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Tietze Extension Theorem for n-dimensional Spaces[1]

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Summary. In this article we prove the Tietze extension theorem for an arbitrary convex compact subset of $\mathcal{E}^n$ with a non-empty interior. This theorem states that, if T is a normal topological space, X is a closed subset of T, and A is a convex compact subset of $\mathcal{E}^n$ with a non-empty interior, then a continuous function $f : X \to A$ can be extended to a continuous function $g : T \to \mathcal{E}^n$. Additionally we show that a subset A is replaceable by an arbitrary subset of a topological space that is homeomorphic with a convex compact subset of $\mathcal{E}^n$ with a non-empty interior. This article is based on [20]; [23] and [22] can also serve as reference books.

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The notation and terminology used in this paper have been introduced in the following articles: [8], [36], [24], [30], [1], [15], [21], [16], [25], [6], [9], [17], [37], [10], [11], [3], [34], [5], [12], [26], [33], [35], [41], [42], [13], [40], [19], [31], [28], [43], [18], [44], [29], and [14].

1. Closed Hypercube

From now on n, m, i denote natural numbers, p, q denote points of $\mathcal{E}^n_{\mathrm{T}}$, r, s denote real numbers, and R denotes a real-valued finite sequence.

Note that every finite sequence which is empty is also non-negative yielding.

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Let n be a non zero natural number, X be a set, and F be an element of ((the carrier of $\mathbb{R}^{\mathbf{1}})^X)^n$. Let us note that the functor $\prod^* F$ yields a function from X into $\mathcal{E}^n_{\mathrm{T}}$. Now we state the proposition:

(1) Let us consider sets X, Y, a function yielding function F, and objects x, y. Suppose

(i) F is (Y^X)-valued, or

(ii) $y \in \operatorname{dom} \prod^* F$.

Then $F(x)(y) = (\prod^* F)(y)(x)$.

Let us consider n, p, and r. The functor OpenHypercube(p, r) yielding an open subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by

(Def. 1) There exists a point e of $\mathcal{E}^n$ such that

(i) $p = e$, and

(ii) $it =$ OpenHypercube(e, r).

Now we state the propositions:

(2) If $q \in$ OpenHypercube(p, r) and $s \in \,]p(i) - r, p(i) + r[$, then $q +\cdot (i, s) \in$ OpenHypercube(p, r). Proof: Consider e being a point of $\mathcal{E}^n$ such that $p = e$ and OpenHypercube$(p, r) =$ OpenHypercube(e, r). Set $I =$ Intervals(e, r). Set $q_3 = q +\cdot (i, s)$. For every object x such that $x \in \operatorname{dom} I$ holds $q_3(x) \in I(x)$ by [2, (9)], [7, (31), (32)]. □

(3) If $i \in \operatorname{Seg} n$, then $(\mathrm{PROJ}(n, i))^\circ(\text{OpenHypercube}(p, r)) = \,]p(i) - r, p(i) + r[$. The theorem is a consequence of (2).

(4) $q \in$ OpenHypercube(p, r) if and only if for every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in \,]p(i) - r, p(i) + r[$. The theorem is a consequence of (3).

Let us consider n, p, and R. The functor ClosedHypercube(p, R) yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by

(Def. 2) $q \in it$ if and only if for every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in [p(i) - R(i), p(i) + R(i)]$.

Now we state the propositions:

(5) If there exists i such that $i \in \operatorname{Seg} n \cap \operatorname{dom} R$ and $R(i) < 0$, then ClosedHypercube(p, R) is empty.

(6) If for every i such that $i \in \operatorname{Seg} n \cap \operatorname{dom} R$ holds $R(i) \geqslant 0$, then $p \in$ ClosedHypercube(p, R).

Let us consider n and p. Let R be a non-negative yielding real-valued finite sequence. One can check that ClosedHypercube(p, R) is non empty.

Let us consider R. Let us observe that ClosedHypercube(p, R) is convex and compact.

