Commutativeness of Fundamental Groups of Topological Groups

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Summary. In this article we prove that fundamental groups based at the unit point of topological groups are commutative [11].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [19], [9], [10], [20], [4], [5], [22], [23], [21], [1], [6], [17], [18], [2], [25], [26], [24], [13], [12], [13], [8], [14], and [7].

Let $A$ be a non empty set, $x$ be an element, and $a$ be an element of $A$. Let us observe that $(A \mapsto x)(a)$ reduces to $x$.

Let $A$, $B$ be non empty topological spaces, $C$ be a set, and $f$ be a function from $A \times B$ into $C$. Let $b$ be an element of $B$. Let us note that the functor $f(a, b)$ yields an element of $C$. Let $G$ be a multiplicative magma and $g$ be an element of $G$. We say that $g$ is unital if and only if

(Def. 1) $g = 1_G$.

One can check that $1_G$ is unital.

Let $G$ be a unital multiplicative magma. Let us note that there exists an element of $G$ which is unital.

Let $g$ be an element of $G$ and $h$ be a unital element of $G$. One can check that $g \cdot h$ reduces to $g$. One can check that $h \cdot g$ reduces to $g$.

Let $G$ be a group. One can verify that $(1_G)^{-1}$ reduces to $1_G$.

The scheme $TopFuncEx$ deals with non empty topological spaces $S$, $T$ and a non empty set $X$ and a binary functor $F$ yielding an element of $X$ and states that
(Sch. 1) There exists a function $f$ from $S \times T$ into $X$ such that for every point $s$ of $S$ for every point $t$ of $T$, $f(s, t) = F(s, t)$.

The scheme TopFuncEq deals with non empty topological spaces $S$, $T$ and a non empty set $X$ and a binary functor $F$ yielding an element of $X$ and states that

(Sch. 2) For every functions $f$, $g$ from $S \times T$ into $X$ such that for every point $s$ of $S$ for every point $t$ of $T$, $f(s, t) = F(s, t)$ and for every point $s$ of $S$ and for every point $t$ of $T$, $g(s, t) = F(s, t)$ holds $f = g$.

Let $X$ be a non empty set, $T$ be a non empty multiplicative magma, and $f$, $g$ be functions from $X$ into $T$. The functor $f \cdot g$ yielding a function from $X$ into $T$ is defined by

(Def. 2) Let us consider an element $x$ of $X$. Then $it(x) = f(x) \cdot g(x)$.

Now we state the proposition:

(1) Let us consider a non empty set $X$, an associative non empty multiplicative magma $T$, and functions $f$, $g$, $h$ from $X$ into $T$. Then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Let $X$ be a non empty set, $T$ be a commutative non empty multiplicative magma, and $f$, $g$ be functions from $X$ into $T$. Observe that the functor $f \cdot g$ is commutative.

Let $T$ be a non empty topological group structure, $t$ be a point of $T$, and $f$, $g$ be loops of $t$. The functor $f \bullet g$ yielding a function from $I$ into $T$ is defined by the term

(Def. 3) $f \cdot g$.

In this paper $T$ denotes a continuous unital topological space-like non empty topological group structure, $x$, $y$ denote points of $I$, $s$, $t$ denote unital points of $T$, $f$, $g$ denote loops of $t$, and $e$ denotes a constant loop of $t$.

Let us consider $T$, $t$, $f$, and $g$. One can check that the functor $f \bullet g$ yields a loop of $t$. Let $T$ be an inverse-continuous semi topological group. Observe that $-1_T$ is continuous.

Let $T$ be a semi topological group, $t$ be a point of $T$, and $f$ be a loop of $t$. The functor $f^{-1}$ yielding a function from $I$ into $T$ is defined by the term

(Def. 4) $-1_T \cdot f$.

Let us consider a semi topological group $T$, a point $t$ of $T$, and a loop $f$ of $t$. Now we state the propositions:

(2) $(f^{-1})(x) = f(x)^{-1}$.
(3) $(f^{-1})(x) \cdot f(x) = 1_T$.
(4) $f(x) \cdot (f^{-1})(x) = 1_T$.

