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Coproducts in Categories without Uniqueness of cod and dom

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Summary. The paper introduces coproducts in categories without uniqueness of **cod** and **dom**. It is proven that set-theoretical disjoint union is the coproduct in the category Ens [9].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [7], [6], [1], [11], [2], [3], [8], [4], [12], [14], [13], and [5].

From now on I denotes a set and E denotes a non empty set.

Let I be a non empty set, A be a many sorted set indexed by I, and i be an element of I. Let us observe that $\mathrm{coprod}(i, A)$ is relation-like and function-like.

Let C be a non empty category structure, o be an object of C, I be a set, and f be an objects family of I and C. A morphisms family of f and o is a many sorted set indexed by I and is defined by

(Def. 1) Let us consider an element i. Suppose $i \in I$. Then there exists an object o_1 of C such that

(i) $o_1 = f(i)$, and

(ii) $it(i)$ is a morphism from o_1 to o.

Let I be a non empty set. Let us note that a morphisms family of f and o can equivalently be formulated as follows:

(Def. 2) Let us consider an element i of I. Then $it(i)$ is a morphism from $f(i)$ to o.

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Let M be a morphisms family of f and o and i be an element of I. Note that the functor $M(i)$ yields a morphism from $f(i)$ to o. Let C be a functional non empty category structure. Let I be a set. Let us note that every morphisms family of f and o is function yielding.

Now we state the proposition:

(1) Let us consider a non empty category structure C, an object o of C, and an objects family f of $\emptyset$ and C. Then $\emptyset$ is a morphisms family of f and o.

Let C be a non empty category structure, I be a set, A be an objects family of I and C, B be an object of C, and P be a morphisms family of A and B. We say that P is feasible if and only if

(Def. 3) Let us consider a set i. Suppose $i \in I$. Then there exists an object o of C such that

(i) $o = A(i)$, and

(ii) $P(i) \in \langle o, B \rangle$.

Let I be a non empty set. Let us observe that P is feasible if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let us consider an element i of I. Then $P(i) \in \langle A(i), B \rangle$.

Let C be a category and I be a set. We say that P is coprojection morphisms if and only if

(Def. 5) Let us consider an object X of C and a morphisms family F of A and X. Suppose F is feasible. Then there exists a morphism f from B to X such that

(i) $f \in \langle B, X \rangle$, and

(ii) for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$, and

(iii) for every morphism f_1 from B to X such that for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f_1 \cdot P_i$ holds $f = f_1$.

Let I be a non empty set. Let us note that P is coprojection morphisms if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let us consider an object X of C and a morphisms family F of A and X. Suppose F is feasible. Then there exists a morphism f from B to X such that

(i) $f \in \langle B, X \rangle$, and

(ii) for every element i of I, $F(i) = f \cdot P(i)$, and

(iii) for every morphism f_1 from B to X such that for every element i of I, $F(i) = f_1 \cdot P(i)$ holds $f = f_1$.

Let A be an objects family of $\emptyset$ and C. Note that every morphisms family of A and B is feasible.

Now we state the propositions:

(2) Let us consider a category C, an objects family A of $\emptyset$ and C, and an object B of C. Suppose B is initial. Then there exists a morphisms family P of A and B such that P is empty and coprojection morphisms. The theorem is a consequence of (1).

(3) Let us consider an objects family A of I and $\mathrm{Ens}_{\{\emptyset\}}$ and an object o of $\mathrm{Ens}_{\{\emptyset\}}$. Then $I \longmapsto \emptyset$ is a morphisms family of A and o.

