We show that L^p integrable functions form the L^p space. We also prove Minkowski's inequality, Hölder's inequality and that L^p space is Banach space ([15], [27]).

MML identifier: LPSPACE2, version: 7.11.07 4.156.1112

The notation and terminology used in this paper have been introduced in the following papers: [7], [8], [9], [10], [4], [1], [31], [6], [19], [20], [13], [28], [14], [2], [24], [3], [11], [25], [22], [21], [16], [32], [29], [23], [18], [17], [26], [30], [5], and [12].

1. Preliminaries on Powers of Numbers and Operations on Real Sequences

For simplicity, we follow the rules: X denotes a non empty set, x denotes an element of X, S denotes a σ-field of subsets of X, M denotes a σ-measure on S, f, g, f_1, g_1 denote partial functions from X to $\mathbb{R}$, and a, b, c denote real numbers.

The following propositions are true:

(1) For all positive real numbers m, n such that $\frac{1}{m} + \frac{1}{n} = 1$ holds $m > 1$.

ISSN 1426–2630(p), 1898-9934(e)

(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $\int f \, \mathrm{d}M \in \mathbb{R}$ if and only if f is integrable on M.

Let r be a real number. We say that r is great or equal to 1 if and only if:

(Def. 1) $1 \leq r$.

Let us note that every real number which is great or equal to 1 is also positive.

One can verify that there exists a real number which is great or equal to 1.

In the sequel k denotes a positive real number.

We now state several propositions:

(3) For all real numbers a, b, p such that $0 < p$ and $0 \leq a < b$ holds $a^p < b^p$.

(4) If $a \geq 0$ and $b > 0$, then $a^b \geq 0$.

(5) If $a \geq 0$ and $b \geq 0$ and $c > 0$, then $(a \cdot b)^c = a^c \cdot b^c$.

(6) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $(f^a)^b = f^{a \cdot b}$.

(7) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $f^a \, f^b = f^{a+b}$.

(8) $f^1 = f$.

(9) Let s_1, s_2 be sequences of real numbers and k be a positive real number. Suppose that for every element n of $\mathbb{N}$ holds $s_1(n) = s_2(n)^k$ and $s_2(n) \geq 0$. Then s_1 is convergent if and only if s_2 is convergent.

(10) Let s_3 be a sequence of real numbers and n, m be elements of $\mathbb{N}$. If $m \leq n$, then $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(m)$ and $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(11) Let s_3, s_2 be sequences of real numbers and k be a positive real number. Suppose s_3 is convergent and for every element n of $\mathbb{N}$ holds $s_2(n) = |\lim s_3 - s_3(n)|^k$. Then s_2 is convergent and $\lim s_2 = 0$.

2. Real Linear Space of L^p Integrable Functions

Next we state two propositions:

(12) For every positive real number k and for every non empty set X holds $(X \longmapsto 0)^k = X \longmapsto 0$.

(13) For every partial function f from X to $\mathbb{R}$ and for every set D holds $|f{\restriction}D| = |f|{\restriction}D$.

Let us consider X and let f be a partial function from X to $\mathbb{R}$. Observe that $|f|$ is non-negative.

One can prove the following two propositions:

(14) For every partial function f from X to $\mathbb{R}$ such that f is non-negative holds $|f| = f$.

(15) If $X = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $0 = f(x)$, then f is integrable on M and $\int f \,\mathrm{d}M = 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p functions(M, k) yielding a non empty subset of $\mathrm{PFunct}_{\mathrm{RLS}}\, X$ is defined by the condition (Def. 2).

(Def. 2) L^p functions$(M, k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $\bigvee_{E_1 : \text{element of } S} (M({E_1}^{\mathrm{c}}) = 0 \ \wedge\ \operatorname{dom} f = E_1 \ \wedge\ f$ is measurable on $E_1 \ \wedge\ |f|^k$ is integrable on $M)\}$.

Next we state a number of propositions:

(16) For all real numbers a, b, k such that $k > 0$ holds $|a + b|^k \leq (|a| + |b|)^k$ and $(|a| + |b|)^k \leq (2 \cdot \max(|a|, |b|))^k$ and $|a + b|^k \leq (2 \cdot \max(|a|, |b|))^k$.

(17) For all real numbers a, b, k such that $a \geq 0$ and $b \geq 0$ and $k > 0$ holds $(\max(a, b))^k \leq a^k + b^k$.

(18) For every partial function f from X to $\mathbb{R}$ and for all real numbers a, b such that $b > 0$ holds $|a|^b\, |f|^b = |a\, f|^b$.

(19) Let f be a partial function from X to $\mathbb{R}$ and a, b be real numbers. If $a > 0$ and $b > 0$, then $a^b\, |f|^b = (a\, |f|)^b$.

(20) For every partial function f from X to $\mathbb{R}$ and for every real number k and for every set E holds $(f{\restriction}E)^k = f^k{\restriction}E$.

(21) For all real numbers a, b, k such that $k > 0$ holds $|a+b|^k \leq 2^k \cdot (|a|^k + |b|^k)$.

(22) Let k be a positive real number and f, g be partial functions from X to $\mathbb{R}$. Suppose f, $g \in L^p$ functions(M, k). Then $|f|^k$ is integrable on M and $|g|^k$ is integrable on M and $|f|^k + |g|^k$ is integrable on M.

(23) $X \longmapsto 0$ is a partial function from X to $\mathbb{R}$ and $X \longmapsto 0 \in L^p$ functions(M, k).

(24) Let k be a real number. Suppose $k > 0$. Let f, g be partial functions from X to $\mathbb{R}$ and x be an element of X. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$, then $|f + g|^k(x) \leq (2^k\, (|f|^k + |g|^k))(x)$.

(25) If f, $g \in L^p$ functions(M, k), then $f + g \in L^p$ functions(M, k).

(26) If $f \in L^p$ functions(M, k), then $a\, f \in L^p$ functions(M, k).

(27) If f, $g \in L^p$ functions(M, k), then $f - g \in L^p$ functions(M, k).

(28) If $f \in L^p$ functions(M, k), then $|f| \in L^p$ functions(M, k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Note that L^p functions(M, k) is multiplicatively-closed and add closed.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that $\langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$ is Abelian, add-associative, and real linear space-like.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSp LpFunct(M,k) yields a strict Abelian add-associative real linear space-like non empty RLS structure and is defined by:

(Def. 3) $\mathrm{RLSp\ LpFunct}(M,k) = \langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$.

3. Preliminaries on Real Normed Space of L^p Integrable Functions

In the sequel v, u are vectors of RLSp LpFunct(M,k).

