Introduction to Rational Functions

Christoph Schwarzweller  
Institute of Computer Science  
University of Gdańsk  
Wita Stwosza 57, 80-952 Gdańsk, Poland

Summary. In this article we formalize rational functions as pairs of polynomials and define some basic notions including the degree and evaluation of rational functions [8]. The main goal of the article is to provide properties of rational functions necessary to prove a theorem on the stability of networks.

MML identifier: RATFUNC1, version: 7.12.02 4.181.1147

The notation and terminology used in this paper are introduced in the following articles: [14], [3], [4], [5], [18], [20], [16], [17], [1], [15], [2], [6], [12], [10], [11], [22], [19], [21], [9], [13], [23], and [7].

1. Preliminaries

One can prove the following three propositions:

(1) Let \( L \) be an add-associative right zeroed right complementable right distributive non empty double loop structure, \( a \) be an element of \( L \), and \( p, q \) be finite sequences of elements of \( L \). Suppose \( \text{len} \ p = \text{len} \ q \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom} \ p \) holds \( q_i = a \cdot p_i \). Then \( \sum q = a \cdot \sum p \).

(2) Let \( L \) be an add-associative right zeroed right complementable right distributive non empty double loop structure, \( f \) be a finite sequence of elements of \( L \), and \( i, j \) be elements of \( \mathbb{N} \). If \( i \in \text{dom} \ f \) and \( j = i - 1 \), then \( \text{Ins}(f|_{i\downarrow} ; j, f_i) = f \).

(3) Let \( L \) be an add-associative right zeroed right complementable associative unital right distributive commutative non empty double loop structure, \( f \) be a finite sequence of elements of \( L \), and \( i \) be an element of \( \mathbb{N} \). If \( i \in \text{dom} \ f \), then \( \prod f = f_i \cdot \prod(f|_{\uparrow i}) \).
Let $L$ be an add-associative right zeroed right complementable well unital associative left distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $x, y$ be non zero elements of $L$. Note that $\frac{x}{y}$ is non zero.

Let us note that every add-associative right zeroed right complementable right distributive non empty double loop structure which is integral domain-like is also almost left cancelable and every add-associative right zeroed right complementable left distributive non empty double loop structure which is integral domain-like is also almost right cancelable.

Let $x, y$ be integers. Note that $\max(x, y)$ is integer and $\min(x, y)$ is integer.

One can prove the following proposition

(4) For all integers $x, y, z$ holds $\max(x + y, x + z) = x + \max(y, z)$.

2. More on Polynomials

Let $L$ be a non empty zero structure and let $p$ be a polynomial of $L$. We say that $p$ is zero if and only if:

(Def. 1) \(p = 0_L\).

We say that $p$ is constant if and only if:

(Def. 2) \(\deg p \leq 0\).

Let $L$ be a non trivial zero structure. One can verify that there exists a polynomial of $L$ which is non zero.

Let $L$ be a non empty zero structure. One can verify that $0_L$ is zero and constant.

Let $L$ be a non degenerated multiplicative loop with zero structure. Note that $1_L$ is non zero.

Let $L$ be a non empty multiplicative loop with zero structure. Note that $1_L$ is constant.

Let $L$ be a non empty zero structure. One can verify that every polynomial of $L$ which is zero is also constant. Note that every polynomial of $L$ which is non constant is also non zero.

Let $L$ be a non trivial zero structure. One can verify that there exists a polynomial of $L$ which is non constant.

Let $L$ be a well unital non degenerated non empty double loop structure, let $z$ be an element of $L$, and let $k$ be an element of $\mathbb{N}$. Observe that $r\text{poly}(k, z)$ is non zero.

Let $L$ be an add-associative right zeroed right complementable distributive non degenerated double loop structure. One can check that Polynom-Ring $L$ is non degenerated.
Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure. Observe that Polynom-Ring $L$ is integral domain-like.

Next we state two propositions:

(5) Let $L$ be an add-associative right zeroed right complementable right distributive associative non empty double loop structure, $p, q$ be polynomials of $L$, and $a$ be an element of $L$. Then $(a \cdot p) \ast q = a \cdot (p \ast q)$.

