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Cayley's Theorem

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Summary. The article formalizes the Cayley's theorem saying that every group G is isomorphic to a subgroup of the symmetric group on G.

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The notation and terminology used in this paper have been introduced in the following papers: [3], [6], [4], [5], [10], [11], [7], [2], [1], [9], and [8].

In this paper X, Y denote sets, G denotes a group, and n denotes a natural number.

Let us consider X. Note that $\emptyset_{X,\emptyset}$ is onto.

Let us observe that every set which is permutational is also functional.

Let us consider X. The functor permutations X is defined as follows:

(Def. 1) permutations $X = \{f : f$ ranges over permutations of $X\}$.

Next we state three propositions:

(1) For every set f such that $f \in$ permutations X holds f is a permutation of X.

(2) permutations $X \subseteq X^X$.

(3) permutations $\mathrm{Seg}\, n =$ the permutations of n.

Let us consider X. One can verify that permutations X is non empty and functional.

Let X be a finite set. One can verify that permutations X is finite.

Next we state the proposition

(4) permutations $\emptyset = 1$.

Let us consider X. The functor SymGroup X yields a strict constituted functions multiplicative magma and is defined by:

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(Def. 2) The carrier of SymGroup X = permutations X and for all elements x, y of SymGroup X holds $x \cdot y = (y$ **qua** function$)\ \cdot x$.

One can prove the following proposition

(5) Every element of SymGroup X is a permutation of X.

Let us consider X. Note that SymGroup X is non empty, associative, and group-like.

The following propositions are true:

(6) $\mathbf{1}_{\mathrm{SymGroup}\,X} = \mathrm{id}_X$.

(7) For every element x of SymGroup X holds $x^{-1} = (x$ **qua** function$)\ ^{-1}$.

Let us consider n. One can verify that A_n is constituted functions.

One can prove the following proposition

(8) SymGroup Seg $n = A_n$.

Let X be a finite set. Observe that SymGroup X is finite.

We now state the proposition

(9) SymGroup $\emptyset$ = Trivial-multMagma .

Let us note that SymGroup $\emptyset$ is trivial.

Let us consider X, Y and let p be a function from X into Y. Let us assume that $X \neq \emptyset$ and $Y \neq \emptyset$ and p is bijective. The functor SymGroupsIso p yielding a function from SymGroup X into SymGroup Y is defined by:

(Def. 3) For every element x of SymGroup X holds (SymGroupsIso $p)(x) = p \cdot x \cdot p^{-1}$.

We now state four propositions:

(10) For all non empty sets X, Y and for every function p from X into Y such that p is bijective holds SymGroupsIso p is multiplicative.

(11) For all non empty sets X, Y and for every function p from X into Y such that p is bijective holds SymGroupsIso p is one-to-one.

(12) For all non empty sets X, Y and for every function p from X into Y such that p is bijective holds SymGroupsIso p is onto.

(13) If $X \approx Y$, then SymGroup X and SymGroup Y are isomorphic.

Let us consider G. The functor CayleyIso G yields a function from G into SymGroup (the carrier of G) and is defined as follows:

(Def. 4) For every element g of G holds (CayleyIso $G)(g) = \cdot g$.

Let us consider G. One can verify that CayleyIso G is multiplicative.

Let us consider G. One can verify that CayleyIso G is one-to-one.

One can prove the following proposition

(14) G and Im CayleyIso G are isomorphic.

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Borel-Cantelli Lemma[1]

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Summary. This article is about the Borel-Cantelli Lemma in probability theory. Necessary definitions and theorems are given in [10] and [7].

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The notation and terminology used here have been introduced in the following papers: [17], [3], [4], [8], [13], [1], [2], [5], [15], [14], [21], [9], [12], [11], [16], [6], [20], [19], and [18].

For simplicity, we adopt the following rules: O_1 is a non empty set, S_1 is a σ-field of subsets of O_1, P_1 is a probability on S_1, A is a sequence of subsets of S_1, and n is an element of $\mathbb{N}$.

Let D be a set, let x, y be extended real numbers, and let a, b be elements of D. Then $(x > y \rightarrow a, b)$ is an element of D.

We now state two propositions:

(1) For every element k of $\mathbb{N}$ and for every element x of $\mathbb{R}$ such that k is odd and $x > 0$ and $x \leq 1$ holds $(-x\,\mathrm{ExpSeq}_{\mathbb{R}})(k{+}1){+}(-x\,\mathrm{ExpSeq}_{\mathbb{R}})(k{+}2) \geq 0$.

(2) For every element x of $\mathbb{R}$ holds $1 + x \leq (\text{the function exp})(x)$.

Let s be a sequence of real numbers. The functor ExpFuncWithElementOf s yielding a sequence of real numbers is defined as follows:

(Def. 1) For every natural number d holds $(\mathrm{ExpFuncWithElementOf}\, s)(d) = \sum -s(d)\,\mathrm{ExpSeq}_{\mathbb{R}}$.

Next we state two propositions:

(3) (The partial product of $\mathrm{ExpFuncWithElementOf}(P_1 \cdot A))(n) = $ (the function exp)$(-(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa \in \mathbb{N}}(n))$.

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(4) (The partial product of $P_1 \cdot A^{\mathbf{c}})(n) \leq$ (the partial product of ExpFuncWithElementOf$(P_1 \cdot A))(n)$.

Let n_1, n_2 be elements of $\mathbb{N}$. The functor SeqOfIFGT1(n_1, n_2) yielding a sequence of $\mathbb{N}$ is defined by:

(Def. 2) For every element n of $\mathbb{N}$ holds (SeqOfIFGT1$(n_1, n_2))(n) = (n > n_1 \to n + n_2, n)$.