We now state three propositions:

(29) $(v) + (u) = v + u$.

(30) $a\,(u) = a \cdot u$.

(31) Suppose $f = u$. Then

(i) $u + (-1) \cdot u = (X \longmapsto 0){\restriction}\mathrm{dom}\,f$, and

(ii) there exist partial functions v, g from X to $\mathbb{R}$ such that $v, g \in L^p\,\mathrm{functions}(M,k)$ and $v = u + (-1) \cdot u$ and $g = X \longmapsto 0$ and $v =^{M}_{\mathrm{a.e.}} g$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor AlmostZeroLpFunctions(M,k) yielding a non empty subset of RLSp LpFunct(M,k) is defined by:

(Def. 4) $\mathrm{AlmostZeroLpFunctions}(M,k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in L^p\,\mathrm{functions}(M,k) \wedge f =^{M}_{\mathrm{a.e.}} X \longmapsto 0\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that AlmostZeroLpFunctions(M,k) is add closed and multiplicatively-closed.

Next we state the proposition

(32) $0_{\mathrm{RLSp\ LpFunct}(M,k)} = X \longmapsto 0$ and $0_{\mathrm{RLSp\ LpFunct}(M,k)} \in \mathrm{AlmostZeroLpFunctions}(M,k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSpAlmostZeroLpFunctions(M,k) yielding a non empty RLS structure is defined by:

(Def. 5) $\mathrm{RLSpAlmostZeroLpFunctions}(M,k) = \langle \mathrm{AlmostZeroLpFunctions}(M,k), 0_{\mathrm{RLSp\ LpFunct}(M,k)}(\in \mathrm{AlmostZeroLpFunctions}(M,k)), \mathrm{add}\,|(\mathrm{AlmostZeroLp}$

Functions(M, k), RLSp LpFunct$(M, k))$, $\cdot_{\rm AlmostZeroLpFunctions(M,k)}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that RLSp LpFunct(M, k) is strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel v, u are vectors of RLSpAlmostZeroLpFunctions(M, k).

One can prove the following two propositions:

(33) $(v) + (u) = v + u$.

(34) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{R}$, and let k be a positive real number. The functor a.e-eq-class $L^p(f, M, k)$ yields a subset of L^p functions(M, k) and is defined as follows:

(Def. 6) a.e-eq-class $L^p(f, M, k) = \{h; h$ ranges over partial functions from X to $\mathbb{R}$: $h \in L^p$ functions$(M, k) \ \wedge\ f =^{M}_{\rm a.e.} h\}$.

Next we state a number of propositions:

(35) If $f \in L^p$ functions(M, k), then there exists an element E of S such that $M(E^{\rm c}) = 0$ and dom $f = E$ and f is measurable on E.

(36) If $g \in L^p$ functions(M, k) and $g =^{M}_{\rm a.e.} f$, then $g \in$ a.e-eq-class $L^p(f, M, k)$.

(37) Suppose there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom f and f is measurable on E and $g \in$ a.e-eq-class $L^p(f, M, k)$. Then $g =^{M}_{\rm a.e.} f$ and $f \in L^p$ functions(M, k).

(38) If $f \in L^p$ functions(M, k), then $f \in$ a.e-eq-class $L^p(f, M, k)$.

(39) Suppose there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom g and g is measurable on E and a.e-eq-class $L^p(f, M, k) \neq \emptyset$ and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$. Then $f =^{M}_{\rm a.e.} g$.

(40) Suppose $f \in L^p$ functions(M, k) and there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom g and g is measurable on E and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$. Then $f =^{M}_{\rm a.e.} g$.

(41) If $f =^{M}_{\rm a.e.} g$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(42) If $f =^{M}_{\rm a.e.} g$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(43) If $f \in L^p$ functions(M, k) and $g \in$ a.e-eq-class $L^p(f, M, k)$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(44) Suppose that there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom f and f is measurable on E and there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom f_1 and f_1 is measurable on E and there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom g and g is measurable on E and there exists an element E of S such that $M(E^{\rm c}) = 0$ and $E =$ dom g_1 and g_1 is measurable on

E and a.e-eq-class $L^p(f, M, k)$ is non empty and a.e-eq-class $L^p(g, M, k)$ is non empty and a.e-eq-class $L^p(f, M, k)$ = a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k)$ = a.e-eq-class $L^p(g_1, M, k)$. Then a.e-eq-class $L^p(f + g, M, k)$ = a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(45) If f, f_1, g, $g_1 \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k)$ = a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k)$ = a.e-eq-class $L^p(g_1, M, k)$, then a.e-eq-class $L^p(f + g, M, k)$ = a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(46) Suppose that

(i) there exists an element E of S such that $M(E^c) = 0$ and dom $f = E$ and f is measurable on E,

(ii) there exists an element E of S such that $M(E^c) = 0$ and dom $g = E$ and g is measurable on E,

(iii) a.e-eq-class $L^p(f, M, k)$ is non empty, and

(iv) a.e-eq-class $L^p(f, M, k)$ = a.e-eq-class $L^p(g, M, k)$.

Then a.e-eq-class $L^p(a\,f, M, k)$ = a.e-eq-class $L^p(a\,g, M, k)$.

(47) If f, $g \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k)$ = a.e-eq-class $L^p(g, M, k)$, then a.e-eq-class $L^p(a\,f, M, k)$ = a.e-eq-class $L^p(a\,g, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor CosetSet(M, k) yielding a non empty family of subsets of L^p functions(M, k) is defined by:

(Def. 7) CosetSet(M, k) = {a.e-eq-class $L^p(f, M, k)$; f ranges over partial functions from X to $\mathbb{R}$: $f \in L^p$ functions(M, k)}.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor addCoset(M, k) yields a binary operation on CosetSet(M, k) and is defined by the condition (Def. 8).

(Def. 8) Let A, B be elements of CosetSet(M, k) and a, b be partial functions from X to $\mathbb{R}$. If $a \in A$ and $b \in B$, then (addCoset(M, k))(A, B) = a.e-eq-class $L^p(a + b, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor zeroCoset(M, k) yields an element of CosetSet(M, k) and is defined as follows:

(Def. 9) zeroCoset(M, k) = a.e-eq-class $L^p(X \longmapsto 0, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor lmultCoset(M, k) yielding a function from $\mathbb{R} \times$ CosetSet(M, k) into CosetSet(M, k) is defined by the condition (Def. 10).