Now we state the propositions:

(7) If $i \in \operatorname{Seg} n$ and $q \in \text{ClosedHypercube}(p, R)$ and $r \in [p(i) - R(i), p(i) + R(i)]$, then $q +\cdot (i, r) \in \text{ClosedHypercube}(p, R)$. PROOF: Set $p_4 = q +\cdot (i, r)$. For every natural number j such that $j \in \operatorname{Seg} n$ holds $p_4(j) \in [p(j) - R(j), p(j) + R(j)]$ by [7, (32), (31)]. $\square$

(8) Suppose $i \in \operatorname{Seg} n$ and ClosedHypercube(p, R) is not empty. Then $(\text{PROJ}(n, i))^\circ(\text{ClosedHypercube}(p, R)) = [p(i) - R(i), p(i) + R(i)]$. The theorem is a consequence of (5), (7), and (6).

(9) If $n \leqslant \operatorname{len} R$ and $r \leqslant \inf \operatorname{rng} R$, then $\text{OpenHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, R)$.

(10) $q \in \operatorname{Fr} \text{ClosedHypercube}(p, R)$ if and only if $q \in \text{ClosedHypercube}(p, R)$ and there exists i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$. PROOF: Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. If $q \in \operatorname{Fr} \text{ClosedHypercube}(p, R)$, then $q \in \text{ClosedHypercube}(p, R)$ and there exists i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$ by [16, (22)], [32, (105)], [14, (33)], [6, (3)]. For every subset S of T_4 such that S is open and $q \in S$ holds ClosedHypercube(p, R) meets S and $(\text{ClosedHypercube}(p, R))^{\mathrm{c}}$ meets S by [16, (67)], [43, (23)], [38, (5)], [31, (13)]. $\square$

(11) If $r \geqslant 0$, then $p \in \text{ClosedHypercube}(p, n \mapsto r)$.

(12) If $r > 0$, then $\operatorname{Int} \text{ClosedHypercube}(p, n \mapsto r) = \text{OpenHypercube}(p, r)$. PROOF: Set $O = \text{OpenHypercube}(p, r)$. Set $C = \text{ClosedHypercube}(p, n \mapsto r)$. Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. Set $R = n \mapsto r$. Consider e being a point of $\mathcal{E}^n$ such that $p = e$ and $\text{OpenHypercube}(p, r) = \text{OpenHypercube}(e, r)$. $\operatorname{Int} C \subseteq O$ by [43, (39)], [9, (57)], (10), [39, (29)]. Reconsider $q = x$ as a point of T_4. For every i such that $i \in \operatorname{Seg} n$ holds $q(i) \in [p(i) - R(i), p(i) + R(i)]$ by [9, (57)], (3). Consider i such that $i \in \operatorname{Seg} n$ and $q(i) = p(i) - R(i)$ or $q(i) = p(i) + R(i)$. $(\text{PROJ}(n, i))^\circ O =]e(i) - r, e(i) + r[$. $\square$

(13) $\text{OpenHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, n \mapsto r)$.

(14) If $r < s$, then $\text{ClosedHypercube}(p, n \mapsto r) \subseteq \text{OpenHypercube}(p, s)$. The theorem is a consequence of (4).

Let us consider n and p. Let r be a positive real number. Let us note that ClosedHypercube$(p, n \mapsto r)$ is non boundary.

2. PROPERTIES OF THE PRODUCT OF CLOSED HYPERCUBE

From now on T_1, T_2, S_1, S_2 denote non empty topological spaces, t_1 denotes a point of T_1, t_2 denotes a point of T_2, p_2, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$, and p_1, q_1 denote points of $\mathcal{E}^m_{\mathrm{T}}$.

Now we state the propositions:

(15) Let us consider a function f from T_1 into T_2 and a function g from S_1 into S_2. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then $f \times g$ is a homeomorphism.

(16) Suppose $r > 0$ and $s > 0$. Then there exists a function h from $(\mathcal{E}^n_{\mathrm{T}} \restriction \text{ClosedHypercube}(p_2, n \mapsto r)) \times (\mathcal{E}^m_{\mathrm{T}} \restriction \text{ClosedHypercube}(p_1, m \mapsto s))$ into $\mathcal{E}^{n+m}_{\mathrm{T}} \restriction \text{ClosedHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, (n+m) \mapsto 1)$ such that

(i) h is a homeomorphism, and

(ii) $h^{\circ}(\text{OpenHypercube}(p_2, r) \times \text{OpenHypercube}(p_1, s)) = \text{OpenHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$.

