FORMALIZED MATHEMATICS
Vol. 20, No. 3, Pages 193–197, 2012
DOI: 10.2478/v10037-012-0022-0

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# Transition of Consistency and Satisfiability under Language Extensions[1]

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**Summary.** This article is the first in a series of two Mizar articles constituting a formal proof of the Gödel Completeness theorem [17] for uncountably large languages. We follow the proof given in [18]. The present article contains the techniques required to expand formal languages. We prove that consistent or satisfiable theories retain these properties under changes to the language they are formulated in.

MML identifier: `QC_TRANS`, version: `7.14.01 4.183.1153`

The notation and terminology used in this paper have been introduced in the following papers: [8], [1], [2], [11], [16], [4], [15], [12], [13], [7], [6], [22], [3], [19], [23], [24], [5], [20], [9], [10], [21], and [14].

[1]This article is part of the first author's Bachelor thesis under the supervision of the second author.




ISSN 1426–2630(p), 1898-9934(e)

## 1. Language Extensions

For simplicity, we adopt the following rules: $A_1$ denotes an alphabet, $P_1$ denotes a consistent subset of CQC-WFF $A_1$, $p$, $r$ denote elements of CQC-WFF $A_1$, $A$ denotes a non empty set, $J$ denotes an interpretation of $A_1$ and $A$, $v$ denotes an element of the valuations in $A_1$ and $A$, $k$ denotes a natural number, $l$ denotes a CQC-variable list of $k$ and $A_1$, $P$ denotes a predicate symbol of $k$ and $A_1$, and $x$, $y$ denote bound variables of $A_1$.

Let us consider $A_1$ and let $A_2$ be an alphabet. We say that $A_2$ is $A_1$-expanding if and only if:

(Def. 1) $A_1 \subseteq A_2$.

Let us consider $A_1$. Note that there exists an alphabet which is $A_1$-expanding.

Let $A_3$, $A_4$ be countable alphabets. One can check that there exists an alphabet which is countable, $A_3$-expanding, and $A_4$-expanding.

Let $A_1$, $A_4$ be alphabets and let $P$ be a subset of CQC-WFF $A_1$. We say that $P$ is $A_4$-consistent if and only if:

(Def. 2) For every subset $S$ of CQC-WFF $A_4$ such that $P = S$ holds $S$ is consistent.

Let us consider $A_1$. One can check that there exists a subset of CQC-WFF $A_1$ which is non empty and consistent.

Let us consider $A_1$. One can check that every subset of CQC-WFF $A_1$ which is consistent is also $A_1$-consistent and every subset of CQC-WFF $A_1$ which is $A_1$-consistent is also consistent.

For simplicity, we follow the rules: $A_4$ is an $A_1$-expanding alphabet, $J_2$ is an interpretation of $A_4$ and $A$, $J_1$ is an interpretation of $A_1$ and $A$, $v_2$ is an element of the valuations in $A_4$ and $A$, and $v_1$ is an element of the valuations in $A_1$ and $A$.

Next we state several propositions:

(1) $\text{Arity}(P) = \text{len}\, l$.

(2) $\text{Symb}\, A_1 \subseteq \text{Symb}\, A_4$.

(3) The predicate symbols of $A_1 \subseteq$ the predicate symbols of $A_4$.

(4) The bound variables of $A_1 \subseteq$ the bound variables of $A_4$.

(5) For every $k$ holds every $l$ is a CQC-variable list of $k$ and $A_4$.

(6) $P$ is a predicate symbol of $k$ and $A_4$.

(7) For every $A_1$-expanding alphabet $A_4$ holds every $p$ is an element of CQC-WFF $A_4$.

Let us consider $A_1$, let $A_4$ be an $A_1$-expanding alphabet, and let $p$ be an element of CQC-WFF $A_1$. The functor $A_4$-Cast $p$ yields an element of CQC-WFF $A_4$ and is defined by:

(Def. 3) $A_4\text{-Cast}\, p = p$.

Let us consider $A_1$, let $A_4$ be an $A_1$-expanding alphabet, and let $x$ be a bound variable of $A_1$. The functor $A_4\text{-Cast}\,x$ yields a bound variable of $A_4$ and is defined as follows:

(Def. 4) $A_4\text{-Cast}\,x = x$.

Let us consider $A_1$, let $A_4$ be an $A_1$-expanding alphabet, let us consider $k$, and let $P$ be a predicate symbol of $k$ and $A_1$. The functor $A_4\text{-Cast}\,P$ yielding a predicate symbol of $k$ and $A_4$ is defined as follows:

(Def. 5) $A_4\text{-Cast}\,P = P$.

Let us consider $A_1$, let $A_4$ be an $A_1$-expanding alphabet, let us consider $k$, and let $l$ be a CQC-variable list of $k$ and $A_1$. The functor $A_4\text{-Cast}\,l$ yielding a CQC-variable list of $k$ and $A_4$ is defined as follows:

(Def. 6) $A_4\text{-Cast}\,l = l$.