(Def. 10) Let z be an element of $\mathbb{R}$, A be an element of CosetSet(M, k), and f be a partial function from X to $\mathbb{R}$. If $f \in A$, then (lmultCoset$(M, k))(z, A) =$ a.e-eq-class $L^p(z\, f, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor Pre- L^p -Space(M, k) yielding a strict RLS structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of Pre- L^p -Space$(M, k) =$ CosetSet(M, k),

(ii) the addition of Pre- L^p -Space$(M, k) =$ addCoset(M, k),

(iii) $0_{\text{Pre-}\, L^p\, \text{-Space}(M,k)} =$ zeroCoset(M, k), and

(iv) the external multiplication of Pre- L^p -Space$(M, k) =$ lmultCoset(M, k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M, k) is non empty.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M, k) is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

4. Real Normed Space of L^p Integrable Functions

The following propositions are true:

(48) If f, $g \in L^p$ functions(M, k) and $f =^{M}_{\text{a.e.}} g$, then $\int |f|^k \,\mathrm{d}M = \int |g|^k \,\mathrm{d}M$.

(49) If $f \in L^p$ functions(M, k), then $\int |f|^k \,\mathrm{d}M \in \mathbb{R}$ and $0 \leq \int |f|^k \,\mathrm{d}M$.

(50) If there exists a vector x of Pre- L^p -Space(M, k) such that f, $g \in x$, then $f =^{M}_{\text{a.e.}} g$ and f, $g \in L^p$ functions(M, k).

(51) Let k be a positive real number. Then there exists a function N_1 from the carrier of Pre- L^p -Space(M, k) into $\mathbb{R}$ such that for every point x of Pre- L^p -Space(M, k) holds there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \,\mathrm{d}M$ and $N_1(x) = r^{\frac{1}{k}}$.

In the sequel x denotes a point of Pre- L^p -Space(M, k).

We now state two propositions:

(52) If $f \in x$, then $|f|^k$ is integrable on M and $f \in L^p$ functions(M, k).

(53) If f, $g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $\int |f|^k \,\mathrm{d}M = \int |g|^k \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Norm(M, k) yielding a function from the carrier of Pre- L^p -Space(M, k) into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x be a point of Pre- L^p -Space(M,k). Then there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \, \mathrm{d}M$ and $(L^p\text{-Norm}(M,k))(x) = r^{\frac{1}{k}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Space(M,k) yields a non empty normed structure and is defined by:

(Def. 13) L^p -Space(M,k) = ⟨the carrier of Pre- L^p -Space(M,k), the zero of Pre- L^p -Space(M,k), the addition of Pre- L^p -Space(M,k), the external multiplication of Pre- L^p -Space(M,k), L^p -Norm(M,k)⟩.

In the sequel x, y denote points of L^p -Space(M,k).

One can prove the following propositions:

(54)(i) There exists a partial function f from X to $\mathbb{R}$ such that $f \in L^p$ functions(M,k) and $x =$ a.e-eq-class $L^p(f,M,k)$, and

(ii) for every partial function f from X to $\mathbb{R}$ such that $f \in x$ there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(55) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a\,f \in a \cdot x$.

(56) If $f \in x$, then $x =$ a.e-eq-class $L^p(f,M,k)$ and there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(57) $X \longmapsto 0 \in$ the L^1 functions of M.

(58) If $f \in L^p$ functions(M,k) and $\int |f|^k \, \mathrm{d}M = 0$, then $f =^{M}_{\text{a.e.}} X \longmapsto 0$.

(59) $\int |X \longmapsto 0|^k \, \mathrm{d}M = 0$.

(60) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M,m) and $g \in L^p$ functions(M,n). Then $f\,g \in$ the L^1 functions of M and $f\,g$ is integrable on M.

(61) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M,m) and $g \in L^p$ functions(M,n). Then there exists a real number r_1 such that $r_1 = \int |f|^m \, \mathrm{d}M$ and there exists a real number r_2 such that $r_2 = \int |g|^n \, \mathrm{d}M$ and $\int |f\,g| \, \mathrm{d}M \leq {r_1}^{\frac{1}{m}} \cdot {r_2}^{\frac{1}{n}}$.

(62) Let m be a positive real number and r_1, r_2, r_3 be elements of $\mathbb{R}$. Suppose $1 \leq m$ and f, $g \in L^p$ functions(M,m) and $r_1 = \int |f|^m \, \mathrm{d}M$ and $r_2 = \int |g|^m \, \mathrm{d}M$ and $r_3 = \int |f+g|^m \, \mathrm{d}M$. Then ${r_3}^{\frac{1}{m}} \leq {r_1}^{\frac{1}{m}} + {r_2}^{\frac{1}{m}}$.

Let k be a great or equal to 1 real number, let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Note that L^p -Space(M,k) is reflexive, discernible, real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

5. Preliminaries on Completeness of L^p Space

The following propositions are true:

(63) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $r = \int |F_1(n)|^k \, \mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(64) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ with the same dom such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $0 \leq r = \int |F_1(n)|^k \, \mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(65) Let X be a real normed space, S_1 be a sequence of X, and S_0 be a point of X. If $\|S_1 - S_0\|$ is convergent and $\lim \|S_1 - S_0\| = 0$, then S_1 is convergent and $\lim S_1 = S_0$.

(66) Let X be a real normed space and S_1 be a sequence of X. Suppose S_1 is Cauchy sequence by norm. Then there exists an increasing function N from $\mathbb{N}$ into $\mathbb{N}$ such that for all elements i, j of $\mathbb{N}$ if $j \geq N(i)$, then $\|S_1(j) - S_1(N(i))\| < 2^{-i}$.

(67) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m) \in L^p$ functions(M, k). Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \in L^p$ functions(M, k).

(68) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m)$ is non-negative. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is non-negative.

(69) Let F be a sequence of partial functions from X into $\mathbb{R}$, x be an element of X, and n, m be natural numbers. Suppose F has the same dom and $x \in \operatorname{dom} F(0)$ and for every natural number k holds $F(k)$ is non-negative and $n \leq m$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$.

(70) For every sequence F of partial functions from X into $\mathbb{R}$ such that F has the same dom holds $|F|$ has the same dom.

(71) Let k be a great or equal to 1 real number and S_1 be a sequence of L^p-Space(M, k). If S_1 is Cauchy sequence by norm, then S_1 is convergent.

Let us consider X, S, M and let k be a great or equal to 1 real number. Observe that L^p-Space(M, k) is complete.

6. Relations between L^1 Space and L^p Space

One can prove the following propositions:

(72) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{CosetSet}\, M = \mathrm{CosetSet}(M, 1)$.