Proof: Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $n_1 = n+m$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $R_2 = \text{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_4 = \text{ClosedHypercube}(p_2, n \mapsto r)$. Set $R_5 = \text{ClosedHypercube}(p_1, m \mapsto s)$. Set $R_1 = \text{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Set $R_3 = \text{ClosedHypercube}(0_{T_7}, n_1 \mapsto 1)$. Reconsider $R_{10} = R_5$, $R_6 = R_1$ as a non empty subset of T_5. Consider h_3 being a function from $T_5 \restriction R_{10}$ into $T_5 \restriction R_6$ such that h_3 is a homeomorphism and $h_3{}^{\circ}(\operatorname{Fr} R_{10}) = \operatorname{Fr} R_6$. Reconsider $R_9 = R_4$, $R_7 = R_2$ as a non empty subset of T_6. Consider h_4 being a function from $T_6 \restriction R_9$ into $T_6 \restriction R_7$ such that h_4 is a homeomorphism and $h_4{}^{\circ}(\operatorname{Fr} R_9) = \operatorname{Fr} R_7$. Set $O_8 = \text{OpenHypercube}(p_2, r)$. Set $O_9 = \text{OpenHypercube}(p_1, s)$. Set $O_6 = \text{OpenHypercube}(0_{T_7}, 1)$. $\operatorname{Int} R_{10} = O_9$. Set $O_5 = \text{OpenHypercube}(0_{T_6}, 1)$. Set $O_7 = \text{OpenHypercube}(0_{T_5}, 1)$. Reconsider $R_8 = R_3$ as a non empty subset of T_7. Consider f being a function from $T_6 \times T_5$ into T_7 such that f is a homeomorphism and for every element f_5 of T_6 and for every element f_6 of T_5, $f(f_5, f_6) = f_5 \frown f_6$. $f^{\circ}(R_7 \times R_6) \subseteq R_8$ by [14, (87)], [9, (57)], [6, (25)]. $R_8 \subseteq f^{\circ}(R_7 \times R_6)$ by [9, (23)], [27, (17)], [4, (11)], [6, (5)]. Set $h_5 = h_4 \times h_3$. h_5 is a homeomorphism. $\operatorname{Int} R_7 = O_5$. Reconsider $f_1 = f \restriction (R_7 \times R_6)$ as a function from $(T_6 \restriction R_7) \times (T_5 \restriction R_6)$ into $T_7 \restriction R_8$. Reconsider $h = f_1 \cdot h_5$ as a function from $(T_6 \restriction R_4) \times (T_5 \restriction R_5)$ into $T_7 \restriction R_3$. $\operatorname{Int} R_6 = O_7$. $\operatorname{Int} R_9 = O_8$. $h^{\circ}(O_8 \times O_9) \subseteq O_6$ by [14, (87)], [10, (12)], [43, (40)], [10, (49)]. Reconsider $p_3 = y$ as a point of T_7. Consider p, q being finite sequences of elements of $\mathbb{R}$ such that $\operatorname{len} p = n$ and $\operatorname{len} q = m$ and $p_3 = p \frown q$. $q \in O_7$. $q \in R_6$. Consider x_2 being an object such that $x_2 \in \operatorname{dom} h_3$ and $h_3(x_2) = q$. $p \in O_5$. $p \in R_7$. Consider x_1 being an object such that $x_1 \in \operatorname{dom} h_4$ and $h_4(x_1) = p$. $\square$

(17) Suppose $r > 0$ and $s > 0$. Let us consider a function f from T_1 into $\mathcal{E}^n_{\mathrm{T}} \restriction \text{ClosedHypercube}(p_2, n \mapsto r)$ and a function g from T_2 into $\mathcal{E}^m_{\mathrm{T}} \restriction \text{ClosedHypercube}(p_1, m \mapsto s)$. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then there exists a function h from $T_1 \times T_2$ into

$\mathcal{E}^{n+m}_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, (n+m) \mapsto 1)$ such that

(iii) h is a homeomorphism, and

(iv) for every t_1 and t_2, $f(t_1) \in \mathrm{OpenHypercube}(p_2, r)$ and $g(t_2) \in \mathrm{OpenHypercube}(p_1, s)$ iff $h(t_1, t_2) \in \mathrm{OpenHypercube}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$.