(4) Let us consider an objects family A of I and $\mathrm{Ens}_{\{\emptyset\}}$, an object o of $\mathrm{Ens}_{\{\emptyset\}}$, and a morphisms family P of A and o. If $P = I \longmapsto \emptyset$, then P is feasible and coprojection morphisms. PROOF: P is feasible by [11, (7)]. Reconsider $f = \emptyset$ as a morphism from o to Y. For every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to o such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$ by [11, (7)]. □

Let C be a category. We say that C has coproducts if and only if

(Def. 7) Let us consider a set I and an objects family A of I and C. Then there exists an object B of C and there exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Note that $\mathrm{Ens}_{\{\emptyset\}}$ has coproducts and there exists a category which is strict and has products and coproducts.

Let C be a category, I be a set, A be an objects family of I and C, and B be an object of C. We say that B is A-category coproduct-like if and only if

(Def. 8) There exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Let C be a category with coproducts. Let us observe that there exists an object of C which is A-category coproduct-like.

Let C be a category and A be an objects family of $\emptyset$ and C. Note that every object of C which is A-category coproduct-like is also initial.

Now we state the propositions:

(5) Let us consider a category C, an objects family A of $\emptyset$ and C, and an object B of C. If B is initial, then B is A-category coproduct-like. The theorem is a consequence of (2).

(6) Let us consider a category C, an objects family A of I and C, and objects C_1, C_2 of C. Suppose

(i) C_1 is A-category coproduct-like, and

(ii) C_2 is A-category coproduct-like.

Then C_1,C_2 are iso.

From now on A denotes an objects family of I and Ens_E.

Let us consider I, E, and A. Assume $\bigcup \mathrm{coprod}(A) \in E$. The functor $\coprod A$ yielding an object of Ens_E is defined by the term

(Def. 9) $\bigcup \mathrm{coprod}(A)$.

The functor $\mathrm{Coprod}(A)$ yielding a many sorted set indexed by I is defined by

(Def. 10) Let us consider an element i. Suppose $i \in I$. Then there exists a function F from $A(i)$ into $\bigcup \mathrm{coprod}(A)$ such that

(i) $it(i) = F$, and

(ii) for every element x such that $x \in A(i)$ holds $F(x) = \langle x, i \rangle$.

Observe that $\mathrm{Coprod}(A)$ is function yielding.

Assume $\bigcup \mathrm{coprod}(A) \in E$. The functor $\coprod_{i \in I} A(i)$ yielding a morphisms family of A and $\coprod A$ is defined by the term

(Def. 11) $\mathrm{Coprod}(A)$.

Now we state the propositions:

(7) If $\bigcup \mathrm{coprod}(A) = \emptyset$, then $\mathrm{Coprod}(A)$ is empty yielding.

(8) If $\bigcup \mathrm{coprod}(A) = \emptyset$, then A is empty yielding.

(9) If $\bigcup \mathrm{coprod}(A) \in E$ and $\bigcup \mathrm{coprod}(A) = \emptyset$, then $\coprod_{i \in I} A(i) = I \longmapsto \emptyset$. The theorem is a consequence of (7).

(10) If $\bigcup \mathrm{coprod}(A) \in E$, then $\coprod_{i \in I} A(i)$ is feasible and coprojection morphisms. The theorem is a consequence of (7) and (8).

(11) If $\bigcup \mathrm{coprod}(A) \in E$, then $\coprod A$ is A-category coproduct-like. The theorem is a consequence of (10).

(12) If for every I and A, $\bigcup \mathrm{coprod}(A) \in E$, then Ens_E has coproducts. The theorem is a consequence of (10).

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[6] Artur Korniłowicz. Products in categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 20(**4**):303–307, 2012. doi:10.2478/v10037-012-0036-7.

[7] Beata Madras. Basic properties of objects and morphisms. *Formalized Mathematics*, 6 (**3**):329–334, 1997.

[8] Beata Perkowska. Free many sorted universal algebra. *Formalized Mathematics*, 5(**1**): 67–74, 1996.

[9] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*, volume 45 of *Biblioteka Matematyczna*. PWN, Warszawa, 1978.

[10] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(**2**):259–267, 1996.
[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.
[12] Andrzej Trybulec. Many sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

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