(73) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{addCoset}\, M = \mathrm{addCoset}(M, 1)$.

(74) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{zeroCoset}\, M = \mathrm{zeroCoset}(M, 1)$.

(75) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{lmultCoset}\, M = \mathrm{lmultCoset}(M, 1)$.

(76) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then pre-L-Space M = Pre-L^p-Space$(M, 1)$.

(77) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then L^1-Norm(M) = L^p-Norm$(M, 1)$.

(78) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then L^1-Space(M) = L^p-Space$(M, 1)$.

References

[1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Formalized Mathematics*, 9(**3**):565–582, 2001.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[13] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006, doi:10.2478/v10037-006-0008-x.
[14] Noboru Endou, Yasunari Shidama, and Keiko Narita. Egoroff's theorem. *Formalized Mathematics*, 16(**1**):57–63, 2008, doi:10.2478/v10037-008-0009-z.
[15] P. R. Halmos. *Measure Theory.* Springer-Verlag, 1987.
[16] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[17] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[18] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[19] Keiko Narita, Noboru Endou, and Yasunari Shidama. Integral of complex-valued measurable function. *Formalized Mathematics*, 16(**4**):319–324, 2008, doi:10.2478/v10037-008-0039-6.
[20] Keiko Narita, Noboru Endou, and Yasunari Shidama. Lebesgue's convergence theorem of complex-valued function. *Formalized Mathematics*, 17(**2**):137–145, 2009, doi: 10.2478/v10037-009-0015-9.
[21] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[22] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[23] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.
[24] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[25] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[26] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[27] Walter Rudin. *Real and Complex Analysis*. Mc Graw-Hill, Inc., 1974.
[28] Yasunari Shidama and Noboru Endou. Integral of real-valued measurable function. *Formalized Mathematics*, 14(**4**):143–152, 2006, doi:10.2478/v10037-006-0018-8.
[29] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[30] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[31] Yasushige Watase, Noboru Endou, and Yasunari Shidama. On L^1 space formed by real-valued partial functions. *Formalized Mathematics*, 16(**4**):361–369, 2008, doi:10.2478/v10037-008-0044-9.
[32] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 4, 2010

FORMALIZED MATHEMATICS
Vol. **18**, No. **3**, Pages 171–174, **2010**

Miscellaneous Facts about Open Functions and Continuous Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. In this article we give definitions of open functions and continuous functions formulated in terms of "balls" of given topological spaces.

MML identifier: `TOPS_4`, version: `7.11.07 4.156.1112`

The notation and terminology used here have been introduced in the following papers: [6], [4], [5], [8], [1], [2], [3], [10], [11], [12], [7], [9], and [13].

1. Open Functions

We adopt the following rules: n, m are elements of $\mathbb{N}$, T is a non empty topological space, and M, M_1, M_2 are non empty metric spaces.

The following propositions are true:

(1) Let A, B, S, T be topological spaces, f be a function from A into S, and g be a function from B into T. Suppose that
  (i) the topological structure of $A =$ the topological structure of B,
  (ii) the topological structure of $S =$ the topological structure of T,
  (iii) $f = g$, and
  (iv) f is open.

  Then g is open.

(2) Let P be a subset of $\mathcal{E}^m_{\mathrm{T}}$. Then P is open if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ such that $p \in P$ there exists a positive real number r such that $\mathrm{Ball}(p, r) \subseteq P$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Let X, Y be non empty topological spaces and f be a function from X into Y. Then f is open if and only if for every point p of X and for every open subset V of X such that $p \in V$ there exists an open subset W of Y such that $f(p) \in W$ and $W \subseteq f^{\circ}V$.

(4) Let f be a function from T into $M_{\rm top}$. Then f is open if and only if for every point p of T and for every open subset V of T and for every point q of M such that $q = f(p)$ and $p \in V$ there exists a positive real number r such that $\mathrm{Ball}(q, r) \subseteq f^{\circ}V$.

(5) Let f be a function from $M_{\rm top}$ into T. Then f is open if and only if for every point p of M and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^{\circ}\,\mathrm{Ball}(p, r)$.

(6) Let f be a function from $(M_1)_{\rm top}$ into $(M_2)_{\rm top}$. Then f is open if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $\mathrm{Ball}(q, s) \subseteq f^{\circ}\,\mathrm{Ball}(p, r)$.

(7) Let f be a function from T into $\mathcal{E}^m_{\rm T}$. Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $\mathrm{Ball}(f(p), r) \subseteq f^{\circ}V$.

(8) Let f be a function from $\mathcal{E}^m_{\rm T}$ into T. Then f is open if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^{\circ}\,\mathrm{Ball}(p, r)$.

(9) Let f be a function from $\mathcal{E}^m_{\rm T}$ into $\mathcal{E}^n_{\rm T}$. Then f is open if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every positive real number r there exists a positive real number s such that $\mathrm{Ball}(f(p), s) \subseteq f^{\circ}\,\mathrm{Ball}(p, r)$.

(10) Let f be a function from T into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $]f(p) - r, f(p) + r[\subseteq f^{\circ}V$.

(11) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into T. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists an open subset V of T such that $f(p) \in V$ and $V \subseteq f^{\circ}]p - r, p + r[$.

(12) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^{\circ}]p - r, p + r[$.

(13) Let f be a function from $\mathcal{E}^m_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^{\circ}\,\mathrm{Ball}(p, r)$.

(14) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathcal{E}^m_{\rm T}$. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $\mathrm{Ball}(f(p), s) \subseteq f^{\circ}]p - r, p + r[$.

2. Continuous Functions

Next we state a number of propositions:

(15) Let f be a function from T into $M_{\rm top}$. Then f is continuous if and only if for every point p of T and for every point q of M and for every positive real number r such that $q = f(p)$ there exists an open subset W of T such that $p \in W$ and $f^{\circ}W \subseteq \mathrm{Ball}(q, r)$.

(16) Let f be a function from $M_{\rm top}$ into T. Then f is continuous if and only if for every point p of M and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^{\circ}\,\mathrm{Ball}(p, s) \subseteq V$.

(17) Let f be a function from $(M_1)_{\rm top}$ into $(M_2)_{\rm top}$. Then f is continuous if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $f^{\circ}\,\mathrm{Ball}(p, s) \subseteq \mathrm{Ball}(q, r)$.

(18) Let f be a function from T into $\mathcal{E}^m_{\rm T}$. Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^{\circ}W \subseteq \mathrm{Ball}(f(p), r)$.