Proof: Set $n_1 = n + m$. Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $R_7 = n \mapsto r$. Set $R_6 = m \mapsto s$. Set $R_8 = n_1 \mapsto 1$. Set $R_4 = \mathrm{ClosedHypercube}(p_2, R_7)$. Set $R_5 = \mathrm{ClosedHypercube}(p_1, R_6)$. Set $C_2 = \mathrm{ClosedHypercube}(0_{T_7}, R_8)$. Reconsider $R_{10} = R_5$ as a non empty subset of T_5. Reconsider $R_9 = R_4$ as a non empty subset of T_6. Set $O_8 = \mathrm{OpenHypercube}(p_2, r)$. Set $O_9 = \mathrm{OpenHypercube}(p_1, s)$. Set $O = \mathrm{OpenHypercube}(0_{T_7}, 1)$. Consider h being a function from $(T_6 \restriction R_9) \times (T_5 \restriction R_{10})$ into $T_7 \restriction C_2$ such that h is a homeomorphism and $h^\circ(O_8 \times O_9) = O$. Reconsider $G = g$ as a function from T_2 into $T_5 \restriction R_{10}$. Reconsider $F = f$ as a function from T_1 into $T_6 \restriction R_9$. Reconsider $f_4 = h \cdot (F \times G)$ as a function from $T_1 \times T_2$ into $T_7 \restriction C_2$. $F \times G$ is a homeomorphism. $O_9 \subseteq R_{10}$. $O_8 \subseteq R_9$. If $f(t_1) \in O_8$ and $g(t_2) \in O_9$, then $f_4(t_1, t_2) \in O$ by [14, (87)], [10, (12)]. Consider x_3 being an object such that $x_3 \in \mathrm{dom}\, h$ and $x_3 \in O_8 \times O_9$ and $h(x_3) = h(\langle f(t_1), g(t_2) \rangle)$. $\square$

Let us consider n. One can check that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non boundary, convex, and compact.

Now we state the propositions:

(18) Let us consider a non boundary convex compact subset A of $\mathcal{E}^n_{\mathrm{T}}$, a non boundary convex compact subset B of $\mathcal{E}^m_{\mathrm{T}}$, a non boundary convex compact subset C of $\mathcal{E}^{n+m}_{\mathrm{T}}$, a function f from T_1 into $\mathcal{E}^n_{\mathrm{T}} \restriction A$, and a function g from T_2 into $\mathcal{E}^m_{\mathrm{T}} \restriction B$. Suppose

(i) f is a homeomorphism, and

(ii) g is a homeomorphism.

Then there exists a function h from $T_1 \times T_2$ into $\mathcal{E}^{n+m}_{\mathrm{T}} \restriction C$ such that

(iii) h is a homeomorphism, and

(iv) for every t_1 and t_2, $f(t_1) \in \mathrm{Int}\, A$ and $g(t_2) \in \mathrm{Int}\, B$ iff $h(t_1, t_2) \in \mathrm{Int}\, C$.

Proof: Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $n_1 = n+m$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $R_7 = \mathrm{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_6 = \mathrm{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Set $R_8 = \mathrm{ClosedHypercube}(0_{T_7}, n_1 \mapsto 1)$. Consider g_1 being a function from $T_5 \restriction B$ into $T_5 \restriction R_6$ such that g_1 is a homeomorphism and $g_1{}^\circ(\mathrm{Fr}\, B) = \mathrm{Fr}\, R_6$. Reconsider $g_2 = g_1 \cdot g$ as a function from T_2 into $T_5 \restriction R_6$. Consider f_7 being a function from $T_6 \restriction A$ into $T_6 \restriction R_7$ such that f_7 is a homeomorphism and $f_7{}^\circ(\mathrm{Fr}\, A) = \mathrm{Fr}\, R_7$. Reconsider $f_8 = f_7 \cdot f$ as a function from T_1 into $T_6 \restriction R_7$. Set $O_3 = \mathrm{OpenHypercube}(0_{T_6}, 1)$. Set $O_2 = \mathrm{OpenHypercube}(0_{T_5}, 1)$. Set $O_4 = \mathrm{OpenHypercube}(0_{T_7}, 1)$. Consider H