Let k be an element of $\mathbb{N}$. The SeqOfIFGT2 k yields a sequence of $\mathbb{N}$ and is defined by:

(Def. 3) For every element n of $\mathbb{N}$ holds (the SeqOfIFGT2 $k)(n) = n + k$.

Let k be an element of $\mathbb{N}$. The SeqOfIFGT3 k yields a sequence of $\mathbb{N}$ and is defined as follows:

(Def. 4) For every element n of $\mathbb{N}$ holds (the SeqOfIFGT3 $k)(n) = (n > k \to 0, 1)$.

Let n_1, n_2 be elements of $\mathbb{N}$. The functor SeqOfIFGT4(n_1, n_2) yielding a sequence of $\mathbb{N}$ is defined as follows:

(Def. 5) For every element n of $\mathbb{N}$ holds (SeqOfIFGT4$(n_1, n_2))(n) = (n > n_1 + 1 \to n + n_2, n)$.

Let n_1, n_2 be elements of $\mathbb{N}$. One can verify that SeqOfIFGT1(n_1, n_2) is one-to-one and SeqOfIFGT4(n_1, n_2) is one-to-one.

Let n be an element of $\mathbb{N}$. Observe that the SeqOfIFGT2 n is one-to-one.

Let X be a set, let s be an element of $\mathbb{N}$, and let A be a sequence of subsets of X. The functor ShiftSeq(A, s) yielding a sequence of subsets of X is defined by:

(Def. 6) ShiftSeq$(A, s) = A \uparrow s$.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let s be an element of $\mathbb{N}$, and let A be a sequence of subsets of S_1. The functor @ShiftSeq(A, s) yields a sequence of subsets of S_1 and is defined by:

(Def. 7) @ShiftSeq$(A, s) =$ ShiftSeq(A, s).

Next we state the proposition

(5)(i) For all sequences A, B of subsets of S_1 such that $n > n_1$ and $B = A \cdot$ SeqOfIFGT1(n_1, n_2) holds (the partial product of $P_1 \cdot B)(n) =$ (the partial product of $P_1 \cdot A)(n_1) \cdot$ (the partial product of $P_1 \cdot$ @ShiftSeq$(A, n_1 + n_2 + 1))(n - n_1 - 1)$, and

(ii) for all sequences A, B, C of subsets of S_1 and for every sequence e of $\mathbb{N}$ such that $n > n_1$ and $C = A \cdot e$ and $B = C \cdot$ SeqOfIFGT1(n_1, n_2) holds (the partial Intersection of $B)(n) =$ (the partial Intersection of $C)(n_1) \cap$ (the partial Intersection of @ShiftSeq$(C, n_1 + n_2 + 1))(n - n_1 - 1)$.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let P_1 be a probability on S_1, and let A be a sequence of subsets of S_1. We say that A is all independent w.r.t. P_1 if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let B be a sequence of subsets of S_1. Given a sequence e of $\mathbb{N}$ such that e is one-to-one and for every element n of $\mathbb{N}$ holds $A(e(n)) = B(n)$. Let n be an element of $\mathbb{N}$. Then (the partial product of $P_1 \cdot B)(n) = P_1(($the partial Intersection of $B)(n))$.

The following propositions are true:

(6) Suppose $n > n_1$ and A is all independent w.r.t. P_1. Then $P_1(($the partial Intersection of $A^{\mathbf{c}})(n_1) \cap ($the partial Intersection of @ShiftSeq$(A, n_1+n_2+1))(n-n_1-1)) = ($the partial product of $P_1 \cdot A^{\mathbf{c}})(n_1) \cdot ($the partial product of $P_1 \cdot$ @ShiftSeq$(A, n_1+n_2+1))(n-n_1-1)$.

(7) (The partial Intersection of $A^{\mathbf{c}})(n) = ($the partial Union of $A)(n)^{\mathrm{c}}$.

(8) $P_1(($the partial Intersection of $A^{\mathbf{c}})(n)) = 1 - P_1(($the partial Union of $A)(n))$.

Let X be a set and let A be a sequence of subsets of X. The UnionShiftSeq A yielding a sequence of subsets of X is defined as follows:

(Def. 9) For every element n of $\mathbb{N}$ holds (the UnionShiftSeq $A)(n) = \bigcup$ShiftSeq(A, n).

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @UnionShiftSeq A yields a sequence of subsets of S_1 and is defined as follows:

(Def. 10) The @UnionShiftSeq $A =$ the UnionShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @lim sup A yielding an event of S_1 is defined as follows:

(Def. 11) The @lim sup $A = \bigcap$(the @UnionShiftSeq A).

Let X be a set and let A be a sequence of subsets of X. The IntersectShiftSeq A yields a sequence of subsets of X and is defined as follows:

(Def. 12) For every element n of $\mathbb{N}$ holds (the IntersectShiftSeq $A)(n) =$ Intersection ShiftSeq(A, n).

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @IntersectShiftSeq A yielding a sequence of subsets of S_1 is defined as follows:

(Def. 13) The @IntersectShiftSeq $A =$ the IntersectShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @lim inf A yielding an event of S_1 is defined by:

(Def. 14) The @lim inf $A = \bigcup$(the @IntersectShiftSeq A).

The following propositions are true:

(9) (The @IntersectShiftSeq $A^{\mathbf{c}})(n) = ($the @UnionShiftSeq $A)(n)^{\mathrm{c}}$.