(19) Let f be a function from $\mathcal{E}^m_{\rm T}$ into T. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^{\circ}\,\mathrm{Ball}(p, s) \subseteq V$.

(20) Let f be a function from $\mathcal{E}^m_{\rm T}$ into $\mathcal{E}^n_{\rm T}$. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every positive real number r there exists a positive real number s such that $f^{\circ}\,\mathrm{Ball}(p, s) \subseteq \mathrm{Ball}(f(p), r)$.

(21) Let f be a function from T into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^{\circ}W \subseteq \,]f(p) - r, f(p) + r[$.

(22) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into T. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^{\circ}]p - s, p + s[\, \subseteq V$.

(23) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $f^{\circ}]p - s, p + s[\, \subseteq \,]f(p) - r, f(p) + r[$.

(24) Let f be a function from $\mathcal{E}^m_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\rm T}$ and for every positive real number r there exists a positive real number s such that $f^{\circ}\,\mathrm{Ball}(p, s) \subseteq \,]f(p) - r, f(p) + r[$.

(25) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathcal{E}^m_{\rm T}$. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $f^{\circ}]p - s, p + s[\, \subseteq \mathrm{Ball}(f(p), r)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[8] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[9] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[11] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[12] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[13] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received February 9, 2010

FORMALIZED MATHEMATICS
Vol. 18, No. 3, Pages 175–183, 2010

On the Continuity of Some Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. We prove that basic arithmetic operations preserve continuity of functions.

MML identifier: `TOPREALC`, version: `7.11.07 4.156.1112`

The terminology and notation used here have been introduced in the following articles: [20], [1], [6], [13], [4], [7], [19], [8], [9], [5], [21], [2], [3], [10], [18], [25], [26], [23], [12], [22], [24], [14], [16], [17], [15], and [11].

1. Preliminaries

For simplicity, we adopt the following rules: x, X are sets, i, n, m are natural numbers, r, s are real numbers, c, c_1, c_2, d are complex numbers, f, g are complex-valued functions, g_1 is an n-element complex-valued finite sequence, f_1 is an n-element real-valued finite sequence, T is a non empty topological space, and p is an element of $\mathcal{E}^n_{\mathrm{T}}$.

Let R be a binary relation and let X be an empty set. Observe that $R^\circ X$ is empty and $R^{-1}(X)$ is empty.

Let A be an empty set. Observe that every element of A is empty.

We now state the proposition

(1) For every trivial set X and for every set Y such that $X \approx Y$ holds Y is trivial.

Let r be a real number. Observe that $r^{\mathbf{2}}$ is non negative.

Let r be a positive real number. Note that $r^{\mathbf{2}}$ is positive.

Let us note that $\sqrt{0}$ is zero.

ISSN 1426–2630(p), 1898-9934(e)

Let f be an empty set. Note that ${}^2 f$ is empty and $|f|$ is zero.

The following propositions are true:

(2) $f\,(c_1 + c_2) = f\,c_1 + f\,c_2$.

(3) $f\,(c_1 - c_2) = f\,c_1 - f\,c_2$.

(4) $f/c + g/c = (f + g)/c$.

(5) $f/c - g/c = (f - g)/c$.

(6) If $c_1 \neq 0$ and $c_2 \neq 0$, then $f/c_1 - g/c_2 = (f\,c_2 - g\,c_1)/(c_1 \cdot c_2)$.

(7) If $c \neq 0$, then $f/c - g = (f - c\,g)/c$.

(8) $(c - d)\,f = c\,f - d\,f$.

(9) $(f - g)^{\mathbf{2}} = (g - f)^{\mathbf{2}}$.

(10) $(f/c)^{\mathbf{2}} = f^{\mathbf{2}}/c^{\mathbf{2}}$.

(11) $|n \mapsto r - n \mapsto s| = \sqrt{n} \cdot |r - s|$.

Let us consider f, x, c. Observe that $f +\cdot\, (x, c)$ is complex-valued.

We now state a number of propositions:

(12) $(\langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, c))^{\mathbf{2}} = \langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, c^{\mathbf{2}})$.

(13) If $x \in \operatorname{Seg} n$, then $|\langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, r)| = |r|$.

(14) $0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, 0) = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(15) $f_1 \bullet (0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, r)) = 0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, f_1(x) \cdot r)$.

(16) $|(f_1, 0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, r))| = f_1(x) \cdot r$.

(17) $(g_1 +\cdot\, (i, c)) - g_1 = \langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (i, c - g_1(i))$.

(18) $|\langle r \rangle| = |r|$.

(19) Every real-valued finite sequence is a finite sequence of elements of $\mathbb{R}$.

(20) For every real-valued finite sequence f such that $|f| \neq 0$ there exists a natural number i such that $i \in \operatorname{dom} f$ and $f(i) \neq 0$.

(21) For every real-valued finite sequence f holds $|\sum f| \leq \sum |f|$.

(22) Let A be a non empty 1-sorted structure, B be a trivial non empty 1-sorted structure, t be a point of B, and f be a function from A into B. Then $f = A \longmapsto t$.

Let n be a non zero natural number, let i be an element of $\operatorname{Seg} n$, and let T be a real-membered non empty topological space. Note that $\operatorname{proj}(\operatorname{Seg} n \longmapsto T, i)$ is real-valued.

Let us consider n, let p be an element of $\mathcal{R}^n$, and let us consider r. Then p/r is an element of $\mathcal{R}^n$.

One can prove the following proposition

(23) For all points p, q of $\mathcal{E}^m_{\mathrm{T}}$ holds $p \in \operatorname{Ball}(q, r)$ iff $-p \in \operatorname{Ball}(-q, r)$.

Let S be a 1-sorted structure. We say that S is complex-functions-membered if and only if:

(Def. 1) The carrier of S is complex-functions-membered.

We say that S is real-functions-membered if and only if:

(Def. 2) The carrier of S is real-functions-membered.

Let us consider n. One can verify that $\mathcal{E}^n_{\mathrm{T}}$ is real-functions-membered.

Let us observe that $\mathcal{E}^0_{\mathrm{T}}$ is real-membered.

One can check that $\mathcal{E}^0_{\mathrm{T}}$ is trivial.

Let us observe that every 1-sorted structure which is real-functions-membered is also complex-functions-membered.

Let us mention that there exists a 1-sorted structure which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered 1-sorted structure. One can check that the carrier of S is complex-functions-membered.

Let S be a real-functions-membered 1-sorted structure. Note that the carrier of S is real-functions-membered.

Let us observe that there exists a topological space which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered topological space. Observe that every subspace of S is complex-functions-membered.

