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Banach Algebra of Bounded Complex-Valued Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let V be a complex algebra. A complex algebra is called a complex subalgebra of V if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the addition of it = (the addition of V) $\restriction$ (the carrier of it),

(iii) the multiplication of it = (the multiplication of V) $\restriction$ (the carrier of it),

(iv) the external multiplication of it = (the external multiplication of V)$\restriction$($\mathbb{C}$ $\times$ the carrier of it),

(v) $1_{\rm it} = 1_V$, and

(vi) $0_{\rm it} = 0_V$.

We now state the proposition

(1) Let X be a non empty set, V be a complex algebra, V_1 be a non empty subset of V, d_1, d_2 be elements of X, A be a binary operation on X, M be a function from $X \times X$ into X, and M_1 be a function from $\mathbb{C} \times X$ into X. Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the multiplication of V) $\restriction$ (V_1) and $M_1 =$ (the external multiplication of V)$\restriction$($\mathbb{C} \times V_1$) and V_1 has inverse. Then $\langle X, M, A, M_1, d_2, d_1\rangle$ is a complex subalgebra of V.

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Let V be a complex algebra. One can check that there exists a complex subalgebra of V which is strict.

Let V be a complex algebra and let V_1 be a subset of V. We say that V_1 is $\mathbb{C}$-additively-linearly-closed if and only if:

(Def. 2) V_1 is add closed and has inverse and for every complex number a and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let V be a complex algebra and let V_1 be a subset of V. Let us assume that V_1 is $\mathbb{C}$-additively-linearly-closed and non empty. The functor $\mathrm{Mult}(V_1, V)$ yielding a function from $\mathbb{C} \times V_1$ into V_1 is defined as follows:

(Def. 3) $\mathrm{Mult}(V_1, V) = (\text{the external multiplication of } V)\restriction(\mathbb{C} \times V_1)$.

Let X be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctions}\, X$ yielding a non empty subset of $\mathrm{CAlgebra}(X)$ is defined by:

(Def. 4) $\mathbb{C}\text{-BoundedFunctions}\, X = \{f : X \to \mathbb{C}\colon f\restriction X \text{ is bounded}\}$.

Let X be a non empty set. Note that $\mathrm{CAlgebra}(X)$ is scalar unital.

Let X be a non empty set. One can verify that $\mathbb{C}\text{-BoundedFunctions}\, X$ is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let V be a complex algebra. Observe that there exists a non empty subset of V which is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let V be a non empty CLS structure. We say that V is scalar-multiplcation-cancelable if and only if:

(Def. 5) For every complex number a and for every element v of V such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following two propositions:

(2) Let V be a complex algebra and V_1 be a $\mathbb{C}$-additively-linearly-closed multiplicatively-closed non empty subset of V. Then $\langle V_1, \mathrm{mult}(V_1, V), \mathrm{Add}(V_1, V), \mathrm{Mult}(V_1, V), \mathrm{One}(V_1, V), \mathrm{Zero}(V_1, V)\rangle$ is a complex subalgebra of V.

(3) Let V be a complex algebra and V_1 be a complex subalgebra of V. Then

(i) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element v_1 of V_1 and for every element v of V and for every complex number a such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and

(v) $0_{(V_1)} = 0_V$.

Let X be a non empty set. The $\mathbb{C}$-algebra of bounded functions of X yielding a complex algebra is defined by:

(Def. 6) The $\mathbb{C}$-algebra of bounded functions of $X =$ $\langle \mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$

$\mathrm{Add}(\mathbb{C}\text{-BoundedFunctions}\,X, \mathrm{CAlgebra}(X))$,
$\mathrm{Mult}(\mathbb{C}\text{-BoundedFunctions}\,X, \mathrm{CAlgebra}(X))$,
$\mathrm{One}(\mathbb{C}\text{-BoundedFunctions}\,X, \mathrm{CAlgebra}(X))$,
$\mathrm{Zero}(\mathbb{C}\text{-BoundedFunctions}\,X, \mathrm{CAlgebra}(X))\rangle$.

One can prove the following proposition

(4) For every non empty set X holds the $\mathbb{C}$-algebra of bounded functions of X is a complex subalgebra of $\mathrm{CAlgebra}(X)$.

Let X be a non empty set. Note that the $\mathbb{C}$-algebra of bounded functions of X is vector distributive and scalar unital.

