

Free \mathbb{Z} -module¹

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Summary. In this article we formalize a free \mathbb{Z} -module and its rank. We formally prove that for a free finite rank \mathbb{Z} -module V , the number of elements in its basis, that is a rank of the \mathbb{Z} -module, is constant regardless of the selection of its basis. \mathbb{Z} -module is necessary for lattice problems, LLL(Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattice [15]. Some theorems in this article are described by translating theorems in [21] and [8] into theorems of \mathbb{Z} -module.

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The papers [17], [1], [3], [9], [4], [5], [23], [20], [14], [18], [16], [19], [2], [6], [12], [27], [28], [25], [26], [13], [24], [22], [7], [10], and [11] provide the terminology and notation for this paper.

1. FREE \mathbb{Z} -MODULE

In this paper V is a \mathbb{Z} -module, v is a vector of V , and W is a submodule of V . Let us note that there exists a \mathbb{Z} -module which is non trivial.

Let V be a \mathbb{Z} -module. One can verify that there exists a finite subset of V which is linearly independent.

Let K be a field, let V be a non empty vector space structure over K , let L be a linear combination of V , and let v be a vector of V . Then $L(v)$ is an element of K .

Next we state two propositions:

- (1) Let u be a vector of V . Then there exists a z linear combination l of V such that $l(u) = 1$ and for every vector v of V such that $v \neq u$ holds $l(v) = 0$.

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- (2) Let G be a \mathbb{Z} -module, i be an element of \mathbb{Z} , w be an element of \mathbb{Z} , and v be an element of G . Suppose $G = \langle \text{the carrier of } (\mathbb{Z}^{\mathbb{R}}), \text{ the zero of } (\mathbb{Z}^{\mathbb{R}}), \text{ the addition of } (\mathbb{Z}^{\mathbb{R}}), \text{ the left integer multiplication of } (\mathbb{Z}^{\mathbb{R}}) \rangle$ and $v = w$. Then $i \cdot v = i \cdot w$.

Let I_1 be a \mathbb{Z} -module. We say that I_1 is free if and only if:

- (Def. 1) There exists a subset A of I_1 such that A is linearly independent and $\text{Lin}(A) = \text{the } \mathbb{Z}\text{-module structure of } I_1$.

Let us consider V . One can check that $\mathbf{0}_V$ is free.

One can verify that there exists a \mathbb{Z} -module which is strict and free.

Let V be a \mathbb{Z} -module. One can verify that there exists a submodule of V which is strict and free.

Let V be a free \mathbb{Z} -module. A subset of V is called a basis of V if:

- (Def. 2) It is linearly independent and $\text{Lin}(it) = \text{the } \mathbb{Z}\text{-module structure of } V$.

One can verify that every free \mathbb{Z} -module inherits cancelable on multiplication.

Let us observe that there exists a non trivial \mathbb{Z} -module which is free.

In the sequel K_1, K_2 denote z linear combinations of V and X denotes a subset of V .

We now state a number of propositions:

- (3) If X is linearly independent and the support of $K_1 \subseteq X$ and the support of $K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.
- (4) Let V be a free \mathbb{Z} -module and A be a subset of V . Suppose A is linearly independent. Then there exists a subset B of V such that $A \subseteq B$ and B is linearly independent and for every vector v of V there exists an integer a such that $a \cdot v \in \text{Lin}(B)$.
- (5) Let L be a z linear combination of V , F, G be finite sequences of elements of V , and P be a permutation of $\text{dom } F$. If $G = F \cdot P$, then $\sum(L \cdot F) = \sum(L \cdot G)$.
- (6) Let L be a z linear combination of V and F be a finite sequence of elements of V . If the support of L misses $\text{rng } F$, then $\sum(L \cdot F) = \mathbf{0}_V$.
- (7) Let F be a finite sequence of elements of V . Suppose F is one-to-one. Let L be a z linear combination of V . If the support of $L \subseteq \text{rng } F$, then $\sum(L \cdot F) = \sum L$.
- (8) Let L be a z linear combination of V and F be a finite sequence of elements of V . Then there exists a z linear combination K of V such that the support of $K = \text{rng } F \cap (\text{the support of } L)$ and $L \cdot F = K \cdot F$.
- (9) Let L be a z linear combination of V , A be a subset of V , and F be a finite sequence of elements of V . Suppose $\text{rng } F \subseteq \text{the carrier of } \text{Lin}(A)$. Then there exists a z linear combination K of A such that $\sum(L \cdot F) = \sum K$.