(6) Let $L$ be an add-associative right zeroed right complementable right distributive commutative associative non empty double loop structure, $p, q$ be polynomials of $L$, and $a$ be an element of $L$. Then $p \ast (a \cdot q) = a \cdot (p \ast q)$.

Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, let $p$ be a non zero polynomial of $L$, and let $a$ be a non zero element of $L$. Note that $a \cdot p$ is non zero.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let $p_1, p_2$ be non zero polynomials of $L$. Observe that $p_1 \ast p_2$ is non zero.

One can prove the following proposition

(7) Let $L$ be an add-associative right zeroed right complementable distributive Abelian integral domain-like non trivial double loop structure, $p_1, p_2$ be polynomials of $L$, and $p_3$ be a non zero polynomial of $L$. If $p_1 \ast p_3 = p_2 \ast p_3$, then $p_1 = p_2$.

Let $L$ be a non trivial zero structure and let $p$ be a non zero polynomial of $L$. One can check that degree($p$) is natural.

Next we state several propositions:

(8) Let $L$ be an add-associative right zeroed right complementable unital right distributive non empty double loop structure and $p$ be a polynomial of $L$. If $\deg p = 0$, then for every element $x$ of $L$ holds eval($p, x$) $\neq 0_L$.

(9) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure, $p$ be a polynomial of $L$, and $x$ be an element of $L$. Then eval($p, x$) = $0_L$ if and only if $rpoly(1, x) | p$.

(10) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure, $p, q$ be polynomials of $L$, and $x$ be an element of $L$. If $rpoly(1, x) | p \ast q$, then $rpoly(1, x) | p$ or $rpoly(1, x) | q$.

(11) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure and $f$ be a finite sequence of elements
of Polynom-Ring $L$. Suppose that for every natural number $i$ such that $i \in \text{dom } f$ there exists an element $z$ of $L$ such that $f(i) = \text{rpoly}(1, z)$. Let $p$ be a polynomial of $L$. If $p = \prod f$, then $p \neq 0$. Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure.

(12) Suppose that for every natural number $i$ such that $i \in \text{dom } f$ there exists an element $z$ of $L$ such that $f(i) = \text{rpoly}(1, z)$. Let $p$ be a polynomial of $L$. Suppose $p = \prod f$. Let $x$ be an element of $L$. Then eval($p, x$) = $0_L$ if and only if there exists a natural number $i$ such that $i \in \text{dom } f$ and $f(i) = \text{rpoly}(1, x)$.

3. Common Roots of Polynomials

Let $L$ be a unital non empty double loop structure, let $p_1, p_2$ be polynomials of $L$, and let $x$ be an element of $L$. We say that $x$ is a common root of $p_1$ and $p_2$ if and only if:

(Def. 3) $x$ is a root of $p_1$ and $x$ is a root of $p_2$.

Let $L$ be a unital non empty double loop structure and let $p_1, p_2$ be polynomials of $L$. We say that $p_1$ and $p_2$ have a common root if and only if:

(Def. 4) There exists an element of $L$ which is a common root of $p_1$ and $p_2$.

Let $L$ be a unital non empty double loop structure and let $p_1, p_2$ be polynomials of $L$. We introduce $p_1$ and $p_2$ have common roots as a synonym of $p_1$ and $p_2$ have a common root. We introduce $p_1$ and $p_2$ have no common roots as an antonym of $p_1$ and $p_2$ have a common root.

Next we state several propositions:

(13) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p$ be a polynomial of $L$, and $x$ be an element of $L$. If $x$ is a root of $p$, then $x$ is a root of $-p$.

(14) Let $L$ be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, $p_1, p_2$ be polynomials of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p_1$ and $p_2$, then $x$ is a root of $p_1 + p_2$.

(15) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p_1, p_2$ be polynomials of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p_1$ and $p_2$, then $x$ is a root of $-(p_1 + p_2)$.

(16) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p, q$ be polynomials
of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p$ and $q$, then $x$ is a root of $p + q$.