Next we state several propositions:

(5) Let X be a non empty set, F, G, H be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g, h be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(6) Let X be a non empty set, a be a complex number, F, G be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(7) Let X be a non empty set, F, G, H be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g, h be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(8) For every non empty set X holds $0_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 0$.

(9) For every non empty set X holds $\mathbf{1}_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 1_{\mathbb{C}}$.

Let X be a non empty set and let F be a set. Let us assume that $F \in \mathbb{C}\text{-BoundedFunctions}\,X$. The functor $\mathrm{modetrans}(F, X)$ yields a function from X into $\mathbb{C}$ and is defined by:

(Def. 7) $\mathrm{modetrans}(F, X) = F$ and $\mathrm{modetrans}(F, X) \restriction X$ is bounded.

Let X be a non empty set and let f be a function from X into $\mathbb{C}$. The functor $\mathrm{PreNorms}(f)$ yields a non empty subset of $\mathbb{R}$ and is defined by:

(Def. 8) $\mathrm{PreNorms}(f) = \{|f(x)| : x$ ranges over elements of $X\}$.

We now state two propositions:

(10) For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f \restriction X$ is bounded holds $\mathrm{PreNorms}(f)$ is upper bounded.

(11) Let X be a non empty set and f be a function from X into $\mathbb{C}$. Then $f \restriction X$ is bounded if and only if $\mathrm{PreNorms}(f)$ is upper bounded.

Let X be a non empty set. The functor $\mathbb{C}$-BoundedFunctionsNorm X yields a function from $\mathbb{C}$-BoundedFunctions X into $\mathbb{R}$ and is defined by:

(Def. 9) For every set x such that $x \in$ $\mathbb{C}$-BoundedFunctions X holds $(\mathbb{C}\text{-BoundedFunctionsNorm}\, X)(x) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(x, X))$.

One can prove the following two propositions:

(13)[1] For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f \restriction X$ is bounded holds $\mathrm{modetrans}(f, X) = f$.

(14) For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f \restriction X$ is bounded holds $(\mathbb{C}\text{-BoundedFunctionsNorm}\, X)(f) = \sup \mathrm{PreNorms}(f)$.

Let X be a non empty set. The $\mathbb{C}$-normed algebra of bounded functions of X yielding a normed complex algebra structure is defined by:

(Def. 10) The $\mathbb{C}$-normed algebra of bounded functions of $X =$ $\langle \mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$ $\mathrm{Add}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$ $\mathrm{Mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$ $\mathrm{One}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$ $\mathrm{Zero}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)), \mathbb{C}\text{-BoundedFunctionsNorm}\, X\rangle$.

Let X be a non empty set. One can verify that the $\mathbb{C}$-normed algebra of bounded functions of X is non empty.

Let X be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of X is unital.

We now state a number of propositions:

(15) Let W be a normed complex algebra structure and V be a complex algebra. Suppose $\langle$the carrier of W, the multiplication of W, the addition of W, the external multiplication of W, the one of W, the zero of $W\rangle = V$. Then W is a complex algebra.

(16) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex algebra.

(17) Let X be a non empty set and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X. Then $(\mathrm{Mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)))(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex linear space.

(19) For every non empty set X holds $X \longmapsto 0 = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$.

(20) Let X be a non empty set, x be an element of X, f be a function from X into $\mathbb{C}$, and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X. If $f = F$ and $f \restriction X$ is bounded, then $|f(x)| \leq \|F\|$.

[1]The proposition (12) has been removed.

(21) For every non empty set X and for every point F of the $\mathbb{C}$-normed algebra of bounded functions of X holds $0 \leq \|F\|$.

(22) Let X be a non empty set and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$. Then $0 = \|F\|$.

(23) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(24) Let X be a non empty set, a be a complex number, f, g be functions from X into $\mathbb{C}$, and F, G be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(25) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(26) Let X be a non empty set, a be a complex number, and F, G be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Then

- (i) if $\|F\| = 0$, then $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$,
- (ii) if $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$, then $\|F\| = 0$,
- (iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and
- (iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let X be a non empty set. Note that the $\mathbb{C}$-normed algebra of bounded functions of X is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

We now state two propositions:

(27) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of X holds $h(x) = f(x) - g(x)$.

(28) Let X be a non empty set and s_1 be a sequence of the $\mathbb{C}$-normed algebra of bounded functions of X. If s_1 is CCauchy, then s_1 is convergent.

Let X be a non empty set. Observe that the $\mathbb{C}$-normed algebra of bounded functions of X is complete.

Next we state the proposition

(29) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex Banach algebra.

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