(10) Suppose A is all independent w.r.t. P_1. Then P_1((the partial Intersection of $A^{\mathbf{c}}$)(n)) = (the partial product of $P_1 \cdot A^{\mathbf{c}}$)(n).

(11) Let X be a set and A be a sequence of subsets of X. Then

(i) the superior setsequence A = the UnionShiftSeq A, and

(ii) the inferior setsequence A = the IntersectShiftSeq A.

(12)(i) The superior setsequence A = the @UnionShiftSeq A, and

(ii) the inferior setsequence A = the @IntersectShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let P_1 be a probability on S_1, and let A be a sequence of subsets of S_1. The functor SumShiftSeq(P_1, A) yields a sequence of real numbers and is defined by:

(Def. 15) For every element n of $\mathbb{N}$ holds (SumShiftSeq(P_1, A))(n) = $\sum(P_1 \cdot$ @ShiftSeq$(A, n))$.

We now state several propositions:

(13) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent, then P_1(the @lim sup A) = 0 and lim SumShiftSeq$(P_1, A) = 0$ and SumShiftSeq(P_1, A) is convergent.

(14)(i) For every set X and for every sequence A of subsets of X and for every element n of $\mathbb{N}$ and for every set x holds there exists an element k of $\mathbb{N}$ such that $x \in$ (ShiftSeq(A, n))(k) iff there exists an element k of $\mathbb{N}$ such that $k \geq n$ and $x \in A(k)$,

(ii) for every set X and for every sequence A of subsets of X and for every set x holds $x \in$ Intersection (the UnionShiftSeq A) iff for every element m of $\mathbb{N}$ there exists an element n of $\mathbb{N}$ such that $n \geq m$ and $x \in A(n)$,

(iii) for every sequence A of subsets of S_1 and for every set x holds $x \in \bigcap$(the @UnionShiftSeq A) iff for every element m of $\mathbb{N}$ there exists an element n of $\mathbb{N}$ such that $n \geq m$ and $x \in A(n)$,

(iv) for every set X and for every sequence A of subsets of X and for every set x holds $x \in \bigcup$(the IntersectShiftSeq A) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \in A(k)$,

(v) for every sequence A of subsets of S_1 and for every set x holds $x \in \bigcup$(the @IntersectShiftSeq A) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \in A(k)$, and

(vi) for every sequence A of subsets of S_1 and for every element x of O_1 holds $x \in \bigcup$(the @IntersectShiftSeq $A^{\mathbf{c}}$) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \notin A(k)$.

(15)(i) $\limsup A$ = the @lim sup A,

(ii) $\liminf A$ = the @lim inf A,

(iii) the @lim inf $A^{\mathbf{c}}$ = (the @lim sup $A)^{\mathrm{c}}$,

(iv) P_1(the @lim inf $A^{\mathbf{c}}$) + P_1(the @lim sup A) = 1, and

(v) $P_1(\liminf(A^{\mathbf{c}})) + P_1(\limsup A) = 1$.

(16)(i) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent, then $P_1(\limsup A) = 0$ and $P_1(\liminf(A^{\mathbf{c}})) = 1$, and

(ii) if A is all independent w.r.t. P_1 and $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is divergent to $+\infty$, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ and $P_1(\limsup A) = 1$.

(17) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is not convergent and A is all independent w.r.t. P_1, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ and $P_1(\limsup A) = 1$.

(18) If A is all independent w.r.t. P_1, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ or $P_1(\liminf(A^{\mathbf{c}})) = 1$ but $P_1(\limsup A) = 0$ or $P_1(\limsup A) = 1$.

(19) $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot @\mathrm{ShiftSeq}(A, n_1+1))(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}(n_1 + 1 + n) - (\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}(n_1)$.

(20) P_1((the @IntersectShiftSeq $A^{\mathbf{c}})(n)) = 1 - P_1$((the @UnionShiftSeq $A)(n))$.

(21)(i) If $A^{\mathbf{c}}$ is all independent w.r.t. P_1, then P_1((the partial Intersection of $A)(n))$ = (the partial product of $P_1 \cdot A)(n)$, and

(ii) if A is all independent w.r.t. P_1, then $1 - P_1$((the partial Union of $A)(n))$ = (the partial product of $P_1 \cdot A^{\mathbf{c}})(n)$.

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More on the Continuity of Real Functions[1]

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Summary. In this article we demonstrate basic properties of the continuous functions from $\mathbb{R}$ to $\mathcal{R}^n$ which correspond to state space equations in control engineering.

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The terminology and notation used here have been introduced in the following articles: [3], [7], [17], [2], [4], [12], [13], [14], [16], [1], [5], [9], [15], [18], [10], [8], [20], [21], [19], [11], [22], and [6].

For simplicity, we use the following convention: n, i denote elements of $\mathbb{N}$, X, X_1 denote sets, r, p, s, x_0, x_1, x_2 denote real numbers, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathcal{R}^n$, and h denotes a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let us consider n, f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 1) There exists a partial function g from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $f = g$ and g is continuous in x_0.

We now state four propositions:

(1) If $h = f$, then f is continuous in x_0 iff h is continuous in x_0.

(2) If $x_0 \in X$ and f is continuous in x_0, then $f{\restriction}X$ is continuous in x_0.

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(3) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(4) Let r be a real number, z be an element of $\mathcal{R}^n$, and w be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $z = w$. Then $\{y \in \mathcal{R}^n\colon |y - z| < r\} = \{y; y$ ranges over points of $\langle \mathcal{E}^n, \|\cdot\| \rangle\colon \|y - w\| < r\}$.

