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Morley's Trisector Theorem

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Summary. Morley's trisector theorem states that "The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle" [10].

There are many proofs of Morley's trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

MSC: 51M04 03B35

Keywords: Euclidean geometry; Morley's trisector theorem; equilateral triangle

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The notation and terminology used in this paper have been introduced in the following articles: [1], [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. Preliminaries

From now on A, B, C, D, E, F, G denote points of $\mathcal{E}^2_{\mathrm{T}}$.

Now we state the propositions:

(1) $\measuredangle(A, B, A) = 0$.

(2) $0 \leqslant \measuredangle(A, B, C) < 2 \cdot \pi$.

(3) (i) $0 \leqslant \measuredangle(A, B, C) < \pi$, or

(ii) $\measuredangle(A, B, C) = \pi$, or

(iii) $\pi < \measuredangle(A, B, C) < 2 \cdot \pi$.

The theorem is a consequence of (2).

(4) $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \measuredangle(E, A, F)$.

(5) If A, B, C are mutually different and $0 < \measuredangle(A, B, C) < \pi$, then $0 < \measuredangle(B, C, A) < \pi$ and $0 < \measuredangle(C, A, B) < \pi$.

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(6) Suppose A, B, C are mutually different and $\measuredangle(A,B,C) = 0$. Then
  (i) $\measuredangle(B,C,A) = 0$ and $\measuredangle(C,A,B) = \pi$, or
  (ii) $\measuredangle(B,C,A) = \pi$ and $\measuredangle(C,A,B) = 0$ and $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = \pi$.

(7) Suppose A, B, C are mutually different and $\measuredangle(A,B,C) = \pi$. Then
  (i) $\measuredangle(B,C,A) = 0$, and
  (ii) $\measuredangle(C,A,B) = 0$, and
  (iii) $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = \pi$.

(8) If A, B, C are mutually different and $\measuredangle(A,B,C) > \pi$, then $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = 5 \cdot \pi$.

Let us assume that $\measuredangle(C,B,A) < \pi$. Now we state the propositions:

(9) $0 \leqslant$ area of $\triangle(A,B,C)$. The theorem is a consequence of (2).

(10) $0 \leqslant \varnothing_{\lhd}(A,B,C)$. The theorem is a consequence of (9).

2. Morley's Theorem

Now we state the propositions:

(11) Suppose A, F, C form a triangle and $\measuredangle(C,F,A) < \pi$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $(\measuredangle(A,C,B)/3) + (\measuredangle(B,A,C)/3) + (\measuredangle(C,B,A)/3) = \pi/3$.
Then $|A - F| \cdot \sin((\pi/3) - (\measuredangle(C,B,A)/3)) = |A - C| \cdot \sin(\measuredangle(A,C,B)/3)$.

(12) Suppose A, B, C form a triangle and A, F, C form a triangle and $\measuredangle(C,F,A) < \pi$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $(\measuredangle(A,C,B)/3) + (\measuredangle(B,A,C)/3) + (\measuredangle(C,B,A)/3) = \pi/3$ and $\sin((\pi/3) - (\measuredangle(C,B,A)/3)) \neq 0$. Then $|A - F| = 4 \cdot \varnothing_{\lhd}(A,B,C) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin((\pi/3) + (\measuredangle(C,B,A)/3)) \cdot \sin(\measuredangle(A,C,B)/3)$. The theorem is a consequence of (11).

(13) Suppose C, A, B form a triangle and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then $\measuredangle(E,A,F) = \measuredangle(B,A,C)/3$. Proof: $\measuredangle(E,A,F) \neq 4 \cdot \pi + (\measuredangle(B,A,C)/3)$ by [17, (5)], (2), [7, (30)]. $\measuredangle(E,A,F) \neq 2 \cdot \pi + (\measuredangle(B,A,C)/3)$ by (2), [7, (30)]. $\square$

(14) Suppose C, A, B form a triangle and $\measuredangle(A,C,B) < \pi$ and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then $(\pi/3) + (\measuredangle(A,C,B)/3) + ((\pi/3) + (\measuredangle(C,B,A)/3)) + \measuredangle(E,A,F) = \pi$. The theorem is a consequence of (13).

(15) If A, C, B form a triangle, then $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. The theorem is a consequence of (2).

(16) Suppose A, B, C form a triangle and A, B, E form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and A, F, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,B) < \pi$. Then $|F - E| = 4 \cdot \varnothing_{\text{◖}}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$.

