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The C^k Space[1]

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Summary. In this article, we formalize continuous differentiability of real-valued functions on n-dimensional real normed linear spaces. Next, we give a definition of the C^k space according to [23].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [10], [3], [5], [11], [17], [6], [7], [19], [18], [2], [8], [14], [12], [15], [13], [21], [22], [16], [20], and [9].

1. Definition of Continuously Differentiable Functions and Some Properties

Let m be a non zero element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, k be an element of $\mathbb{N}$, and Z be a set. We say that f is continuously differentiable up to order of k and Z if and only if

(Def. 1) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) f is partial differentiable up to order k and Z, and

(iii) for every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $f{\restriction}^I Z$ is continuous on Z.

Now we state the propositions:

(1) Let us consider a non zero element m of $\mathbb{N}$, a set Z, a non empty finite sequence I of elements of $\mathbb{N}$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partially differentiable on Z w.r.t. I. Then $\operatorname{dom}(f{\restriction}^I Z) = Z$.

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(2) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) X is open, and

(ii) $X \subseteq \operatorname{dom} f$.

Then f is continuously differentiable up to order of 1 and X if and only if f is differentiable on X and for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of $\mathcal{R}^m$, $|f'(x_1)(v) - f'(x_0)(v)| \leqslant r \cdot |v|$.

(3) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) X is open, and

(ii) $X \subseteq \operatorname{dom} f$, and

(iii) f is continuously differentiable up to order of 1 and X.

Then f is continuous on X. The theorem is a consequence of (2).

(4) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) g is continuously differentiable up to order of k and X, and

(iii) X is open.

Then $f + g$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $(f+g){\restriction}^I X$ is continuous on X. $\square$

(5) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a real number r, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) X is open.

Then $r \cdot f$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $r \cdot f{\restriction}^I X$ is continuous on X. $\square$

(6) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) g is continuously differentiable up to order of k and X, and

(iii) X is open.

Then $f-g$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $(f-g){\upharpoonright}^I X$ is continuous on X. $\square$

Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and non empty finite sequences I, G of elements of $\mathbb{N}$. Now we state the propositions:

(7) $f{\upharpoonright}^{G^\frown I} Z = (f{\upharpoonright}^G Z){\upharpoonright}^I Z$.

(8) $f{\upharpoonright}^{G^\frown I} Z$ is continuous on Z if and only if $(f{\upharpoonright}^G Z){\upharpoonright}^I Z$ is continuous on Z.

Now we state the propositions:

(9) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, elements i, j of $\mathbb{N}$, and a non empty finite sequence I of elements of $\mathbb{N}$. Suppose

(i) f is continuously differentiable up to order of $i+j$ and Z, and

(ii) $\operatorname{rng} I \subseteq \operatorname{Seg} m$, and

(iii) $\operatorname{len} I = j$.

Then $f{\upharpoonright}^I Z$ is continuously differentiable up to order of i and Z. The theorem is a consequence of (1) and (7).

(10) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and elements i, j of $\mathbb{N}$. Suppose

(i) f is continuously differentiable up to order of i and Z, and

(ii) $j \leqslant i$.

Then f is continuously differentiable up to order of j and Z.

(11) Let us consider a non zero element m of $\mathbb{N}$ and a non empty subset Z of $\mathcal{R}^m$. Suppose Z is open. Let us consider an element k of $\mathbb{N}$ and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and Z, and

(ii) g is continuously differentiable up to order of k and Z.

Then $f \cdot g$ is continuously differentiable up to order of k and Z. The theorem is a consequence of (10), (1), (3), (9), and (7). PROOF: Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv$ for every partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$ such that f is continuously differentiable up to order of $\$_1$ and Z and g is continuously differentiable up to order of $\$_1$ and Z holds $f \cdot g$ is continuously differentiable up to order of $\$_1$ and Z. Set $Z0 =$ (0 **qua** natural number). $\mathcal{P}[0]$. For every element k of $\mathbb{N}$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\square$

(12) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Let us consider an element x of $\mathcal{R}^m$. If $x \in X$, then f is differentiable in x and $f'(x) = \mathcal{R}^m \longmapsto 0$.