Let n be an element of $\mathbb{N}$, let Z be a set, and let f be a partial function from Z to $\mathcal{R}^n$. The functor $|f|$ yielding a partial function from Z to $\mathbb{R}$ is defined by:

(Def. 2) $\operatorname{dom} |f| = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom} |f|$ holds $|f|_x = |f_x|$.

Let n be an element of $\mathbb{N}$, let Z be a non empty set, and let f be a partial function from Z to $\mathcal{R}^n$. The functor $-f$ yields a partial function from Z to $\mathcal{R}^n$ and is defined by:

(Def. 3) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every set c such that $c \in \operatorname{dom}(-f)$ holds $(-f)_c = -f_c$.

One can prove the following propositions:

(5) Let f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1, g_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(6) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and a be a real number. If $f_1 = g_1$, then $a \cdot f_1 = a \cdot g_1$.

(7) For every partial function f_1 from $\mathbb{R}$ to $\mathcal{R}^n$ holds $(-1) \cdot f_1 = -f_1$.

(8) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $-f_1 = -g_1$.

(9) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $\|f_1\| = |g_1|$.

(10) Let f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1, g_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(11) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset N_1 of $\mathcal{R}^n$ such that there exists a real number r such that $0 < r$ and $\{y \in \mathcal{R}^n\colon |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.

(12) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset N_1 of $\mathcal{R}^n$ such that there exists a real number r such that $0 < r$ and $\{y \in \mathcal{R}^n: |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_1$.

(13) If there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(14) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0.

(15) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 - f_2$ is continuous in x_0.

(16) If f is continuous in x_0, then $r \cdot f$ is continuous in x_0.

(17) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $|f|$ is continuous in x_0.

(18) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $-f$ is continuous in x_0.

(19) Let S be a real normed space, z be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, f_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and f_2 be a partial function from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to the carrier of S. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and $z = (f_1)_{x_0}$ and f_2 is continuous in z. Then $f_2 \cdot f_1$ is continuous in x_0.

(20) Let S be a real normed space, f_1 be a partial function from $\mathbb{R}$ to the carrier of S, and f_2 be a partial function from the carrier of S to $\mathbb{R}$. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider n, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x_0 be an element of $\mathcal{R}^n$. We say that f is continuous in x_0 if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a point y_0 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a partial function g from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$ such that $x_0 = y_0$ and $f = g$ and g is continuous in y_0.

One can prove the following two propositions:

(21) Let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, h be a partial function from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$, x_0 be an element of $\mathcal{R}^n$, and y_0 be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = h$ and $x_0 = y_0$. Then f is continuous in x_0 if and only if h is continuous in y_0.

(22) Let f_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and f_2 be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider n, f. We say that f is continuous if and only if:

(Def. 5) For every x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0.

One can prove the following propositions:

(23) Let g be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $g = f$, then g is continuous iff f is continuous.

(24) Suppose $X \subseteq \operatorname{dom} f$. Then $f{\restriction}X$ is continuous if and only if for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

Let us consider n. Observe that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also continuous.

Let us consider n. Observe that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is continuous.

Let us consider n, let f be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let X be a set. One can verify that $f{\restriction}X$ is continuous.

One can prove the following proposition

(25) If $f{\restriction}X$ is continuous and $X_1 \subseteq X$, then $f{\restriction}X_1$ is continuous.

Let us consider n. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also continuous.

Let us consider n, f and let X be a trivial set. One can verify that $f{\restriction}X$ is continuous.

Let us consider n and let f_1, f_2 be continuous partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. One can check that $f_1 + f_2$ is continuous.

The following propositions are true:

(26) If $X \subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1{\restriction}X$ is continuous and $f_2{\restriction}X$ is continuous, then $(f_1 + f_2){\restriction}X$ is continuous and $(f_1 - f_2){\restriction}X$ is continuous.

(27) If $X \subseteq \operatorname{dom} f_1$ and $X_1 \subseteq \operatorname{dom} f_2$ and $f_1{\restriction}X$ is continuous and $f_2{\restriction}X_1$ is continuous, then $(f_1 + f_2){\restriction}(X \cap X_1)$ is continuous and $(f_1 - f_2){\restriction}(X \cap X_1)$ is continuous.

Let us consider n, let f be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider r. Observe that $r \cdot f$ is continuous.

The following propositions are true:

(28) If $X \subseteq \operatorname{dom} f$ and $f{\restriction}X$ is continuous, then $(r \cdot f){\restriction}X$ is continuous.

(29) If $X \subseteq \operatorname{dom} f$ and $f{\restriction}X$ is continuous, then $|f|{\restriction}X$ is continuous and $(-f){\restriction}X$ is continuous.

(30) If f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then $f{\restriction}\mathbb{R}$ is continuous.

(31) For every subset Y of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $\operatorname{dom} f$ is compact and $f{\restriction}\operatorname{dom} f$ is continuous and $Y = \operatorname{rng} f$ holds Y is compact.

(32) Let Y be a subset of $\mathbb{R}$ and Z be a subset of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $Y \subseteq \operatorname{dom} f$ and $Z = f^{\circ}Y$ and Y is compact and $f{\restriction}Y$ is continuous. Then Z is compact.

Let us consider n, f. We say that f is Lipschitzian if and only if:

(Def. 6) There exists a partial function g from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $g = f$ and g is Lipschitzian.

The following propositions are true:

(33) f is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

(34) If $f = h$, then f is Lipschitzian iff h is Lipschitzian.

(35) $f \restriction X$ is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f \restriction X)$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

Let us consider n. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also Lipschitzian.

Let us consider n. Note that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty.

Let us consider n, let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let X be a set. Note that $f \restriction X$ is Lipschitzian.