- (10) Let L be a \mathbb{Z} -linear combination of V and A be a subset of V . Suppose the support of $L \subseteq$ the carrier of $\text{Lin}(A)$. Then there exists a \mathbb{Z} -linear combination K of A such that $\sum L = \sum K$.
- (11) Let L be a \mathbb{Z} -linear combination of V . Suppose the support of $L \subseteq$ the carrier of W . Let K be a \mathbb{Z} -linear combination of W . Suppose $K = L$ on the carrier of W . Then the support of $L =$ the support of K and $\sum L = \sum K$.
- (12) Let K be a \mathbb{Z} -linear combination of W . Then there exists a \mathbb{Z} -linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.
- (13) Let L be a \mathbb{Z} -linear combination of V . Suppose the support of $L \subseteq$ the carrier of W . Then there exists a \mathbb{Z} -linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.
- (14) For every free \mathbb{Z} -module V and for every basis I of V and for every vector v of V holds $v \in \text{Lin}(I)$.
- (15) For every subset A of W such that A is linearly independent holds A is a linearly independent subset of V .
- (16) Let A be a subset of V . Suppose A is linearly independent and $A \subseteq$ the carrier of W . Then A is a linearly independent subset of W .
- (17) Let V be a \mathbb{Z} -module and A be a subset of V . Suppose A is linearly independent. Let v be a vector of V . If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \text{Lin}(B)$.
- (18) Let V be a free \mathbb{Z} -module, I be a basis of V , and A be a non empty subset of V . Suppose A misses I . Let B be a subset of V . If $B = I \cup A$, then B is linearly dependent.
- (19) For every subset A of V such that $A \subseteq$ the carrier of W holds $\text{Lin}(A)$ is a submodule of W .
- (20) For every subset A of V and for every subset B of W such that $A = B$ holds $\text{Lin}(A) = \text{Lin}(B)$.

Let V be a \mathbb{Z} -module and let A be a linearly independent subset of V . One can check that $\text{Lin}(A)$ is free.

Let V be a free \mathbb{Z} -module. Observe that Ω_V is strict and free.

2. FINITE RANK FREE \mathbb{Z} -MODULE

Let I_1 be a free \mathbb{Z} -module. We say that I_1 is finite-rank if and only if:

(Def. 3) There exists a finite subset of I_1 which is a basis of I_1 .

Let us consider V . Note that $\mathbf{0}_V$ is finite-rank.

Let us note that there exists a free \mathbb{Z} -module which is strict and finite-rank.

Let V be a \mathbb{Z} -module. Note that there exists a free submodule of V which is strict and finite-rank.

Let V be a \mathbb{Z} -module and let A be a finite linearly independent subset of V . One can check that $\text{Lin}(A)$ is finite-rank.

Let V be a \mathbb{Z} -module. We say that V is finitely-generated if and only if:

(Def. 4) There exists a finite subset A of V such that $\text{Lin}(A) =$ the \mathbb{Z} -module structure of V .

Let us consider V . One can verify that $\mathbf{0}_V$ is finitely-generated.

Let us mention that there exists a \mathbb{Z} -module which is strict, finitely-generated, and free.

Let V be a finite-rank free \mathbb{Z} -module. Observe that every basis of V is finite.

3. RANK OF A FINITE RANK FREE \mathbb{Z} -MODULE

The following propositions are true:

- (21) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , and u_1, u_2 be vectors of V . If $u_1 \neq u_2$ and $u_1, u_2 \in I$, then $\text{ZMtoMQV}(V, p, u_1) \neq \text{ZMtoMQV}(V, p, u_2)$.
- (22) Let p be a prime number, V be a \mathbb{Z} -module, Z_1 be a vector space over $\text{GF}(p)$, and v_1 be a vector of Z_1 . If $Z_1 = \text{Z}_M\text{QvectSp}(V, p)$, then there exists a vector v of V such that $v_1 = \text{ZMtoMQV}(V, p, v)$.
- (23) Let p be a prime number, V be a \mathbb{Z} -module, I be a subset of V , and l_1 be a linear combination of $\text{Z}_M\text{QvectSp}(V, p)$. Then there exists a z linear combination l of I such that for every vector v of V if $v \in I$, then there exists a vector w of V such that $w \in I$ and $w \in \text{ZMtoMQV}(V, p, v)$ and $l(w) = l_1(\text{ZMtoMQV}(V, p, v))$.
- (24) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , and l_1 be a linear combination of $\text{Z}_M\text{QvectSp}(V, p)$. Then there exists a z linear combination l of I such that for every vector v of V if $v \in I$, then $l(v) = l_1(\text{ZMtoMQV}(V, p, v))$.
- (25) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , and X be a non empty subset of $\text{Z}_M\text{QvectSp}(V, p)$. Suppose $X = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$. Then there exists a function F from X into the carrier of V such that for every vector u of V such that $u \in I$ holds $F(\text{ZMtoMQV}(V, p, u)) = u$ and F is one-to-one and $\text{dom } F = X$ and $\text{rng } F = I$.
- (26) Let p be a prime number, V be a free \mathbb{Z} -module, and I be a basis of V . Then $\overline{\{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}} = \overline{I}$.
- (27) For every prime number p and for every free \mathbb{Z} -module V holds $\text{ZMtoMQV}(V, p, \mathbf{0}_V) = \mathbf{0}_{\text{Z}_M\text{QvectSp}(V, p)}$.
- (28) Let p be a prime number, V be a free \mathbb{Z} -module, and s, t be elements of V . Then $\text{ZMtoMQV}(V, p, s) + \text{ZMtoMQV}(V, p, t) = \text{ZMtoMQV}(V, p, s+t)$.

- (29) Let p be a prime number, V be a free \mathbb{Z} -module, s be a finite sequence of elements of V , and t be a finite sequence of elements of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$. Suppose $\text{len } s = \text{len } t$ and for every element i of \mathbb{N} such that $i \in \text{dom } s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = \text{ZMtoMQV}(V, p, s_1)$. Then $\sum t = \text{ZMtoMQV}(V, p, \sum s)$.
- (30) Let p be a prime number, V be a free \mathbb{Z} -module, s be an element of V , a be an integer, and b be an element of $\text{GF}(p)$. If $a = b$, then $b \cdot \text{ZMtoMQV}(V, p, s) = \text{ZMtoMQV}(V, p, a \cdot s)$.
- (31) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , l be a z linear combination of I , I_2 be a subset of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$, and l_1 be a linear combination of I_2 . Suppose $I_2 = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$ and for every vector v of V such that $v \in I$ holds $l(v) = l_1(\text{ZMtoMQV}(V, p, v))$. Then $\sum l_1 = \text{ZMtoMQV}(V, p, \sum l)$.
- (32) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , and I_2 be a subset of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$. If $I_2 = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$, then I_2 is linearly independent.
- (33) Let p be a prime number, V be a free \mathbb{Z} -module, I be a subset of V , and I_2 be a subset of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$. Suppose $I_2 = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$. Let s be a finite sequence of elements of V . Suppose that for every element i of \mathbb{N} such that $i \in \text{dom } s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $\text{ZMtoMQV}(V, p, s_1) \in \text{Lin}(I_2)$. Then $\text{ZMtoMQV}(V, p, \sum s) \in \text{Lin}(I_2)$.
- (34) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , I_2 be a subset of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$, and l be a z linear combination of I . If $I_2 = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$, then $\text{ZMtoMQV}(V, p, \sum l) \in \text{Lin}(I_2)$.
- (35) Let p be a prime number, V be a free \mathbb{Z} -module, I be a basis of V , and I_2 be a subset of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$. If $I_2 = \{\text{ZMtoMQV}(V, p, u); u \text{ ranges over vectors of } V: u \in I\}$, then I_2 is a basis of $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$.

Let p be a prime number and let V be a finite-rank free \mathbb{Z} -module. Observe that $\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p)$ is finite dimensional.

Next we state the proposition

- (36) For every finite-rank free \mathbb{Z} -module V and for all bases A, B of V holds $\overline{A} = \overline{B}$.

Let V be a finite-rank free \mathbb{Z} -module. The functor $\text{rank } V$ yields a natural number and is defined as follows:

- (Def. 5) For every basis I of V holds $\text{rank } V = \overline{I}$.

The following proposition is true

- (37) For every prime number p and for every finite-rank free \mathbb{Z} -module V holds $\text{rank } V = \text{dim}(\mathbb{Z}_M\mathbb{Q}\text{VectSp}(V, p))$.

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