(13) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Let us consider an element x_0 of $\mathcal{R}^m$ and a real number r. Suppose

(iii) $x_0 \in X$, and

(iv) $0 < r$.

Then there exists a real number s such that

(v) $0 < s$, and

(vi) for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of $\mathcal{R}^m$, $|f'(x_1)(v) - f'(x_0)(v)| \leqslant r \cdot |v|$.

The theorem is a consequence of (12).

(14) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Then

(iii) f is differentiable on X, and

(iv) $\operatorname{dom} f'_{\restriction X} = X$, and

(v) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f'_{\restriction X})_x = \mathcal{R}^m \longmapsto 0$.

The theorem is a consequence of (12).

(15) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, a real number d, and an element i of $\mathbb{N}$. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$, and

(iii) $1 \leqslant i \leqslant m$.

Then

(iv) f is partially differentiable on X w.r.t. i, and

(v) $f{\restriction^i}X$ is continuous on X.

The theorem is a consequence of (14) and (13).

(16) Let us consider a non zero element m of $\mathbb{N}$, an element i of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$, and

(iii) $1 \leqslant i \leqslant m$.

Then $f{\restriction}^i X = X \longmapsto 0$. The theorem is a consequence of (15) and (12).

Let us consider a non zero element m of $\mathbb{N}$, a non empty finite sequence I of elements of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number d. Now we state the propositions:

(17) Suppose X is open and $f = X \longmapsto d$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$. Then

(i) $(\mathrm{PartDiffSeq}(f, X, I))(0) = X \longmapsto d$, and

(ii) for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $(\mathrm{PartDiffSeq}(f, X, I))(i) = X \longmapsto 0$.

(18) Suppose X is open and $f = X \longmapsto d$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$. Then

(i) f is partially differentiable on X w.r.t. I, and

(ii) $f{\restriction}^I X$ is continuous on X.

Now we state the proposition:

(19) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Then f is continuously differentiable up to order of k and X. The theorem is a consequence of (18).

Let m be a non zero element of $\mathbb{N}$. Observe that there exists a non empty subset of $\mathcal{R}^m$ which is open.

2. Definition of the C^k Space

Let m be a non zero element of $\mathbb{N}$, k be an element of $\mathbb{N}$, and X be a non empty open subset of $\mathcal{R}^m$. The functor the $\mathbb{C}^k$ functions of k and X yielding a non empty subset of RAlgebra X is defined by the term

(Def. 2) $\{f$ where f is a partial function from $\mathcal{R}^m$ to $\mathbb{R}$: f is continuously differentiable up to order of k and X and $\operatorname{dom} f = X\}$.

Let us note that the $\mathbb{C}^k$ functions of k and X is additively linearly closed and multiplicatively closed.

The functor the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X yielding a subalgebra of RAlgebra X is defined by the term

(Def. 3) $\langle$the $\mathbb{C}^k$ functions of k and X, mult(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Add(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Mult(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), One(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Zero(the $\mathbb{C}^k$ functions of k and X, RAlgebra X)$\rangle$.

Let us note that the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital commutative associative right unital right distributive and vector associative.

Now we state the propositions:

(20) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, and partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and

(ii) $g = G$, and

(iii) $h = H$.

Then $H = F+G$ if and only if for every element x of X, $h(x) = f(x)+g(x)$.

(21) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number a. Suppose

(i) $f = F$, and

(ii) $g = G$.

Then $G = a \cdot F$ if and only if for every element x of X, $g(x) = a \cdot f(x)$.

(22) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, and partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and

(ii) $g = G$, and

(iii) $h = H$.

Then $H = F \cdot G$ if and only if for every element x of X, $h(x) = f(x) \cdot g(x)$.

Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, and a non empty open subset X of $\mathcal{R}^m$. Now we state the propositions:

(23) $0_\alpha = X \longmapsto 0$, where α is the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X.

(24) $\mathbf{1}_\alpha = X \longmapsto 1$, where α is the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X.

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