We now state the proposition

(36) If $f \restriction X$ is Lipschitzian and $X_1 \subseteq X$, then $f \restriction X_1$ is Lipschitzian.

Let us consider n and let f_1, f_2 be Lipschitzian partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

We now state two propositions:

(37) If $f_1 \restriction X$ is Lipschitzian and $f_2 \restriction X_1$ is Lipschitzian, then $(f_1 + f_2) \restriction (X \cap X_1)$ is Lipschitzian.

(38) If $f_1 \restriction X$ is Lipschitzian and $f_2 \restriction X_1$ is Lipschitzian, then $(f_1 - f_2) \restriction (X \cap X_1)$ is Lipschitzian.

Let us consider n, let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider p. Observe that $p \cdot f$ is Lipschitzian.

Next we state the proposition

(39) If $f \restriction X$ is Lipschitzian and $X \subseteq \operatorname{dom} f$, then $(p \cdot f) \restriction X$ is Lipschitzian.

Let us consider n and let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $|f|$ is Lipschitzian.

Next we state the proposition

(40) If $f \restriction X$ is Lipschitzian, then $-f \restriction X$ is Lipschitzian and $|f| \restriction X$ is Lipschitzian and $(-f) \restriction X$ is Lipschitzian.

Let us consider n. One can check that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also Lipschitzian.

Let us consider n. One can verify that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is Lipschitzian is also continuous.

The following propositions are true:

(41) For all elements r, p of $\mathcal{R}^n$ such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f \restriction X$ is continuous.

(42) For every element x_0 of $\mathcal{R}^n$ such that $1 \leq i \leq n$ holds $\mathrm{proj}(i,n)$ is continuous in x_0.

(43) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then h is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \mathrm{dom}\, h$, and

(ii) for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i,n) \cdot h$ is continuous in x_0.

(44) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then h is continuous if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i,n) \cdot h$ is continuous.

(45) For every point x_0 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i,n)$ is continuous in x_0.

(46) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous in x_0 if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i,n) \cdot h$ is continuous in x_0.

(47) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i,n) \cdot h$ is continuous.

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Representation Theorem for Stacks

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Summary. In the paper the concept of stacks is formalized. As the main result the Theorem of Representation for Stacks is given. Formalization is done according to [13].

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The papers [6], [15], [14], [2], [4], [7], [16], [8], [9], [10], [5], [1], [17], [11], [19], [21], [20], [3], [18], and [12] provide the terminology and notation for this paper.

1. Introductions

In this paper i is a natural number and x is a set.

Let A be a set and let s_1, s_2 be finite sequences of elements of A. Then $s_1 \frown s_2$ is an element of A^*.

Let A be a set, let i be a natural number, and let s be a finite sequence of elements of A. Then $s_{\restriction i}$ is an element of A^*.

The following two propositions are true:

(1) $\emptyset_{\restriction i} = \emptyset$.

(2) Let D be a non empty set and s be a finite sequence of elements of D. Suppose $s \neq \emptyset$. Then there exists a finite sequence w of elements of D and there exists an element n of D such that $s = \langle n \rangle \frown w$.

The scheme *IndSeqD* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every finite sequence p of elements of $\mathcal{A}$ holds $\mathcal{P}[p]$

provided the following conditions are met:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$, and

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- For every finite sequence p of elements of $\mathcal{A}$ and for every element x of $\mathcal{A}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[\langle x\rangle \frown p]$.

Let C, D be non empty sets and let R be a binary relation. A function from $C \times D$ into D is said to be a binary operation of C and D being congruence w.r.t. R if:

(Def. 1) For every element x of C and for all elements y_1, y_2 of D such that $\langle y_1, y_2\rangle \in R$ holds $\langle \mathrm{it}(x, y_1), \mathrm{it}(x, y_2)\rangle \in R$.

The scheme *LambdaD2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

There exists a function M from $\mathcal{A} \times \mathcal{B}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

Let C, D be non empty sets, let R be an equivalence relation of D, and let b be a function from $C \times D$ into D. Let us assume that b is a binary operation of C and D being congruence w.r.t. R. The functor $b_{/R}$ yielding a function from $C \times \mathrm{Classes}\, R$ into $\mathrm{Classes}\, R$ is defined as follows:

(Def. 2) For all sets x, y, y_1 such that $x \in C$ and $y \in \mathrm{Classes}\, R$ and $y_1 \in y$ holds $b_{/R}(x, y) = [b(x, y_1)]_R$.

Let A, B be non empty sets, let C be a subset of A, let D be a subset of B, let f be a function from A into B, and let g be a function from C into D. Then $f{+}\!\cdot g$ is a function from A into B.

2. Stack Algebra

We introduce stack systems which are extensions of 2-sorted and are systems

⟨ a carrier, a carrier', empty stacks, a push function, a pop function, a top function ⟩,

where the carrier is a set, the carrier' is a set, the empty stacks constitute a subset of the carrier', the push function is a function from the carrier×the carrier' into the carrier', the pop function is a function from the carrier' into the carrier', and the top function is a function from the carrier' into the carrier.

Let a_1 be a non empty set, let a_2 be a set, let a_3 be a subset of a_2, let a_4 be a function from $a_1 \times a_2$ into a_2, let a_5 be a function from a_2 into a_2, and let a_6 be a function from a_2 into a_1. Observe that stack system$\langle a_1, a_2, a_3, a_4, a_5, a_6\rangle$ is non empty.