(17) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non trivial double loop structure and $p_1, p_2$ be polynomials of $L$. If $p_1 | p_2$ and $p_1$ has roots, then $p_1$ and $p_2$ have common roots.

Let $L$ be a unital non empty double loop structure and let $p, q$ be polynomials of $L$. The common roots of $p$ and $q$ yields a subset of $L$ and is defined by:

(Def. 5) The common roots of $p$ and $q = \{ x \in L : x$ is a common root of $p$ and $q \}$.

4. Normalized Polynomials

Let $L$ be a non empty zero structure and let $p$ be a polynomial of $L$. The leading coefficient of $p$ yields an element of $L$ and is defined by:

(Def. 6) The leading coefficient of $p = p(\text{len } p - 1)$.

We introduce $\text{LC}_p$ as a synonym of the leading coefficient of $p$.

Let $L$ be a non trivial double loop structure and let $p$ be a non zero polynomial of $L$. One can check that $\text{LC}_p$ is non zero.

One can prove the following proposition

(18) Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non empty double loop structure, $p$ be a polynomial of $L$, and $a$ be an element of $L$.

Then $\text{LC}(a \cdot p) = a \cdot \text{LC}_p$.

Let $L$ be a non empty double loop structure and let $p$ be a polynomial of $L$. We say that $p$ is normalized if and only if:

(Def. 7) $\text{LC}_p = 1_L$.

Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure and let $p$ be a non zero polynomial of $L$. One can check that $\frac{1_{L}}{\text{LC}_p} \cdot p$ is normalized.

Let $L$ be a field and let $p$ be a non zero polynomial of $L$. One can verify that $\text{NormPolynomial} p$ is normalized.

5. Rational Functions

Let $L$ be a non trivial multiplicative loop with zero structure. Rational function of $L$ is defined by:

(Def. 8) There exists a polynomial $p_1$ of $L$ and there exists a non zero polynomial $p_2$ of $L$ such that $i = \langle p_1, p_2 \rangle$. 
Let $L$ be a non trivial multiplicative loop with zero structure, let $p_1$ be a polynomial of $L$, and let $p_2$ be a non zero polynomial of $L$. Then $\langle p_1, p_2 \rangle$ is a rational function of $L$.

Let $L$ be a non trivial multiplicative loop with zero structure and let $z$ be a rational function of $L$. Then $z_1$ is a polynomial of $L$. Then $z_2$ is a non zero polynomial of $L$.

Let $L$ be a non trivial multiplicative loop with zero structure and let $z$ be a rational function of $L$. We say that $z$ is zero if and only if:

\[ z_1 = 0. L. \]

Let $L$ be a non trivial multiplicative loop with zero structure. One can check that there exists a rational function of $L$ which is non zero.

Next we state the proposition

(19) Let $L$ be a non trivial multiplicative loop with zero structure and $z$ be a rational function of $L$. Then $z = \langle z_1, z_2 \rangle$.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We say that $z$ is irreducible if and only if:

\[ z_1 \text{ and } z_2 \text{ have no common roots.} \]

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We introduce $z$ is reducible as an antonym of $z$ is irreducible.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We say that $z$ is normalized if and only if:

\[ z \text{ is irreducible and } z_2 \text{ is normalized.} \]

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure. Observe that every rational function of $L$ which is normalized is also irreducible.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. Note that $\text{LC}(z_2)$ is non zero.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. The norm rational function of $z$ yields a rational function of $L$ and is defined by:

\[ \text{(Def. 12)} \quad \text{The norm rational function of } \frac{LC(z_2)}{z_11} \cdot z_1, \frac{LC(z_2)}{z_21} \cdot z_2. \]

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral
domain-like non trivial double loop structure and let $z$ be a rational function of $L$. We introduce NormRatF $z$ as a synonym of the norm rational function of $z$.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a non zero rational function of $L$. Observe that the norm rational function of $z$ is non zero.