being a function from $T_7 \restriction R_8$ into $T_7 \restriction C$ such that H is a homeomorphism and $H^\circ(\operatorname{Fr} R_8) = \operatorname{Fr} C$. $\operatorname{Int} R_6 = O_2$. Consider P being a function from $T_1 \times T_2$ into $T_7 \restriction R_8$ such that P is a homeomorphism and for every t_1 and t_2, $f_8(t_1) \in O_3$ and $g_2(t_2) \in O_2$ iff $P(t_1, t_2) \in O_4$. Reconsider $H_1 = H \cdot P$ as a function from $T_1 \times T_2$ into $T_7 \restriction C$. $\operatorname{Int} R_8 = O_4$. If $f(t_1) \in \operatorname{Int} A$ and $g(t_2) \in \operatorname{Int} B$, then $H_1(t_1, t_2) \in \operatorname{Int} C$ by [10, (11), (12)], (12). $P(\langle t_1, t_2 \rangle) \in \operatorname{Int} R_8$. $P(t_1, t_2) \in O_4$. $\operatorname{Int} R_7 = O_3$. $f(t_1) \in \operatorname{Int} A$ by [43, (40)]. $\square$

(19) Let us consider a point p_2 of $\mathcal{E}^n_{\mathrm{T}}$, a point p_1 of $\mathcal{E}^m_{\mathrm{T}}$, r, and s. Suppose

(i) $r > 0$, and

(ii) $s > 0$.

Then there exists a function h from $\mathrm{Tdisk}(p_2, r) \times \mathrm{Tdisk}(p_1, s)$ into $\mathrm{Tdisk}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$ such that

(iii) h is a homeomorphism, and

(iv) $h^\circ(\mathrm{Ball}(p_2, r) \times \mathrm{Ball}(p_1, s)) = \mathrm{Ball}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$.

Proof: Set $T_6 = \mathcal{E}^n_{\mathrm{T}}$. Set $T_5 = \mathcal{E}^m_{\mathrm{T}}$. Set $n_1 = n + m$. Set $T_7 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Reconsider $C_4 = \overline{\mathrm{Ball}}(p_2, r)$ as a non empty subset of T_6. Reconsider $C_3 = \overline{\mathrm{Ball}}(p_1, s)$ as a non empty subset of T_5. Reconsider $C_5 = \overline{\mathrm{Ball}}(0_{T_7}, 1)$ as a non empty subset of T_7. Set $R_7 = \mathrm{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_6 = \mathrm{ClosedHypercube}(0_{T_5}, m \mapsto 1)$. Consider f_7 being a function from $T_6 \restriction C_4$ into $T_6 \restriction R_7$ such that f_7 is a homeomorphism and ${f_7}^\circ(\operatorname{Fr} C_4) = \operatorname{Fr} R_7$. Consider g_1 being a function from $T_5 \restriction C_3$ into $T_5 \restriction R_6$ such that g_1 is a homeomorphism and ${g_1}^\circ(\operatorname{Fr} C_3) = \operatorname{Fr} R_6$. Consider P being a function from $\mathrm{Tdisk}(p_2, r) \times \mathrm{Tdisk}(p_1, s)$ into $\mathrm{Tdisk}(0_{T_7}, 1)$ such that P is a homeomorphism and for every point t_1 of $T_6 \restriction C_4$ and for every point t_2 of $T_5 \restriction C_3$, $f_7(t_1) \in \operatorname{Int} R_7$ and $g_1(t_2) \in \operatorname{Int} R_6$ iff $P(t_1, t_2) \in \operatorname{Int} C_5$. $P^\circ(\mathrm{Ball}(p_2, r) \times \mathrm{Ball}(p_1, s)) \subseteq \mathrm{Ball}(0_{T_7}, 1)$ by [30, (3)], [43, (40)]. Consider x being an object such that $x \in \operatorname{dom} P$ and $P(x) = y$. Consider y_1, y_2 being objects such that $y_1 \in C_4$ and $y_2 \in C_3$ and $x = \langle y_1, y_2 \rangle$. $\square$