Let a_1 be a set, let a_2 be a non empty set, let a_3 be a subset of a_2, let a_4 be a function from $a_1 \times a_2$ into a_2, let a_5 be a function from a_2 into a_2, and let a_6 be a function from a_2 into a_1. One can verify that stack system$\langle a_1, a_2, a_3, a_4, a_5, a_6\rangle$ is non void.

Let us note that there exists a stack system which is non empty, non void, and strict.

Let X be a stack system. A stack of X is an element of the carrier' of X.

Let X be a non empty non void stack system and let s be a stack of X. The predicate empty(s) is defined by:

(Def. 3) $s \in$ the empty stacks of X.

The functor pop s yields a stack of X and is defined by:

(Def. 4) $\mathrm{pop}\, s =$ (the pop function of X)(s).

The functor top s yields an element of X and is defined by:

(Def. 5) $\mathrm{top}\, s =$ (the top function of X)(s).

Let e be an element of X. The functor push(e, s) yields a stack of X and is defined by:

(Def. 6) $\mathrm{push}(e, s) =$ (the push function of X)(e,s).

Let A be a non empty set. Standard stack system over A yielding a non empty non void strict stack system is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of standard stack system over $A = A$,

(ii) the carrier' of standard stack system over $A = A^*$, and

(iii) for every stack s of standard stack system over A holds empty(s) iff s is empty and for every finite sequence g such that $g = s$ holds if not empty(s), then $\mathrm{top}\, s = g(1)$ and $\mathrm{pop}\, s = g_{\restriction 1}$ and if empty(s), then $\mathrm{top}\, s =$ the element of standard stack system over A and $\mathrm{pop}\, s = \emptyset$ and for every element e of standard stack system over A holds $\mathrm{push}(e, s) = \langle e \rangle ^\frown g$.

In the sequel A denotes a non empty set, c denotes an element of standard stack system over A, and m denotes a stack of standard stack system over A.

Let us consider A. Note that every stack of standard stack system over A is relation-like and function-like.

Let us consider A. Observe that every stack of standard stack system over A is finite sequence-like.

We adopt the following convention: X denotes a non empty non void stack system, s, s_1 denote stacks of X, and e, e_1, e_2 denote elements of X.

Let us consider X. We say that X is pop-finite if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let f be a function from $\mathbb{N}$ into the carrier' of X. Then there exists a natural number i and there exists s such that $f(i) = s$ and if not empty(s), then $f(i+1) \neq \mathrm{pop}\, s$.

We say that X is push-pop if and only if:

(Def. 9) If not empty(s), then $s = \mathrm{push}(\mathrm{top}\, s, \mathrm{pop}\, s)$.

We say that X is top-push if and only if:

(Def. 10) $e = \mathrm{top}\, \mathrm{push}(e, s)$.

We say that X is pop-push if and only if:

(Def. 11) $s = \mathrm{pop}\, \mathrm{push}(e, s)$.

We say that X is push-non-empty if and only if:

(Def. 12) not empty(push(e, s)).

Let A be a non empty set. One can verify the following observations:

* standard stack system over A is pop-finite,
* standard stack system over A is push-pop,
* standard stack system over A is top-push,
* standard stack system over A is pop-push, and
* standard stack system over A is push-non-empty.

Let us observe that there exists a non empty non void stack system which is pop-finite, push-pop, top-push, pop-push, push-non-empty, and strict.

A stack algebra is a pop-finite push-pop top-push pop-push push-non-empty non empty non void stack system.

Next we state the proposition

(3) For every non empty non void stack system X such that X is pop-finite there exists a stack s of X such that empty(s).

Let X be a pop-finite non empty non void stack system. Note that the empty stacks of X is non empty.

We now state two propositions:

(4) If X is top-push and pop-push and push(e_1, s_1) = push(e_2, s_2), then $e_1 = e_2$ and $s_1 = s_2$.

(5) If X is push-pop and not empty(s_1) and not empty(s_2) and pop s_1 = pop s_2 and top s_1 = top s_2, then $s_1 = s_2$.

3. Schemes of Induction

Now we present three schemes. The scheme *INDsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are satisfied:

- For every stack s of $\mathcal{A}$ such that empty(s) holds $\mathcal{P}[s]$, and
- For every stack s of $\mathcal{A}$ and for every element e of $\mathcal{A}$ such that $\mathcal{P}[s]$ holds $\mathcal{P}[\text{push}(e, s)]$.

The scheme *EXsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

There exists an element a of $\mathcal{C}$ and there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that

(i) $a = F(\mathcal{B})$,

(ii) for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$, and

(iii) for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$

for all values of the parameters.

The scheme *UNIQsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

Let a_1, a_2 be elements of $\mathcal{C}$. Suppose that

(i) there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that $a_1 = F(\mathcal{B})$ and for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$, and

(ii) there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that $a_2 = F(\mathcal{B})$ and for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$.

Then $a_1 = a_2$

for all values of the parameters.

4. Stack Congruence

We adopt the following rules: X is a stack algebra, s, s_1, s_2, s_3 are stacks of X, and e, e_1, e_2, e_3 are elements of X.

Let us consider X, s. The functor $|s|$ yielding an element of (the carrier of $X)^*$ is defined by the condition (Def. 13).

(Def. 13) There exists a function F from the carrier' of X into (the carrier of $X)^*$ such that $|s| = F(s)$ and for every s_1 such that empty(s_1) holds $F(s_1) = \emptyset$ and for all s_1, e holds $F(\text{push}(e, s_1)) = \langle e \rangle \frown F(s_1)$.

Next we state several propositions:

(6) If empty(s), then $|s| = \emptyset$.

(7) If not empty(s), then $|s| = \langle \text{top}\, s \rangle \frown |\text{pop}\, s|$.