Let $L$ be a non degenerated multiplicative loop with zero structure. The functor $0.L$ yields a rational function of $L$ and is defined by:

(Def. 13) $0.L = \langle 0.L, 1.L \rangle$.

The functor $1.L$ yields a rational function of $L$ and is defined as follows:

(Def. 14) $1.L = \langle 1.L, 1.L \rangle$.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $0.L$ is normalized.

Let $L$ be a non degenerated multiplicative loop with zero structure. Note that $1.L$ is non zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that $1.L$ is irreducible.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. Observe that there exists a rational function of $L$ which is irreducible and non zero.

Let $L$ be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure and let $x$ be an element of $L$. One can check that $\langle \text{rpoly}(1, x), \text{rpoly}(1, x) \rangle$ is reducible and non zero as a rational function of $L$.

Let $L$ be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure. Observe that there exists a rational function of $L$ which is reducible and non zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that there exists a rational function of $L$ which is normalized.

Let $L$ be a non degenerated multiplicative loop with zero structure. One can verify that $0.L$ is zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $1.L$ is normalized.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let $p$, $q$ be rational functions of $L$. The functor $p + q$ yields a rational function of $L$ and is defined by:
(Def. 15) \[ p + q = \langle p_1 * q_2 + p_2 * q_1, p_2 * q_2 \rangle. \]

Let \( L \) be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let \( p, q \) be rational functions of \( L \). The functor \( p * q \) yielding a rational function of \( L \) is defined by:

(Def. 16) \[ p * q = \langle \langle p_1 * q_1, p_2 * q_2 \rangle \rangle. \]

One can prove the following proposition

(20) Let \( L \) be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, \( p \) be a rational function of \( L \), and \( a \) be a non zero element of \( L \). Then \( \langle a * p_1, a * p_2 \rangle \) is irreducible if and only if \( p \) is irreducible.

6. Normalized Rational Functions

We now state the proposition

(21) Let \( L \) be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative integral domain-like non trivial double loop structure and \( z \) be a rational function of \( L \). Then there exists a rational function \( z_1 \) of \( L \) and there exists a non zero polynomial \( z_2 \) of \( L \) such that

(i) \[ z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle, \]

(ii) \( z_1 \) is irreducible, and

(iii) there exists a finite sequence \( f \) of elements of Polynom-Ring \( L \) such that \( z_2 = \prod f \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom } f \) there exists an element \( x \) of \( L \) such that \( x \) is a common root of \( z_1 \) and \( z_2 \) and \( f(i) = rpoly(1, x) \).

Let \( L \) be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let \( z \) be a rational function of \( L \). The functor \( NF z \) yielding a rational function of \( L \) is defined by:

(Def. 17)(i) There exists a rational function \( z_1 \) of \( L \) and there exists a non zero polynomial \( z_2 \) of \( L \) such that \( z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle \) and \( z_1 \) is irreducible and \( NF z = \) the norm rational function of \( z_1 \) and there exists a finite sequence \( f \) of elements of Polynom-Ring \( L \) such that \( z_2 = \prod f \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom } f \) there exists an element \( x \) of \( L \) such that \( x \) is a common root of \( z_1 \) and \( z_2 \) and \( f(i) = rpoly(1, x) \) if \( z \) is non zero,

(ii) \( NF z = 0. L, \) otherwise.

Let \( L \) be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral
Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a non zero rational function of $L$. One can verify that $NF z$ is non zero.

One can prove the following propositions:

(22) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, $z$ be a non zero rational function of $L$, $z_1$ be a rational function of $L$, and $z_2$ be a non zero polynomial of $L$. Suppose that

(i) $z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle$,
(ii) $z_1$ is irreducible, and
(iii) there exists a finite sequence $f$ of elements of Polynom-Ring $L$ such that $z_2 = \prod f$ and for every element $i$ of $\mathbb{N}$ such that $i \in \text{dom } f$ there exists an element $x$ of $L$ such that $x$ is a common root of $z_1$ and $z_2$ and $f(i) = \text{rpoly}(1, x)$.

Then $NF z = \text{the norm rational function of } z_1$.

(23) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $NF 0 \cdot L = 0 \cdot L$.