Let S be a real-functions-membered topological space. One can verify that every subspace of S is real-functions-membered.

Let X be a complex-functions-membered set. The functor $(-)X$ yields a complex-functions-membered set and is defined as follows:

(Def. 3) For every complex-valued function f holds $-f \in (-)X$ iff $f \in X$.

Let us observe that the functor $(-)X$ is involutive.

Let X be an empty set. One can verify that $(-)X$ is empty.

Let X be a non empty complex-functions-membered set. Observe that $(-)X$ is non empty.

The following proposition is true

(24) Let X be a complex-functions-membered set and f be a complex-valued function. Then $-f \in X$ if and only if $f \in (-)X$.

Let X be a real-functions-membered set. One can verify that $(-)X$ is real-functions-membered.

Next we state the proposition

(25) For every subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $-X = (-)X$.

Let us consider n and let X be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $(-)X$ is a subset of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n and let X be an open subset of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $(-)X$ is open.

Let us consider n, p, x. Then $p(x)$ is an element of $\mathbb{R}$.

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T, and let x be a point of $R \times S$. Then $f(x)$ is a point of T.

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T, let r be a point of R, and let s be a point of S. Then $f(r, s)$ is a point of T.

Let us consider n, p, r. Then $p + r$ is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n, p, r. Then $p - r$ is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n, p, r. Then $p\,r$ is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n, p, r. Then p/r is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Then $p_1\,p_2$ is a point of $\mathcal{E}^n_{\rm T}$. Let us note that the functor $p_1\,p_2$ is commutative.

Let us consider n and let p be a point of $\mathcal{E}^n_{\rm T}$. Then ${}^2p$ is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Then p_1/p_2 is a point of $\mathcal{E}^n_{\rm T}$.

Let us consider n, p, x, r. Then $p +\cdot (x, r)$ is a point of $\mathcal{E}^n_{\rm T}$.

Next we state the proposition

(26) For all points a, o of $\mathcal{E}^n_{\rm T}$ such that $n \neq 0$ and $a \in \mathrm{Ball}(o, r)$ holds $|\sum(a - o)| < n \cdot r$.

Let us consider n. Note that $\mathcal{E}^n$ is real-functions-membered.

One can prove the following propositions:

(27) Let V be an add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V. Then $(v+u)-u = v$.

(28) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V. Then $(v - u) + u = v$.

(29) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f + c = f + (\mathrm{dom}\, f \longmapsto c)$.

(30) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f - c = f - (\mathrm{dom}\, f \longmapsto c)$.

(31) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f \cdot c = f \cdot (\mathrm{dom}\, f \longmapsto c)$.

(32) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f/c = f/(\mathrm{dom}\, f \longmapsto c)$.

Let D be a complex-functions-membered set and let f, g be finite sequences of elements of D. One can verify the following observations:

* $f + g$ is finite sequence-like,
* $f - g$ is finite sequence-like,
* $f \cdot g$ is finite sequence-like, and
* f/g is finite sequence-like.

Next we state a number of propositions:

(33) For every function f from X into $\mathcal{E}^n_{\rm T}$ holds $-f$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(34) For every function f from $\mathcal{E}^i_{\rm T}$ into $\mathcal{E}^n_{\rm T}$ holds $f \circ -$ is a function from $\mathcal{E}^i_{\rm T}$ into $\mathcal{E}^n_{\rm T}$.

(35) For every function f from X into $\mathcal{E}^n_{\rm T}$ holds $f + r$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(36) For every function f from X into $\mathcal{E}^n_{\rm T}$ holds $f - r$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(37) For every function f from X into $\mathcal{E}^n_{\rm T}$ holds $f \cdot r$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(38) For every function f from X into $\mathcal{E}^n_{\rm T}$ holds f/r is a function from X into $\mathcal{E}^n_{\rm T}$.

(39) For all functions f, g from X into $\mathcal{E}^n_{\rm T}$ holds $f + g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(40) For all functions f, g from X into $\mathcal{E}^n_{\rm T}$ holds $f - g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(41) For all functions f, g from X into $\mathcal{E}^n_{\rm T}$ holds $f \cdot g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(42) For all functions f, g from X into $\mathcal{E}^n_{\rm T}$ holds f/g is a function from X into $\mathcal{E}^n_{\rm T}$.

(43) Let f be a function from X into $\mathcal{E}^n_{\rm T}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f + g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(44) Let f be a function from X into $\mathcal{E}^n_{\rm T}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f - g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(45) Let f be a function from X into $\mathcal{E}^n_{\rm T}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f \cdot g$ is a function from X into $\mathcal{E}^n_{\rm T}$.

(46) Let f be a function from X into $\mathcal{E}^n_{\rm T}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then f/g is a function from X into $\mathcal{E}^n_{\rm T}$.

Let n be a natural number, let T be a non empty set, let R be a real-membered set, and let f be a function from T into R. The functor ${\rm incl}(f, n)$ yields a function from T into $\mathcal{E}^n_{\rm T}$ and is defined by:

(Def. 4) For every element t of T holds $({\rm incl}(f, n))(t) = n \mapsto f(t)$.

We now state several propositions:

(47) Let R be a real-membered set, f be a function from T into R, and t be a point of T. If $x \in {\rm Seg}\, n$, then $({\rm incl}(f, n))(t)(x) = f(t)$.

(48) For every non empty set T and for every real-membered set R and for every function f from T into R holds ${\rm incl}(f, 0) = T \longmapsto 0$.

(49) For every function f from T into $\mathcal{E}^n_{\rm T}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f + g = f + {\rm incl}(g, n)$.

(50) For every function f from T into $\mathcal{E}^n_{\rm T}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f - g = f - \mathrm{incl}(g, n)$.

(51) For every function f from T into $\mathcal{E}^n_{\rm T}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f \cdot g = f \cdot \mathrm{incl}(g, n)$.

(52) For every function f from T into $\mathcal{E}^n_{\rm T}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f/g = f/\,\mathrm{incl}(g, n)$.

Let us consider n. The functor $\bigotimes_n$ yields a function from $\mathcal{E}^n_{\rm T} \times \mathcal{E}^n_{\rm T}$ into $\mathcal{E}^n_{\rm T}$ and is defined by:

(Def. 5) For all points x, y of $\mathcal{E}^n_{\rm T}$ holds $\bigotimes_n(x,\, y) = x\, y$.

Next we state two propositions:

(53) $\bigotimes_0 = \mathcal{E}^0_{\rm T} \times \mathcal{E}^0_{\rm T} \longmapsto 0_{\mathcal{E}^0_{\rm T}}$.

