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Elementary Introduction to Stochastic Finance in Discrete Time

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Summary. This article gives an elementary introduction to stochastic finance (in discrete time). A formalization of random variables is given and some elements of Borel sets are considered. Furthermore, special functions (for buying a present portfolio and the value of a portfolio in the future) and some statements about the relation between these functions are introduced. For details see: [8] (p. 185), [7] (pp. 12, 20), [6] (pp. 3–6).

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The notation and terminology used in this paper have been introduced in the following papers: [15], [2], [1], [3], [4], [11], [10], [9], [5], [14], [12], and [13].

We use the following convention: O_1, O_2 are non empty sets, S_1, F are σ-fields of subsets of O_1, and S_2, F_2 are σ-fields of subsets of O_2.

Let a, r be real numbers. We introduce the halfline finance of a and r as a synonym of $[a, r[$. Then the halfline finance of a and r is a subset of $\mathbb{R}$.

We now state two propositions:

(1) For every real number k holds $\mathbb{R} \setminus [k, +\infty[=]-\infty, k[$.

(2) For every real number k holds $\mathbb{R} \setminus]-\infty, k[= [k, +\infty[$.

Let a, b be real numbers. The half open sets of a and b yields a sequence of subsets of $\mathbb{R}$ and is defined by the conditions (Def. 1).

(Def. 1)(i) (The half open sets of a and b)(0) = the halfline finance of a and $b+1$, and

(ii) for every element n of $\mathbb{N}$ holds (the half open sets of a and b)$(n+1)$ = the halfline finance of a and $b + \frac{1}{n+1}$.

A sequence of real numbers is said to be a price function if:

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(Def. 2) It(0) = 1 and for every element n of $\mathbb{N}$ holds it$(n) \geq 0$.

Let p_1, j_1 be sequences of real numbers. We introduce the elements of buy portfolio of p_1 and j_1 as a synonym of $p_1 \cdot j_1$. Then the elements of buy portfolio of p_1 and j_1 is a sequence of real numbers.

Let d be a natural number. The buy portfolio extension of p_1, j_1, and d yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 3) The buy portfolio extension of p_1, j_1, and $d = (\sum_{\alpha=0}^{\kappa}$(the elements of buy portfolio of p_1 and $j_1)(\alpha))_{\kappa \in \mathbb{N}}(d)$.

The buy portfolio of p_1, j_1, and d yielding an element of $\mathbb{R}$ is defined as follows:

(Def. 4) The buy portfolio of p_1, j_1, and $d = (\sum_{\alpha=0}^{\kappa}$((the elements of buy portfolio of p_1 and $j_1) \uparrow 1)(\alpha))_{\kappa \in \mathbb{N}}(d-1)$.

Let O_1, O_2 be sets, let S_1 be a σ-field of subsets of O_1, let S_2 be a σ-field of subsets of O_2, and let X be a function. We say that X is random variable on S_1 and S_2 if and only if:

(Def. 5) For every element x of S_2 holds $\{y \in O_1$: $X(y)$ is an element of $x\}$ is an element of S_1.

Let O_1, O_2 be sets, let F be a σ-field of subsets of O_1, and let F_2 be a σ-field of subsets of O_2. The set of random variables on F and F_2 is defined by:

(Def. 6) The set of random variables on F and $F_2 = \{f : O_1 \to O_2$: f is random variable on F and $F_2\}$.

Let us consider O_1, O_2, F, F_2. One can check that the set of random variables on F and F_2 is non empty.

Let O_1, O_2 be non empty sets, let F be a σ-field of subsets of O_1, let F_2 be a σ-field of subsets of O_2, and let X be a set. Let us assume that $X =$ the set of random variables on F and F_2. Let k be an element of X. The change element to function F, F_2, and k yielding a function from O_1 into O_2 is defined by:

(Def. 7) The change element to function F, F_2, and $k = k$.

Let O_1 be a non empty set, let F be a σ-field of subsets of O_1, let X be a non empty set, and let k be an element of X. The random variables for future elements of portfolio value of F and k yields a function from O_1 into $\mathbb{R}$ and is defined by the condition (Def. 8).

(Def. 8) Let w be an element of O_1. Then (the random variables for future elements of portfolio value of F and $k)(w) =$ (the change element to function F, the Borel sets, and $k)(w)$.

Let p be a natural number, let O_1, O_2 be non empty sets, let F be a σ-field of subsets of O_1, let F_2 be a σ-field of subsets of O_2, and let X be a set. Let us assume that $X =$ the set of random variables on F and F_2. Let G be a function from $\mathbb{N}$ into X. The element of F, F_2, G, and p yields a function from O_1 into O_2 and is defined as follows:

(Def. 9) The element of F, F_2, G, and $p = G(p)$.

Let r be a real number, let O_1 be a non empty set, let F be a σ-field of subsets of O_1, let X be a non empty set, let w be an element of O_1, let G be a function from $\mathbb{N}$ into X, and let p_1 be a sequence of real numbers. The future elements of portfolio value of r, p_1, F, w, and G yields a sequence of real numbers and is defined by the condition (Def. 10).

(Def. 10) Let n be an element of $\mathbb{N}$. Then (the future elements of portfolio value of r, p_1, F, w, and $G)(n) =$ (the random variables for future elements of portfolio value of F and $G(n))(w) \cdot p_1(n)$.

