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The Geometric Interior in Real Linear Spaces

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Summary. We introduce the notions of the geometric interior and the centre of mass for subsets of real linear spaces. We prove a number of theorems concerning these notions which are used in the theory of abstract simplicial complexes.

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The papers [1], [6], [11], [2], [5], [3], [4], [13], [7], [16], [10], [14], [12], [8], [9], and [15] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x denotes a set, r, s denote real numbers, n denotes a natural number, V denotes a real linear space, v, u, w, p denote vectors of V, A, B denote subsets of V, A_1 denotes a finite subset of V, I denotes an affinely independent subset of V, I_1 denotes a finite affinely independent subset of V, F denotes a family of subsets of V, and L_1, L_2 denote linear combinations of V.

Next we state four propositions:

(1) Let L be a linear combination of A. Suppose L is convex and $v \neq \sum L$ and $L(v) \neq 0$. Then there exists p such that $p \in \operatorname{conv} A \setminus \{v\}$ and $\sum L = L(v) \cdot v + (1 - L(v)) \cdot p$ and $\frac{1}{L(v)} \cdot \sum L + (1 - \frac{1}{L(v)}) \cdot p = v$.

(2) Let p_1, p_2, w_1, w_2 be elements of V. Suppose that $v, u \in \operatorname{conv} I$ and $u \notin \operatorname{conv} I \setminus \{p_1\}$ and $u \notin \operatorname{conv} I \setminus \{p_2\}$ and $w_1 \in \operatorname{conv} I \setminus \{p_1\}$ and

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$w_2 \in \operatorname{conv} I \setminus \{p_2\}$ and $r \cdot u + (1 - r) \cdot w_1 = v$ and $s \cdot u + (1 - s) \cdot w_2 = v$ and $r < 1$ and $s < 1$. Then $w_1 = w_2$ and $r = s$.

(3) Let L be a linear combination of A_1. Suppose $A_1 \subseteq \operatorname{conv} I_1$ and sum $L = 1$. Then

(i) $\sum L \in \operatorname{Affin} I_1$, and

(ii) for every element x of V there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence G of elements of V such that $(\sum L \to I_1)(x) = \sum F$ and $\operatorname{len} G = \operatorname{len} F$ and G is one-to-one and $\operatorname{rng} G =$ the support of L and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = L(G(n)) \cdot (G(n) \to I_1)(x)$.

(4) For every subset A_2 of V such that A_2 is affine and $\operatorname{conv} A \cap \operatorname{conv} B \subseteq A_2$ and $\operatorname{conv} A \setminus \{v\} \subseteq A_2$ and $v \notin A_2$ holds $\operatorname{conv} A \setminus \{v\} \cap \operatorname{conv} B = \operatorname{conv} A \cap \operatorname{conv} B$.

2. The Geometric Interior

Let V be a non empty RLS structure and let A be a subset of V. The functor $\operatorname{Int} A$ yields a subset of V and is defined by:

(Def. 1) $x \in \operatorname{Int} A$ iff $x \in \operatorname{conv} A$ and it is not true that there exists a subset B of V such that $B \subset A$ and $x \in \operatorname{conv} B$.

Let V be a non empty RLS structure and let A be an empty subset of V. Observe that $\operatorname{Int} A$ is empty.

We now state a number of propositions:

(5) For every non empty RLS structure V and for every subset A of V holds $\operatorname{Int} A \subseteq \operatorname{conv} A$.

(6) Let V be a real linear space-like non empty RLS structure and A be a subset of V. Then $\operatorname{Int} A = A$ if and only if A is trivial.

(7) If $A \subset B$, then $\operatorname{conv} A$ misses $\operatorname{Int} B$.

(8) $\operatorname{conv} A = \bigcup\{\operatorname{Int} B : B \subseteq A\}$.

(9) $\operatorname{conv} A = \operatorname{Int} A \cup \bigcup\{\operatorname{conv} A \setminus \{v\} : v \in A\}$.

(10) If $x \in \operatorname{Int} A$, then there exists a linear combination L of A such that L is convex and $x = \sum L$.

(11) For every linear combination L of A such that L is convex and $\sum L \in \operatorname{Int} A$ holds the support of $L = A$.

(12) For every linear combination L of I such that L is convex and the support of $L = I$ holds $\sum L \in \operatorname{Int} I$.