PROOF: $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. $\sin((\pi/3) - (\measuredangle(C,B,A)/3)) \neq 0$. $0 < \measuredangle(A,C,B)$. $\measuredangle(C,B,A) < \pi$. $0 < \measuredangle(A,C,B) < \pi$ and A, C, B are mutually different. $\measuredangle(B,A,C) < \pi$. $0 < \measuredangle(B,A,E) < \pi$. $\measuredangle(A,E,B) < \pi$. $0 < \measuredangle(F,A,C) < \pi$. $\measuredangle(C,F,A) < \pi$. F, A, E form a triangle by [19, (4)], (5), [17, (5)], [7, (31)]. $|A - F| = \varnothing_{\text{◖}}(A,B,C) \cdot 4 \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin((\pi/3) + (\measuredangle(C,B,A)/3)) \cdot \sin(\measuredangle(A,C,B)/3)$. $(\pi/3) + (\measuredangle(A,C,B)/3) + ((\pi/3) + (\measuredangle(C,B,A)/3)) + \measuredangle(E,A,F) = \pi$. $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos\measuredangle(E,A,F)$. □

(17) Suppose A, B, C form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$. Then A, B, E form a triangle. The theorem is a consequence of (1) and (2).

(18) Suppose A, B, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then A, F, C form a triangle. The theorem is a consequence of (1) and (2).

(19) Suppose A, B, C form a triangle and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Then C, G, B form a triangle. The theorem is a consequence of (1) and (2).

Let us assume that A, B, C form a triangle and $\measuredangle(A,C,B) < \pi$ and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Now we state the propositions:

(20) (i) $|F - E| = 4 \cdot \varnothing_{\text{◖}}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$, and

(ii) $|G - F| = 4 \cdot \varnothing_{\text{◖}}(C,A,B) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3)$, and

(iii) $|E - G| = 4 \cdot \varnothing_{\text{◖}}(B,C,A) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

(21) (i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.

The theorem is a consequence of (20).

(22) MORLEY'S TRISECTOR THEOREM:
Suppose A, B, C form a triangle and $\measuredangle(A,B,C) < \pi$ and $\measuredangle(E,C,A) = \measuredangle(B,C,A)/3$ and $\measuredangle(C,A,E) = \measuredangle(C,A,B)/3$ and $\measuredangle(A,B,F) = \measuredangle(A,B,C)/3$ and $\measuredangle(F,A,B) = \measuredangle(C,A,B)/3$ and $\measuredangle(B,C,G) = \measuredangle(B,C,A)/3$ and $\measuredangle(G,B,C) = \measuredangle(A,B,C)/3$. Then

(i) $|F-E| = |G-F|$, and

(ii) $|F-E| = |E-G|$, and

(iii) $|G-F| = |E-G|$.

The theorem is a consequence of (21).

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Flexary Operations[1]

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Summary. In this article we introduce necessary notation and definitions to prove the Euler's Partition Theorem according to H.S. Wilf's lecture notes [31]. Our aim is to create an environment which allows to formalize the theorem in a way that is as similar as possible to the original informal proof.

Euler's Partition Theorem is listed as item `#45` from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` [30].

MSC: 11B99 03B35

Keywords: summation method; flexary plus; matrix generalization

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [8], [15], [27], [13], [14], [23], [9], [10], [7], [25], [24], [3], [4], [19], [5], [22], [32], [33], [11], [21], [28], [18], and [12].

1. Auxiliary Facts about Finite Sequences Concatenation

From now on x, y denote objects, D, D_1, D_2 denote non empty sets, i, j, k, m, n denote natural numbers, f, g denote finite sequences of elements of D^*, f_1 denotes a finite sequence of elements of ${D_1}^*$, and f_2 denotes a finite sequence of elements of ${D_2}^*$.

Now we state the propositions:

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(1) Let us consider a function yielding function F, and an object a. Then $a \in \operatorname{Values} F$ if and only if there exists x and there exists y such that $x \in \operatorname{dom} F$ and $y \in \operatorname{dom}(F(x))$ and $a = F(x)(y)$.

(2) Let us consider a set D, and finite sequences f, g of elements of D^*. Then $\operatorname{Values} f \frown g = \operatorname{Values} f \cup \operatorname{Values} g$.

Proof: Set $F = f \frown g$. $\operatorname{Values} f \subseteq \operatorname{Values} F$ by (1), [6, (26)]. $\operatorname{Values} g \subseteq \operatorname{Values} F$ by (1), [6, (28)]. $\operatorname{Values} F \subseteq \operatorname{Values} f \cup \operatorname{Values} g$ by (1), [6, (25)]. □

(3) The concatenation of $D \odot f \frown g$ = (the concatenation of $D \odot f$) $\frown$ (the concatenation of $D \odot g$).

(4) rng(the concatenation of $D \odot f$) $= \operatorname{Values} f$.