Let $T$ be an inverse-continuous semi topological group, $t$ be a unital point of $T$, and $f$ be a loop of $t$. One can check that the functor $f^{-1}$ yields a loop of
Let $s, t$ be points of $I$. One can check that the functor $s \cdot t$ yields a point of $I$. The functor $\otimes_{R^1}$ yielding a function from $R^1 \times R^1$ into $R^1$ is defined by

(Def. 5) Let us consider points $x, y$ of $R^1$. Then $it(x, y) = x \cdot y$.

Observe that $\otimes_{R^1}$ is continuous.

Now we state the proposition:

(5) $(R^1 \times R^1)(I^1[0, 1] \times R^1[0, 1]) = I \times I$.

The functor $\otimes_I$ yielding a function from $I \times I$ into $I$ is defined by the term

(Def. 6) $\otimes_{R^1} \mid R^1[0, 1]$.

Now we state the proposition:

(6) $(\otimes_I)(x, y) = x \cdot y$.

One can verify that $\otimes_{I}$ is continuous.

Now we state the proposition:

(7) Let us consider points $a, b$ of $I$ and a neighbourhood $N$ of $a \cdot b$. Then there exists a neighbourhood $N_1$ of $a$ and there exists a neighbourhood $N_2$ of $b$ such that for every points $x, y$ of $I$ such that $x \in N_1$ and $y \in N_2$ holds $x \cdot y \in N$. The theorem is a consequence of (6).

Let $T$ be a non empty multiplicative magma and $F, G$ be functions from $I \times I$ into $T$. The functor $F \ast G$ yielding a function from $I \times I$ into $T$ is defined by

(Def. 7) Let us consider points $a, b$ of $I$. Then $it(a, b) = F(a, b) \cdot G(a, b)$.

Now we state the proposition:

(8) Let us consider functions $F, G$ from $I \times I$ into $T$ and subsets $M, N$ of $I \times I$. Then $(F \ast G)^0(M \cap N) \subseteq F^0 M \cdot G^0 N$.

Let us consider $T$. Let $F, G$ be continuous functions from $I \times I$ into $T$. Observe that $F \ast G$ is continuous.

Now we state the propositions:

(9) Let us consider loops $f_1, f_2, g_1, g_2$ of $t$. Suppose

(i) $f_1, f_2$ are homotopic, and
(ii) $g_1, g_2$ are homotopic.

Then $f_1 \cdot g_1, f_2 \cdot g_2$ are homotopic.

(10) Let us consider loops $f_1, f_2, g_1, g_2$ of $t$, a homotopy $F$ between $f_1$ and $f_2$, and a homotopy $G$ between $g_1$ and $g_2$. Suppose

(i) $f_1, f_2$ are homotopic, and
(ii) $g_1, g_2$ are homotopic.

Then $F \ast G$ is a homotopy between $f_1 \cdot g_1$ and $f_2 \cdot g_2$. The theorem is a consequence of (9).

(11) $f + g = (f + c) \bullet (c + g)$.
(12) $f \bullet g, (f + c) \bullet (c + g)$ are homotopic. The theorem is a consequence of (9).
Let $T$ be a semi topological group, $t$ be a point of $T$, and $f$, $g$ be loops of $t$.

The functor $\text{HopfHomotopy}(f, g)$ yielding a function from $I \times I$ into $T$ is defined by

(Def. 8) Let us consider points $a, b$ of $I$. Then $it(a, b) = (((f^{-1})(a \cdot b) \cdot f(a)) \cdot g(a)) \cdot f(a \cdot b)$.

Note that $\text{HopfHomotopy}(f, g)$ is continuous.

In the sequel $T$ denotes a topological group, $t$ denotes a unital point of $T$, and $f$, $g$ denote loops of $t$.

Now we state the proposition:

(13) $f \cdot g$, $g \cdot f$ are homotopic.

Let us consider $T$, $t$, $f$, and $g$. Let us note that the functor $\text{HopfHomotopy}(f, g)$ yields a homotopy between $f \cdot g$ and $g \cdot f$.

Now we are at the position where we can present the Main Theorem of the paper: $\pi_1(T, t)$ is commutative.

References

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