Let r be a real number, let d be a natural number, let p_1 be a sequence of real numbers, let O_1 be a non empty set, let F be a σ-field of subsets of O_1, let X be a non empty set, let G be a function from $\mathbb{N}$ into X, and let w be an element of O_1. The future portfolio value extension of r, d, p_1, F, G, and w yields an element of $\mathbb{R}$ and is defined by the condition (Def. 11).

(Def. 11) The future portfolio value extension of r, d, p_1, F, G, and w $=$ $(\sum_{\alpha=0}^{\kappa}$ (the future elements of portfolio value of r, p_1, F, w, and $G)(\alpha))_{\kappa \in \mathbb{N}}(d)$.

The future portfolio value of r, d, p_1, F, G, and w yields an element of $\mathbb{R}$ and is defined by the condition (Def. 12).

(Def. 12) The future portfolio value of r, d, p_1, F, G, and $w = (\sum_{\alpha=0}^{\kappa}(($the future elements of portfolio value of r, p_1, F, w, and $G) \uparrow 1)(\alpha))_{\kappa \in \mathbb{N}}(d-1)$.

Let us observe that there exists an element of the Borel sets which is non empty.

One can prove the following propositions:

(3) For every real number k holds $[k, +\infty[$ is an element of the Borel sets and $]-\infty, k[$ is an element of the Borel sets.

(4) For all real numbers k_1, k_2 holds $[k_2, k_1[$ is an element of the Borel sets.

(5) For all real numbers a, b holds Intersection (the half open sets of a and b) is an element of the Borel sets.

(6) For all real numbers a, b holds Intersection (the half open sets of a and $b) = [a, b]$.

(7) Let a, b be real numbers and n be a natural number. Then (the partial intersections of the half open sets of a and $b)(n)$ is an element of the Borel sets.

(8) For all real numbers k_1, k_2 holds $[k_2, k_1]$ is an element of the Borel sets.

(9) Let X be a function from O_1 into $\mathbb{R}$. Suppose X is random variable on S_1 and the Borel sets. Then for every real number k holds $\{w \in O_1\colon X(w) \geq k\}$ is an element of S_1 and $\{w \in O_1\colon X(w) < k\}$ is an element of S_1 and for all real numbers k_1, k_2 such that $k_1 < k_2$ holds $\{w \in O_1\colon k_1 \leq X(w) \ \wedge\ X(w) < k_2\}$ is an element of S_1 and for all real numbers k_1, k_2 such that $k_1 \leq k_2$ holds $\{w \in O_1\colon k_1 \leq X(w) \ \wedge\ X(w) \leq k_2\}$ is an

element of S_1 and for every real number r holds LE-dom$(X, r) = \{w \in O_1: X(w) < r\}$ and for every real number r holds GTE-dom$(X, r) = \{w \in O_1: X(w) \geq r\}$ and for every real number r holds EQ-dom$(X, r) = \{w \in O_1: X(w) = r\}$ and for every real number r holds EQ-dom(X, r) is an element of S_1.

(10) For every real number s holds $O_1 \longmapsto s$ is random variable on S_1 and the Borel sets.

(11) Let p_1 be a sequence of real numbers, j_1 be a price function, and d be a natural number. Suppose $d > 0$. Then the buy portfolio extension of p_1, j_1, and $d = p_1(0) +$ the buy portfolio of p_1, j_1, and d.

(12) Let d be a natural number. Suppose $d > 0$. Let r be a real number, p_1 be a sequence of real numbers, and G be a function from $\mathbb{N}$ into the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = O_1 \longmapsto 1 + r$. Let w be an element of O_1. Then the future portfolio value extension of r, d, p_1, F, G, and $w = (1+r) \cdot p_1(0) +$the future portfolio value of r, d, p_1, F, G, and w.

(13) Let d be a natural number. Suppose $d > 0$. Let r be a real number. Suppose $r > -1$. Let p_1 be a sequence of real numbers, j_1 be a price function, and G be a function from $\mathbb{N}$ into the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = O_1 \longmapsto 1 + r$. Let w be an element of O_1. Suppose the buy portfolio extension of p_1, j_1, and $d \leq 0$. Then the future portfolio value extension of r, d, p_1, F, G, and $w \leq$ (the future portfolio value of r, d, p_1, F, G, and w) $- (1 + r) \cdot$ the buy portfolio of p_1, j_1, and d.

(14) Let d be a natural number. Suppose $d > 0$. Let r be a real number. Suppose $r > -1$. Let p_1 be a sequence of real numbers, j_1 be a price function, and G be a function from $\mathbb{N}$ into the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = O_1 \longmapsto 1+r$. Suppose the buy portfolio extension of p_1, j_1, and $d \leq 0$. Then

(i) $\{w \in O_1$: the future portfolio value extension of r, d, p_1, F, G, and $w \geq 0\} \subseteq \{w \in O_1$: the future portfolio value of r, d, p_1, F, G, and $w \geq (1 + r) \cdot$ the buy portfolio of p_1, j_1, and $d\}$, and

(ii) $\{w \in O_1$: the future portfolio value extension of r, d, p_1, F, G, and $w > 0\} \subseteq \{w \in O_1$: the future portfolio value of r, d, p_1, F, G, and $w > (1 + r) \cdot$ the buy portfolio of p_1, j_1, and $d\}$.

(15) Let f be a function from O_1 into $\mathbb{R}$. Suppose f is random variable on S_1 and the Borel sets. Then f is measurable on $\Omega_{(S_1)}$ and f is a real-valued random variable on S_1.

(16) The set of random variables on S_1 and the Borel sets $\subseteq$ the real-valued random variables set on S_1.

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