Proof: Set D_3 = the concatenation of D. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f of elements of D^* such that $\operatorname{len} f = \$_1$ holds $\operatorname{rng}(D_3 \odot f) = \operatorname{Values} f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(5) If $f_1 = f_2$, then the concatenation of $D_1 \odot f_1$ = the concatenation of $D_2 \odot f_2$.

Proof: Set C = the concatenation of D_2. Set N = the concatenation of D_1. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f_4 of elements of ${D_1}^*$ for every finite sequence f_3 of elements of ${D_2}^*$ such that $\$_1 = \operatorname{len} f_4$ and $f_4 = f_3$ holds $N \odot f_4 = C \odot f_3$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(6) $i \in$ dom(the concatenation of $D \odot f$) if and only if there exists n and there exists k such that $n+1 \in \operatorname{dom} f$ and $k \in \operatorname{dom}(f(n+1))$ and $i = k + $ len(the concatenation of $D \odot f \restriction n$).

Proof: Set D_3 = the concatenation of D. Define $\mathcal{P}$[natural number] $\equiv$ for every i for every finite sequence f of elements of D^* such that $\operatorname{len} f = \$_1$ holds $i \in \operatorname{dom}(D_3 \odot f)$ iff there exists n and there exists k such that $n+1 \in \operatorname{dom} f$ and $k \in \operatorname{dom}(f(n+1))$ and $i = k + \operatorname{len}(D_3 \odot f \restriction n)$. $\mathcal{P}[0]$. If $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[j]$ from [4, Sch. 2]. □

(7) Suppose $i \in$ dom(the concatenation of $D \odot f$). Then

   (i) (the concatenation of $D \odot f$)(i) = (the concatenation of $D \odot f \frown g$)(i), and

   (ii) (the concatenation of $D \odot f$)(i) = (the concatenation of $D \odot g \frown f$)$(i+$ len(the concatenation of $D \odot g$)).

The theorem is a consequence of (3).

(8) Suppose $k \in \operatorname{dom}(f(n+1))$. Then $f(n+1)(k)$ = (the concatenation of

$D \odot f)(k + \mathrm{len}(\text{the concatenation of } D \odot f {\restriction} n))$. The theorem is a consequence of (3).

2. Flexary Plus

From now on f denotes a complex-valued function and g, h denote complex-valued finite sequences.

Let us consider k and n. Let f, g be complex-valued functions. The functor $(f,k)+\ldots+(g,n)$ yielding a complex number is defined by

(Def. 1) (i) $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$, then $it = \sum(h {\restriction} (n -' k + 1))$, **if** $f = g$ and $k \leqslant n$,

(ii) $it = 0$, **otherwise**.

Now we state the propositions:

(9) Suppose $k \leqslant n$. Then there exists h such that

(i) $(f,k)+\ldots+(f,n) = \sum h$, and

(ii) $\mathrm{len}\, h = n -' k + 1$, and

(iii) $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$.

Proof: Define $\mathcal{P}(\text{natural number}) = f(k + \$_1 - 1)$. Set $n_3 = n -' k + 1$. Consider p being a finite sequence such that $\mathrm{len}\, p = n_3$ and for every i such that $i \in \mathrm{dom}\, p$ holds $p(i) = \mathcal{P}(i)$ from [6, Sch. 2]. $\mathrm{rng}\, p \subseteq \mathbb{C}$. $p(1+0) = f(k+0)$ and ... and $p(1 + (n -' k)) = f(k + (n -' k))$ by [4, (11)], [26, (25)]. □

(10) If $(f,k)+\ldots+(f,n) \neq 0$, then there exists i such that $k \leqslant i \leqslant n$ and $i \in \mathrm{dom}\, f$.

Proof: Consider h such that $(f,k)+\ldots+(f,n) = \sum h$ and $\mathrm{len}\, h = n -' k + 1$ and $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$. $\mathrm{rng}\, h \subseteq \{0\}$ by [26, (25)], [4, (11)]. □

(11) $(f,k)+\ldots+(f,k) = f(k)$. The theorem is a consequence of (9).

(12) If $k \leqslant n+1$, then $(f,k)+\ldots+(f,(n+1)) = ((f,k)+\ldots+(f,n)) + f(n+1)$. The theorem is a consequence of (11) and (9).

(13) If $k \leqslant n$, then $(f,k)+\ldots+(f,n) = f(k) + ((f,(k+1))+\ldots+(f,n))$. The theorem is a consequence of (11) and (9).