(13) If $\operatorname{Int} A$ is non empty, then A is finite.

(14) If $v \in I$ and $u \in \operatorname{Int} I$ and $p \in \operatorname{conv} I \setminus \{v\}$ and $r \cdot v + (1 - r) \cdot p = u$, then $p \in \operatorname{Int}(I \setminus \{v\})$.

3. THE CENTER OF MASS

Let us consider V. The center of mass of V yielding a function from $2_+^{\text{the carrier of } V}$ into the carrier of V is defined by the conditions (Def. 2).

(Def. 2)(i) For every non empty finite subset A of V holds (the center of mass of V)$(A) = \frac{1}{\overline{\overline{A}}} \cdot \sum A$, and

(ii) for every A such that A is infinite holds (the center of mass of V)$(A) = 0_V$.

One can prove the following propositions:

(15) There exists a linear combination L of A_1 such that $\sum L = r \cdot \sum A_1$ and $\mathrm{sum}\, L = r \cdot \overline{\overline{A_1}}$ and $L = \mathbf{0}_{\mathrm{LC}_V} + \cdot (A_1 \longmapsto r)$.

(16) If A_1 is non empty, then (the center of mass of V)$(A_1) \in \mathrm{conv}\, A_1$.

(17) If $\bigcup F$ is finite, then (the center of mass of V)$^\circ F \subseteq \mathrm{conv} \bigcup F$.

(18) If $v \in I_1$, then ((the center of mass of V)$(I_1) \rightarrow I_1)(v) = \frac{1}{\overline{\overline{I_1}}}$.

(19) (The center of mass of V)$(I_1) \in I_1$ iff $\overline{\overline{I_1}} = 1$.

(20) If I_1 is non empty, then (the center of mass of V)$(I_1) \in \mathrm{Int}\, I_1$.

(21) If $A \subseteq I_1$ and (the center of mass of V)$(I_1) \in \mathrm{Affin}\, A$, then $I_1 = A$.

(22) If $v \in A_1$ and $A_1 \setminus \{v\}$ is non empty, then (the center of mass of V)$(A_1) = (1 - \frac{1}{\overline{\overline{A_1}}}) \cdot$ (the center of mass of V)$_{A_1 \setminus \{v\}} + \frac{1}{\overline{\overline{A_1}}} \cdot v$.

(23) If $\mathrm{conv}\, A \subseteq \mathrm{conv}\, I_1$ and I_1 is non empty and $\mathrm{conv}\, A$ misses $\mathrm{Int}\, I_1$, then there exists a subset B of V such that $B \subset I_1$ and $\mathrm{conv}\, A \subseteq \mathrm{conv}\, B$.

(24) If $\sum L_1 \neq \sum L_2$ and $\mathrm{sum}\, L_1 = \mathrm{sum}\, L_2$, then there exists v such that $L_1(v) > L_2(v)$.

(25) Let p be a real number. Suppose $(r \cdot L_1 + (1-r) \cdot L_2)(v) \leq p \leq (s \cdot L_1 + (1-s) \cdot L_2)(v)$. Then there exists a real number r_1 such that $(r_1 \cdot L_1 + (1 - r_1) \cdot L_2)(v) = p$ and if $r \leq s$, then $r \leq r_1 \leq s$ and if $s \leq r$, then $s \leq r_1 \leq r$.

(26) If v, $u \in \mathrm{conv}\, A$ and $v \neq u$, then there exist p, w, r such that $p \in A$ and $w \in \mathrm{conv}\, A \setminus \{p\}$ and $0 \leq r < 1$ and $r \cdot u + (1-r) \cdot w = v$.

(27) $A \cup \{v\}$ is affinely independent iff A is affinely independent but $v \in A$ or $v \notin \mathrm{Affin}\, A$.

(28) If $A_1 \subseteq I$ and $v \in A_1$, then $(I \setminus \{v\}) \cup \{$(the center of mass of V)$(A_1)\}$ is an affinely independent subset of V.

(29) Let F be a $\subseteq$-linear family of subsets of V. Suppose $\bigcup F$ is finite and affinely independent. Then (the center of mass of V)$^\circ F$ is an affinely independent subset of V.

(30) Let F be a $\subseteq$-linear family of subsets of V. Suppose $\bigcup F$ is affinely independent and finite. Then Int((the center of mass of V)$^\circ F) \subseteq \mathrm{Int} \bigcup F$.

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