(24) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $NF 1 \cdot L = 1 \cdot L$.

(25) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be an irreducible non zero rational function of $L$. Then $NF z = \text{the norm rational function of } z$.

(26) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, $z$ be a rational function of $L$, and $x$ be an element of $L$. Then $NF \langle \text{rpoly}(1, x) * z_1, \text{rpoly}(1, x) * z_2 \rangle = NF z$.

(27) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be a rational function of $L$. Then $NF NF z = NF z$.

(28) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible in-
tegal domain-like non degenerated double loop structure and \( z \) be a non zero rational function of \( L \). Then \( z \) is irreducible if and only if there exists an element \( a \) of \( L \) such that \( a \neq 0 \) and \( \langle a \cdot z_1, a \cdot z_2 \rangle = \text{NF} \, z \).

### 7. Degree of Rational Functions

Let \( L \) be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let \( z \) be a rational function of \( L \). The functor \( \text{degree}(z) \) yielding an integer is defined as follows:

\[
\text{(Def. 18)} \quad \text{degree}(z) = \max(\text{degree}((\text{NF} \, z)_1), \text{degree}((\text{NF} \, z)_2)).
\]

Let \( L \) be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let \( z \) be a rational function of \( L \). We introduce \( \text{deg} \, z \) as a synonym of \( \text{degree}(z) \).

Next we state two propositions:

\[
\begin{align*}
(29) \quad & \text{Let } L \text{ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and } z \text{ be a rational function of } L. \text{ Then } \text{degree}(z) \leq \max(\text{degree}(z_1), \text{degree}(z_2)). \\
(30) \quad & \text{Let } L \text{ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and } z \text{ be a non zero rational function of } L. \text{ Then } z \text{ is irreducible if and only if } \text{degree}(z) = \max(\text{degree}(z_1), \text{degree}(z_2)).
\end{align*}
\]

### 8. Evaluation of Rational Functions

Let \( L \) be a field, let \( z \) be a rational function of \( L \), and let \( x \) be an element of \( L \). The functor \( \text{eval}(z, x) \) yielding an element of \( L \) is defined by:

\[
\text{(Def. 19)} \quad \text{eval}(z, x) = \frac{\text{eval}(z_1, x)}{\text{eval}(z_2, x)}.
\]

The following propositions are true:

\[
\begin{align*}
(31) \quad & \text{For every field } L \text{ and for every element } x \text{ of } L \text{ holds } \text{eval}(0, L, x) = 0_L. \\
(32) \quad & \text{For every field } L \text{ and for every element } x \text{ of } L \text{ holds } \text{eval}(1, L, x) = 1_L. \\
(33) \quad & \text{Let } L \text{ be a field, } p, q \text{ be rational functions of } L, \text{ and } x \text{ be an element of } L. \text{ If } \text{eval}(p_2, x) \neq 0_L \text{ and } \text{eval}(q_2, x) \neq 0_L, \text{ then } \text{eval}(p + q, x) = \text{eval}(p, x) + \text{eval}(q, x). \\
(34) \quad & \text{Let } L \text{ be a field, } p, q \text{ be rational functions of } L, \text{ and } x \text{ be an element of } L. \text{ If } \text{eval}(p_2, x) \neq 0_L \text{ and } \text{eval}(q_2, x) \neq 0_L, \text{ then } \text{eval}(p \cdot q, x) = \text{eval}(p, x) \cdot \text{eval}(q, x).
\end{align*}
\]
Let $L$ be a field, $p$ be a rational function of $L$, and $x$ be an element of $L$. If $\text{eval}(p_2, x) \neq 0_L$, then $\text{eval}(\text{the norm rational function of } p, x) = \text{eval}(p, x)$.

Let $L$ be a field, $p$ be a rational function of $L$, and $x$ be an element of $L$. If $\text{eval}(p_2, x) \neq 0_L$, then $x$ is a common root of $p_1$ and $p_2$ or $\text{eval}($\text{NF } p, x) = \text{eval}(p, x)$.

References


Received February 8, 2012