(20) Suppose $r > 0$ and $s > 0$ and T_1 and $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Ball}(p_2, r)$ are homeomorphic and T_2 and $\mathcal{E}^m_{\mathrm{T}} \restriction \mathrm{Ball}(p_1, s)$ are homeomorphic. Then $T_1 \times T_2$ and $\mathcal{E}^{n+m}_{\mathrm{T}} \restriction \mathrm{Ball}(0_{\mathcal{E}^{n+m}_{\mathrm{T}}}, 1)$ are homeomorphic.

3. Tietze Extension Theorem

In the sequel T, S denote topological spaces, A denotes a closed subset of T, and B denotes a subset of S.

Now we state the propositions:

(21) Let us consider a non zero natural number n and an element F of $((\text{the carrier of } \mathbb{R}^{\mathbf{1}})^\alpha)^n$. Suppose If $i \in \operatorname{dom} F$, then for every function

h from T into $\mathbb{R}^{\mathbf{1}}$ such that $h = F(i)$ holds h is continuous. Then $\prod^* F$ is continuous, where α is the carrier of T. PROOF: Set $T_4 = \mathcal{E}^n_{\mathrm{T}}$. Set $F_1 = \prod^* F$. For every subset Y of T_4 such that Y is open holds $F_1{}^{-1}(Y)$ is open by [16, (67)], [11, (2)], (1), [19, (17)]. □

(22) Suppose T is normal. Let us consider a function f from $T \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^n_{\mathrm{T}}}, n \mapsto 1)$. Suppose f is continuous. Then there exists a function g from T into $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{ClosedHypercube}(0_{\mathcal{E}^n_{\mathrm{T}}}, n \mapsto 1)$ such that

(i) g is continuous, and

(ii) $g \restriction A = f$.

The theorem is a consequence of (8), (1), and (21).

(23) Suppose T is normal. Let us consider a subset X of $\mathcal{E}^n_{\mathrm{T}}$. Suppose X is compact, non boundary, and convex. Let us consider a function f from $T \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction X$. Suppose f is continuous. Then there exists a function g from T into $\mathcal{E}^n_{\mathrm{T}} \restriction X$ such that

(i) g is continuous, and

(ii) $g \restriction A = f$.

The theorem is a consequence of (22).

Now we state the proposition:

(24) THE FIRST IMPLICATION OF TIETZE EXTENSION THEOREM FOR n-DIMENSIONAL SPACES:
Suppose T is normal. Let us consider a subset X of $\mathcal{E}^n_{\mathrm{T}}$. Suppose

(i) X is compact, non boundary, and convex, and

(ii) B and X are homeomorphic.

Let us consider a function f from $T \restriction A$ into $S \restriction B$. Suppose f is continuous. Then there exists a function g from T into $S \restriction B$ such that

(iii) g is continuous, and

(iv) $g \restriction A = f$.

The theorem is a consequence of (23).

Now we state the proposition:

(25) THE SECOND IMPLICATION OF TIETZE EXTENSION THEOREM FOR n-DIMENSIONAL SPACES:
Let us consider a non empty topological space T and n. Suppose

(i) $n \geqslant 1$, and

(ii) for every topological space S and for every non empty closed subset A of T and for every subset B of S such that there exists a subset X of $\mathcal{E}^n_{\mathrm{T}}$ such that X is compact, non boundary, and convex and B and

X are homeomorphic for every function f from $T{\restriction}A$ into $S{\restriction}B$ such that f is continuous there exists a function g from T into $S{\restriction}B$ such that g is continuous and $g{\restriction}A = f$.

Then T is normal. PROOF: Set $C_1 = [-1, 1]_{\mathrm{T}}$. For every non empty closed subset A of T and for every continuous function f from $T{\restriction}A$ into C_1, there exists a continuous function g from T into $[-1, 1]_{\mathrm{T}}$ such that $g{\restriction}A = f$ by [19, (18), (17)], [11, (2)], [33, (26)]. $\square$

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