(54) For all functions f, g from T into $\mathcal{E}^n_{\rm T}$ holds $f \cdot g = (\bigotimes_n)^\circ(f,\, g)$.

Let us consider m, n. The functor $\mathrm{PROJ}(m, n)$ yields a function from $\mathcal{E}^m_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined as follows:

(Def. 6) For every element p of $\mathcal{E}^m_{\rm T}$ holds $(\mathrm{PROJ}(m, n))(p) = p_n$.

One can prove the following propositions:

(55) For every point p of $\mathcal{E}^m_{\rm T}$ such that $n \in \mathrm{dom}\, p$ holds $(\mathrm{PROJ}(m, n))^\circ\, \mathrm{Ball}(p, r) =]p_n - r, p_n + r[$.

(56) For every non zero natural number m and for every function f from T into $\mathbb{R}^{\mathbf{1}}$ holds $f = \mathrm{PROJ}(m, m) \cdot \mathrm{incl}(f, m)$.

2. Continuity

Let us consider T. One can check that there exists a function from T into $\mathbb{R}^{\mathbf{1}}$ which is non-empty and continuous.

Next we state two propositions:

(57) If $n \in \mathrm{Seg}\, m$, then $\mathrm{PROJ}(m, n)$ is continuous.

(58) If $n \in \mathrm{Seg}\, m$, then $\mathrm{PROJ}(m, n)$ is open.

Let us consider n, T and let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that $\mathrm{incl}(f, n)$ is continuous.

Let us consider n. One can verify that $\bigotimes_n$ is continuous.

One can prove the following proposition

(59) Let f be a function from $\mathcal{E}^m_{\rm T}$ into $\mathcal{E}^n_{\rm T}$. Suppose f is continuous. Then $f \circ -$ is a continuous function from $\mathcal{E}^m_{\rm T}$ into $\mathcal{E}^n_{\rm T}$.

Let us consider T and let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that $-f$ is continuous.

Let us consider T and let f be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. One can verify that f^{-1} is continuous.

Let us consider T, let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$, and let us consider r. One can check the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous,
* $f\,r$ is continuous, and
* f/r is continuous.

Let us consider T and let f, g be continuous functions from T into $\mathbb{R}^{\mathbf{1}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f\,g$ is continuous.

Let us consider T, let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$, and let g be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that f/g is continuous.

Let us consider n, T and let f, g be continuous functions from T into $\mathcal{E}^n_{\mathrm{T}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider n, T, let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$, and let g be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider n, T, let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$, and let g be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that f/g is continuous.

Let us consider n, T, r and let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$. One can verify the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous,
* $f \cdot r$ is continuous, and
* f/r is continuous.

We now state two propositions:

(60) Let r be a non negative real number, n be a non zero natural number, and p be a point of $\mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$. Then $-p$ is a point of $\mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$.

(61) Let r be a non negative real number and f be a function from $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{n}$. Then $f \circ -$ is a function from $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{n}$.

Let n be a natural number, let r be a non negative real number, and let X be a subset of $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$. Then $(-)X$ is a subset of $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$.

Let us consider m, let r be a non negative real number, and let X be an open subset of $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$. One can verify that $(-)X$ is open.

The following proposition is true

(62) Let r be a non negative real number and f be a continuous function from $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{m}$. Then $f \circ -$ is a continuous function from $\text{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{m}$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[6] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[12] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[13] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[14] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[15] Artur Korniłowicz. Arithmetic operations on functions from sets into functional sets. *Formalized Mathematics*, 17(**1**):43–60, 2009, doi:10.2478/v10037-009-0005-y.
[16] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}_{\mathrm{T}}^{n}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.
[17] Artur Korniłowicz and Yasunari Shidama. Some properties of circles on the plane. *Formalized Mathematics*, 13(**1**):117–124, 2005.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[20] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[21] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[22] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 9, 2010

FORMALIZED MATHEMATICS
Vol. **18**, No. **3**, Pages 185–188, **2010**

The Geometric Interior in Real Linear Spaces

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. We introduce the notions of the geometric interior and the centre of mass for subsets of real linear spaces. We prove a number of theorems concerning these notions which are used in the theory of abstract simplicial complexes.

MML identifier: `RLAFFIN2`, version: `7.11.07 4.156.1112`

The papers [1], [6], [11], [2], [5], [3], [4], [13], [7], [16], [10], [14], [12], [8], [9], and [15] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x denotes a set, r, s denote real numbers, n denotes a natural number, V denotes a real linear space, v, u, w, p denote vectors of V, A, B denote subsets of V, A_1 denotes a finite subset of V, I denotes an affinely independent subset of V, I_1 denotes a finite affinely independent subset of V, F denotes a family of subsets of V, and L_1, L_2 denote linear combinations of V.

Next we state four propositions:

(1) Let L be a linear combination of A. Suppose L is convex and $v \neq \sum L$ and $L(v) \neq 0$. Then there exists p such that $p \in \operatorname{conv} A \setminus \{v\}$ and $\sum L = L(v) \cdot v + (1 - L(v)) \cdot p$ and $\frac{1}{L(v)} \cdot \sum L + (1 - \frac{1}{L(v)}) \cdot p = v$.

(2) Let p_1, p_2, w_1, w_2 be elements of V. Suppose that v, $u \in \operatorname{conv} I$ and $u \notin \operatorname{conv} I \setminus \{p_1\}$ and $u \notin \operatorname{conv} I \setminus \{p_2\}$ and $w_1 \in \operatorname{conv} I \setminus \{p_1\}$ and

ISSN 1426–2630(p), 1898-9934(e)

$w_2 \in \operatorname{conv} I \setminus \{p_2\}$ and $r \cdot u + (1-r) \cdot w_1 = v$ and $s \cdot u + (1-s) \cdot w_2 = v$ and $r < 1$ and $s < 1$. Then $w_1 = w_2$ and $r = s$.

(3) Let L be a linear combination of A_1. Suppose $A_1 \subseteq \operatorname{conv} I_1$ and $\operatorname{sum} L = 1$. Then

(i) $\sum L \in \operatorname{Affin} I_1$, and

(ii) for every element x of V there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence G of elements of V such that $(\sum L \to I_1)(x) = \sum F$ and $\operatorname{len} G = \operatorname{len} F$ and G is one-to-one and $\operatorname{rng} G =$ the support of L and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = L(G(n)) \cdot (G(n) \to I_1)(x)$.