Next we state the proposition

(8) Let given $p$, $r$, $x$, $P$, $l$ and $A_4$ be an $A_1$-expanding alphabet. Then $A_4\text{-Cast}\,\mathrm{VERUM}\,A_1 = \mathrm{VERUM}\,A_4$ and $A_4\text{-Cast}\,P[l] = (A_4\text{-Cast}\,P)[A_4\text{-Cast}\,l]$ and $A_4\text{-Cast}\,\neg p = \neg(A_4\text{-Cast}\,p)$ and $A_4\text{-Cast}(p \wedge r) = (A_4\text{-Cast}\,p) \wedge (A_4\text{-Cast}\,r)$ and $A_4\text{-Cast}\,\forall_x p = \forall_{A_4\text{-Cast}\,x}(A_4\text{-Cast}\,p)$.

## 2. Downward Transfer of Consistency and Satisfiability

The following propositions are true:

(9) Suppose $J_1 = J_2 {\restriction}$the predicate symbols of $A_1$ and $v_1 = v_2 {\restriction}$the bound variables of $A_1$. Then $J_2 \models_{v_2} A_4\text{-Cast}\,r$ if and only if $J_1 \models_{v_1} r$.

(10) Let $A_4$ be an $A_1$-expanding alphabet and $T_1$ be a subset of CQC-WFF $A_4$. Suppose $P_1 \subseteq T_1$. Let $A_2$ be a non empty set, $J_2$ be an interpretation of $A_4$ and $A_2$, and $v_2$ be an element of the valuations in $A_4$ and $A_2$. If $J_2 \models_{v_2} T_1$, then there exist $A$, $J$, $v$ such that $J \models_v P_1$.

(11) Let $f$ be a finite sequence of elements of CQC-WFF $A_4$ and $g$ be a finite sequence of elements of CQC-WFF $A_1$. If $f = g$, then $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$.

(12) For every $p$ holds the still not bound in $p$ = the still not bound in $A_4\text{-Cast}\,p$.

(13) Let $p_2$ be an element of CQC-WFF $A_4$, $S$ be a substitution of $A_1$, $S_2$ be a substitution of $A_4$, $x_2$ be a bound variable of $A_4$, and given $x$, $p$. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{RestrictSub}(x, p, S) = \mathrm{RestrictSub}(x_2, p_2, S_2)$.

(14) Let $p_2$ be an element of CQC-WFF $A_4$, $S$ be a finite substitution of $A_1$, $S_2$ be a finite substitution of $A_4$, and given $p$. If $S = S_2$ and $p = p_2$, then $\mathrm{upVar}(S, p) = \mathrm{upVar}(S_2, p_2)$.

(15) Let $p_2$ be an element of CQC-WFF $A_4$, $S$ be a substitution of $A_1$, $S_2$ be a substitution of $A_4$, $x_2$ be a bound variable of $A_4$, and given $x$, $p$. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{ExpandSub}(x, p, \mathrm{RestrictSub}(x, \forall_x p, S)) = \mathrm{ExpandSub}(x_2, p_2, \mathrm{RestrictSub}(x_2, \forall_{x_2} p_2, S_2))$.

(16) Let $Z$ be an element of CQC-Sub-WFF $A_1$ and $Z_2$ be an element of CQC-Sub-WFF $A_4$. Suppose $Z_{\mathbf{1}}$ is universal and $(Z_2)_{\mathbf{1}}$ is universal and $\mathrm{Bound}(Z_{\mathbf{1}}) = \mathrm{Bound}((Z_2)_{\mathbf{1}})$ and $\mathrm{Scope}(Z_{\mathbf{1}}) = \mathrm{Scope}((Z_2)_{\mathbf{1}})$ and $Z = Z_2$. Then $\text{S-Bound}({}^{@} Z) = \text{S-Bound}({}^{@} Z_2)$.

(17) Let $p_2$ be an element of CQC-WFF $A_4$, $x_2$, $y_2$ be bound variables of $A_4$, and given $p$, $x$, $y$. If $p = p_2$ and $x = x_2$ and $y = y_2$, then $p(x, y) = p_2(x_2, y_2)$.

(18) For every consistent subset $P_1$ of CQC-WFF $A_4$ such that $P_1$ is a subset of CQC-WFF $A_1$ holds $P_1$ is $A_1$-consistent.

## 3. Upward Transfer of Consistency and Satisfiability

Next we state two propositions:

(19) For every $p$ there exists a countable alphabet $A_3$ such that $p$ is an element of CQC-WFF $A_3$ and $A_1$ is $A_3$-expanding.

(20) Let $P_1$ be a finite subset of CQC-WFF $A_1$. Then there exists a countable alphabet $A_3$ such that $P_1$ is a finite subset of CQC-WFF $A_3$ and $A_1$ is $A_3$-expanding.

Let us consider $A_1$ and let $P_1$ be a finite subset of CQC-WFF $A_1$. Note that the still not bound in $P_1$ is finite.

Next we state three propositions:

(21) Let $T_1$ be a subset of CQC-WFF $A_4$. Suppose $P_1 = T_1$. Let given $A$, $J$, $v$. Suppose $J \models_v P_1$. Then there exists a non empty set $A_2$ and there exists an interpretation $J_2$ of $A_4$ and $A_2$ and there exists an element $v_2$ of the valuations in $A_4$ and $A_2$ such that $J_2 \models_{v_2} T_1$.

(22) For every subset $C_1$ of CQC-WFF $A_1$ such that $C_1 \subseteq P_1$ holds $C_1$ is consistent.

(23) $P_1$ is $A_4$-consistent.

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*Received May 7, 2012*