(8) If not empty(s), then $|\text{pop}\, s| = |s|_{\restriction 1}$.

(9) $|\text{push}(e, s)| = \langle e \rangle \frown |s|$.

(10) If not empty(s), then $\text{top}\, s = |s|(1)$.

(11) If $|s| = \emptyset$, then empty(s).

(12) For every stack s of standard stack system over A holds $|s| = s$.

(13) For every element x of (the carrier of $X)^*$ there exists s such that $|s| = x$.

Let us consider X, s_1, s_2. The predicate $s_1 =_G s_2$ is defined as follows:

(Def. 14) $|s_1| = |s_2|$.

Let us notice that the predicate $s_1 =_G s_2$ is reflexive and symmetric.

The following propositions are true:

(14) If $s_1 =_G s_2$ and $s_2 =_G s_3$, then $s_1 =_G s_3$.

(15) If $s_1 =_G s_2$ and empty(s_1), then empty(s_2).

(16) If empty(s_1) and empty(s_2), then $s_1 =_G s_2$.

(17) If $s_1 =_G s_2$, then $\mathrm{push}(e, s_1) =_G \mathrm{push}(e, s_2)$.

(18) If $s_1 =_G s_2$ and not empty(s_1), then $\mathrm{pop}\, s_1 =_G \mathrm{pop}\, s_2$.

(19) If $s_1 =_G s_2$ and not empty(s_1), then $\mathrm{top}\, s_1 = \mathrm{top}\, s_2$.

Let us consider X. We say that X is proper for identity if and only if:

(Def. 15) For all s_1, s_2 such that $s_1 =_G s_2$ holds $s_1 = s_2$.

Let us consider A. Observe that standard stack system over A is proper for identity.

Let us consider X. The functor $==_X$ yields a binary relation on the carrier' of X and is defined as follows:

(Def. 16) $\langle s_1, s_2 \rangle \in ==_X$ iff $s_1 =_G s_2$.

Let us consider X. Observe that $==_X$ is total, symmetric, and transitive.

One can prove the following proposition

(20) If empty(s), then $[s]_{==_X} =$ the empty stacks of X.

Let us consider X, s. The functor coset s yielding a subset of the carrier' of X is defined by the conditions (Def. 17).

(Def. 17)(i) $s \in \mathrm{coset}\, s$,

(ii) for all e, s_1 such that $s_1 \in \mathrm{coset}\, s$ holds $\mathrm{push}(e, s_1) \in \mathrm{coset}\, s$ and if not empty(s_1), then $\mathrm{pop}\, s_1 \in \mathrm{coset}\, s$, and

(iii) for every subset A of the carrier' of X such that $s \in A$ and for all e, s_1 such that $s_1 \in A$ holds $\mathrm{push}(e, s_1) \in A$ and if not empty(s_1), then $\mathrm{pop}\, s_1 \in A$ holds $\mathrm{coset}\, s \subseteq A$.

Next we state three propositions:

(21) If $\mathrm{push}(e, s) \in \mathrm{coset}\, s_1$, then $s \in \mathrm{coset}\, s_1$ and if not empty(s) and $\mathrm{pop}\, s \in \mathrm{coset}\, s_1$, then $s \in \mathrm{coset}\, s_1$.

(22) $s \in \mathrm{coset}\,\mathrm{push}(e, s)$ and if not empty(s), then $s \in \mathrm{coset}\,\mathrm{pop}\, s$.

(23) There exists s_1 such that empty(s_1) and $s_1 \in \mathrm{coset}\, s$.

Let us consider A and let R be a binary relation on A. Note that there exists a reduction sequence w.r.t. R which is A-valued.

Let us consider X. The construction reduction X yielding a binary relation on the carrier' of X is defined as follows:

(Def. 18) $\langle s_1, s_2 \rangle \in$ the construction reduction X iff not empty(s_1) and $s_2 = \mathrm{pop}\, s_1$ or there exists e such that $s_2 = \mathrm{push}(e, s_1)$.

Next we state the proposition

(24) Let R be a binary relation on A and t be a reduction sequence w.r.t. R. Then $t(1) \in A$ if and only if t is A-valued.

The scheme *PathIND* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a binary relation $\mathcal{D}$ on $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{B}]$,
- $\mathcal{D}$ reduces $\mathcal{B}$ to $\mathcal{C}$, and
- For all elements x, y of $\mathcal{A}$ such that $\mathcal{D}$ reduces $\mathcal{B}$ to x and $\langle x, y\rangle \in \mathcal{D}$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

One can prove the following propositions:

(25) For every reduction sequence t w.r.t. the construction reduction X such that $s = t(1)$ holds $\operatorname{rng} t \subseteq \operatorname{coset} s$.

(26) $\operatorname{coset} s = \{s_1 :$ the construction reduction Xreduces s to $s_1\}$.

Let us consider X, s. The functor $\operatorname{core} s$ yields a stack of X and is defined by the conditions (Def. 19).

(Def. 19)(i) empty$(\operatorname{core} s)$, and

(ii) there exists a the carrier' of X-valued reduction sequence t w.r.t. the construction reduction X such that $t(1) = s$ and $t(\operatorname{len} t) = \operatorname{core} s$ and for every i such that $1 \leq i < \operatorname{len} t$ holds not empty(t_i) and $t_{i+1} = \operatorname{pop}(t_i)$.

The following propositions are true:

(27) If empty(s), then $\operatorname{core} s = s$.

(28) $\operatorname{core} \operatorname{push}(e, s) = \operatorname{core} s$.

(29) If not empty(s), then $\operatorname{core} \operatorname{pop} s = \operatorname{core} s$.