(4) For every subset A_2 of V such that A_2 is affine and $\operatorname{conv} A \cap \operatorname{conv} B \subseteq A_2$ and $\operatorname{conv} A \setminus \{v\} \subseteq A_2$ and $v \notin A_2$ holds $\operatorname{conv} A \setminus \{v\} \cap \operatorname{conv} B = \operatorname{conv} A \cap \operatorname{conv} B$.

2. The Geometric Interior

Let V be a non empty RLS structure and let A be a subset of V. The functor $\operatorname{Int} A$ yields a subset of V and is defined by:

(Def. 1) $x \in \operatorname{Int} A$ iff $x \in \operatorname{conv} A$ and it is not true that there exists a subset B of V such that $B \subset A$ and $x \in \operatorname{conv} B$.

Let V be a non empty RLS structure and let A be an empty subset of V. Observe that $\operatorname{Int} A$ is empty.

We now state a number of propositions:

(5) For every non empty RLS structure V and for every subset A of V holds $\operatorname{Int} A \subseteq \operatorname{conv} A$.

(6) Let V be a real linear space-like non empty RLS structure and A be a subset of V. Then $\operatorname{Int} A = A$ if and only if A is trivial.

(7) If $A \subset B$, then $\operatorname{conv} A$ misses $\operatorname{Int} B$.

(8) $\operatorname{conv} A = \bigcup\{\operatorname{Int} B : B \subseteq A\}$.

(9) $\operatorname{conv} A = \operatorname{Int} A \cup \bigcup\{\operatorname{conv} A \setminus \{v\} : v \in A\}$.

(10) If $x \in \operatorname{Int} A$, then there exists a linear combination L of A such that L is convex and $x = \sum L$.

(11) For every linear combination L of A such that L is convex and $\sum L \in \operatorname{Int} A$ holds the support of $L = A$.

(12) For every linear combination L of I such that L is convex and the support of $L = I$ holds $\sum L \in \operatorname{Int} I$.

(13) If $\operatorname{Int} A$ is non empty, then A is finite.

(14) If $v \in I$ and $u \in \operatorname{Int} I$ and $p \in \operatorname{conv} I \setminus \{v\}$ and $r \cdot v + (1-r) \cdot p = u$, then $p \in \operatorname{Int}(I \setminus \{v\})$.

3. The Center of Mass

Let us consider V. The center of mass of V yielding a function from $2_+^{\text{the carrier of } V}$ into the carrier of V is defined by the conditions (Def. 2).

(Def. 2)(i) For every non empty finite subset A of V holds (the center of mass of V)$(A) = \frac{1}{\overline{\overline{A}}} \cdot \sum A$, and

(ii) for every A such that A is infinite holds (the center of mass of V)$(A) = 0_V$.

One can prove the following propositions:

(15) There exists a linear combination L of A_1 such that $\sum L = r \cdot \sum A_1$ and sum $L = r \cdot \overline{\overline{A_1}}$ and $L = \mathbf{0}_{\mathrm{LC}_V} +\cdot (A_1 \longmapsto r)$.

(16) If A_1 is non empty, then (the center of mass of V)$(A_1) \in \operatorname{conv} A_1$.

(17) If $\bigcup F$ is finite, then (the center of mass of V)$^\circ F \subseteq \operatorname{conv} \bigcup F$.

(18) If $v \in I_1$, then ((the center of mass of V)$(I_1) \to I_1)(v) = \frac{1}{\overline{\overline{I_1}}}$.

(19) (The center of mass of V)$(I_1) \in I_1$ iff $\overline{\overline{I_1}} = 1$.

(20) If I_1 is non empty, then (the center of mass of V)$(I_1) \in \operatorname{Int} I_1$.

(21) If $A \subseteq I_1$ and (the center of mass of V)$(I_1) \in \operatorname{Affin} A$, then $I_1 = A$.

(22) If $v \in A_1$ and $A_1 \setminus \{v\}$ is non empty, then (the center of mass of V)$(A_1) = (1 - \frac{1}{\overline{\overline{A_1}}}) \cdot$ (the center of mass of V)$_{A_1 \setminus \{v\}} + \frac{1}{\overline{\overline{A_1}}} \cdot v$.

(23) If $\operatorname{conv} A \subseteq \operatorname{conv} I_1$ and I_1 is non empty and $\operatorname{conv} A$ misses $\operatorname{Int} I_1$, then there exists a subset B of V such that $B \subset I_1$ and $\operatorname{conv} A \subseteq \operatorname{conv} B$.

(24) If $\sum L_1 \neq \sum L_2$ and sum L_1 = sum L_2, then there exists v such that $L_1(v) > L_2(v)$.

(25) Let p be a real number. Suppose $(r \cdot L_1 + (1-r) \cdot L_2)(v) \leq p \leq (s \cdot L_1 + (1-s) \cdot L_2)(v)$. Then there exists a real number r_1 such that $(r_1 \cdot L_1 + (1-r_1) \cdot L_2)(v) = p$ and if $r \leq s$, then $r \leq r_1 \leq s$ and if $s \leq r$, then $s \leq r_1 \leq r$.

(26) If v, $u \in \operatorname{conv} A$ and $v \neq u$, then there exist p, w, r such that $p \in A$ and $w \in \operatorname{conv} A \setminus \{p\}$ and $0 \leq r < 1$ and $r \cdot u + (1-r) \cdot w = v$.

(27) $A \cup \{v\}$ is affinely independent iff A is affinely independent but $v \in A$ or $v \notin \operatorname{Affin} A$.

(28) If $A_1 \subseteq I$ and $v \in A_1$, then $(I \setminus \{v\}) \cup \{$(the center of mass of V)$(A_1)\}$ is an affinely independent subset of V.

(29) Let F be a $\subseteq$-linear family of subsets of V. Suppose $\bigcup F$ is finite and affinely independent. Then (the center of mass of V)$^\circ F$ is an affinely independent subset of V.

(30) Let F be a $\subseteq$-linear family of subsets of V. Suppose $\bigcup F$ is affinely independent and finite. Then Int((the center of mass of V)$^\circ F) \subseteq \operatorname{Int} \bigcup F$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[6] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Convex sets and convex combinations. *Formalized Mathematics*, 11(**1**):53–58, 2003.
[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[9] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[10] Karol Pąk. Affine independence in vector spaces. *Formalized Mathematics*, 18(**1**):87–93, 2010, doi: 10.2478/v10037-010-0012-z.
[11] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[12] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.
[13] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[14] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 9, 2010

Contents

Formaliz. Math. 18 (**3**)