(30) $\operatorname{core} s \in \operatorname{coset} s$.

(31) For every element x of (the carrier of $X)^*$ there exists s_1 such that $|s_1| = x$ and $s_1 \in \operatorname{coset} s$.

(32) If $s_1 \in \operatorname{coset} s$, then $\operatorname{core} s_1 = \operatorname{core} s$.

(33) If s_1, $s_2 \in \operatorname{coset} s$ and $|s_1| = |s_2|$, then $s_1 = s_2$.

(34) There exists s such that $\operatorname{coset} s_1 \cap [s_2]_{==_X} = \{s\}$.

5. Quotient Stack System

Let us consider X. The functor $X_{/==}$ yields a strict stack system and is defined by the conditions (Def. 20).

(Def. 20)(i) The carrier of $X_{/==}$ = the carrier of X,

(ii) the carrier' of $X_{/==} = \text{Classes}{==_X}$,

(iii) the empty stacks of $X_{/==}$ = {the empty stacks of X},

(iv) the push function of $X_{/==}$ = (the push function of $X)_{/==_X}$,

(v) the pop function of $X_{/==}$ =
((the pop function of $X)+\cdot\text{id}_{\text{the empty stacks of } X})_{/==_X}$, and

(vi) for every choice function f of Classes $==_X$ holds the top function of $X_{/==}$ = (the top function of X) $\cdot f +\cdot$ (the empty stacks of X, the element of the carrier of X).

Let us consider X. One can verify that $X_{/==}$ is non empty and non void.

The following propositions are true:

(35) For every stack S of $X_{/==}$ there exists s such that $S = [s]_{==_X}$.

(36) $[s]_{==_X}$ is a stack of $X_{/==}$.

(37) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ holds empty(s) iff empty(S).

(38) For every stack S of $X_{/==}$ holds empty(S) iff S = the empty stacks of X.

(39) For every stack S of $X_{/==}$ and for every element E of $X_{/==}$ such that $S = [s]_{==_X}$ and $E = e$ holds push$(e, s) \in$ push(E, S) and $[\text{push}(e, s)]_{==_X} = \text{push}(E, S)$.

(40) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ and not empty(s) holds pop $s \in$ pop S and $[\text{pop}\, s]_{==_X} = \text{pop}\, S$.

(41) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ and not empty(s) holds top S = top s.

Let us consider X. One can verify the following observations:

* $X_{/==}$ is pop-finite,
* $X_{/==}$ is push-pop,
* $X_{/==}$ is top-push,
* $X_{/==}$ is pop-push, and
* $X_{/==}$ is push-non-empty.

Next we state the proposition

(42) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ holds $|S| = |s|$.

Let us consider X. Note that $X_{/==}$ is proper for identity.

Let us note that there exists a stack algebra which is proper for identity.

6. Representation Theorem for Stacks

Let X_1, X_2 be stack algebras and let F, G be functions. We say that F and G form isomorphism between X_1 and X_2 if and only if the conditions (Def. 21) are satisfied.

(Def. 21) dom F = the carrier of X_1 and rng F = the carrier of X_2 and F is one-to-one and dom G = the carrier' of X_1 and rng G = the carrier' of X_2 and G is one-to-one and for every stack s_1 of X_1 and for every stack s_2 of X_2 such that $s_2 = G(s_1)$ holds empty(s_1) iff empty(s_2) and if not empty(s_1), then pop $s_2 = G(\text{pop}\, s_1)$ and top $s_2 = F(\text{top}\, s_1)$ and for every element

e_1 of X_1 and for every element e_2 of X_2 such that $e_2 = F(e_1)$ holds $\mathrm{push}(e_2, s_2) = G(\mathrm{push}(e_1, s_1))$.

We use the following convention: X_1, X_2, X_3 are stack algebras and F, F_1, F_2, G, G_1, G_2 are functions.

The following propositions are true:

(43) $\mathrm{id}_{\text{the carrier of } X}$ and $\mathrm{id}_{\text{the carrier' of } X}$ form isomorphism between X and X.

(44) If F and G form isomorphism between X_1 and X_2, then F^{-1} and G^{-1} form isomorphism between X_2 and X_1.

(45) Suppose F_1 and G_1 form isomorphism between X_1 and X_2 and F_2 and G_2 form isomorphism between X_2 and X_3. Then $F_2 \cdot F_1$ and $G_2 \cdot G_1$ form isomorphism between X_1 and X_3.

(46) Suppose F and G form isomorphism between X_1 and X_2. Let s_1 be a stack of X_1 and s_2 be a stack of X_2. If $s_2 = G(s_1)$, then $|s_2| = F \cdot |s_1|$.

Let X_1, X_2 be stack algebras. We say that X_1 and X_2 are isomorphic if and only if:

(Def. 22) There exist functions F, G such that F and G form isomorphism between X_1 and X_2.

Let us notice that the predicate X_1 and X_2 are isomorphic is reflexive and symmetric.

We now state four propositions:

(47) If X_1 and X_2 are isomorphic and X_2 and X_3 are isomorphic, then X_1 and X_3 are isomorphic.

(48) If X_1 and X_2 are isomorphic and X_1 is proper for identity, then X_2 is proper for identity.

(49) Let X be a proper for identity stack algebra. Then there exists G such that

(i) for every stack s of X holds $G(s) = |s|$, and

(ii) $\mathrm{id}_{\text{the carrier of } X}$ and G form isomorphism between X and standard stack system over the carrier of X.

(50) Let X be a proper for identity stack algebra. Then X and standard stack system over the carrier of X are isomorphic.

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