(14) If $k \leqslant m \leqslant n$, then $((f,k)+\ldots+(f,m)) + ((f,(m+1))+\ldots+(f,n)) = (f,k)+\ldots+(f,n)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv ((f,k)+\ldots+(f,m)) + ((f,(m+1))+\ldots+(f,(m+\$_1))) = (f,k)+\ldots+(f,(m+\$_1))$. $\mathcal{P}[0]$ by [4, (13)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (11)], (12). $\mathcal{P}[i]$ from [4, Sch. 2]. □

(15) If $k > \operatorname{len} h$, then $(h, k) + \ldots + (h, n) = 0$. The theorem is a consequence of (9).

(16) If $n \geqslant \operatorname{len} h$, then $(h, k) + \ldots + (h, n) = (h, k) + \ldots + (h, \operatorname{len} h)$. The theorem is a consequence of (15) and (12).

(17) $(h, 0) + \ldots + (h, k) = (h, 1) + \ldots + (h, k)$. The theorem is a consequence of (13).

(18) $(h, 1) + \ldots + (h, \operatorname{len} h) = \sum h$. The theorem is a consequence of (9).

(19) $(g \frown h, k) + \ldots + (g \frown h, n) = ((g, k) + \ldots + (g, n)) + ((h, (k -' \operatorname{len} g)) + \ldots + (h, (n -' \operatorname{len} g)))$. The theorem is a consequence of (11), (15), (16), (17), and (14).

Let us consider n and k. Let f be a real-valued finite sequence. One can check that $(f, k) + \ldots + (f, n)$ is real.

Let f be a natural-valued finite sequence. Note that $(f, k) + \ldots + (f, n)$ is natural.

Let f be a complex-valued function. Assume $\operatorname{dom} f \cap \mathbb{N}$ is finite. The functor $(f, n) + \ldots$ yielding a complex number is defined by

(Def. 2) for every k such that for every i such that $i \in \operatorname{dom} f$ holds $i \leqslant k$ holds $it = (f, n) + \ldots + (f, k)$.

Let us consider h. One can check that the functor $(h, n) + \ldots$ yields a complex number and is defined by the term

(Def. 3) $(h, n) + \ldots + (h, \operatorname{len} h)$.

Let n be a natural number and h be a natural-valued finite sequence. Let us note that $(h, n) + \ldots$ is natural.

Now we state the propositions:

(20) Let us consider a finite, complex-valued function f. Then $f(n) + (f, (n + 1)) + \ldots = (f, n) + \ldots$. The theorem is a consequence of (13).

(21) $\sum h = (h, 1) + \ldots$.

(22) $\sum h = h(1) + (h, 2) + \ldots$. The theorem is a consequence of (18) and (20).

The scheme *TT* deals with complex-valued finite sequences f, g and natural numbers a, b and non zero natural numbers n, k and states that

(Sch. 1) $(f, a) + \ldots = (g, b) + \ldots$

provided

- for every j, $(f, (a + j \cdot n)) + \ldots + (f, (a + j \cdot n + (n -' 1))) = (g, (b + j \cdot k)) + \ldots + (g, (b + j \cdot k + (k -' 1)))$.

3. POWER FUNCTION

Let r be a real number and f be a real-valued function. The functor r^f yielding a real-valued function is defined by

(Def. 4) $\operatorname{dom} it = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $it(x) = r^{f(x)}$.

Let n be a natural number and f be a natural-valued function. One can verify that n^f is natural-valued.

Let r be a real number and f be a real-valued finite sequence. One can check that r^f is finite sequence-like and r^f is $(\operatorname{len} f)$-element.

Let f be a one-to-one, natural-valued function. Observe that $(2+n)^f$ is one-to-one.

(23) Let us consider real numbers r, s. Then $r^{\langle s \rangle} = \langle r^s \rangle$.

(24) Let us consider a real number r, and real-valued finite sequences f, g. Then $r^{f \frown g} = r^f \frown r^g$.

PROOF: Set $f_5 = f \frown g$. Set $r_2 = r^f$. Set $r_3 = r^g$. For every i such that $1 \leqslant i \leqslant \operatorname{len} f_5$ holds $r^{f_5}(i) = (r_2 \frown r_3)(i)$ by [26, (25)], [6, (25)]. $\square$

(25) Let us consider a real-valued function f, and a function g. Then $2^f \cdot g = 2^{f \cdot g}$. PROOF: Set $h = 2^f$. Set $f_5 = f \cdot g$. $\operatorname{dom}(h \cdot g) \subseteq \operatorname{dom} 2^{f_5}$ by [9, (11)]. $\operatorname{dom} 2^{f_5} \subseteq \operatorname{dom}(h \cdot g)$ by [9, (11)]. For every x such that $x \in \operatorname{dom} 2^{f_5}$ holds $(h \cdot g)(x) = 2^{f_5}(x)$ by [9, (11), (13)]. $\square$

(26) Let us consider an increasing, natural-valued finite sequence f. If $n > 1$, then $n^f(1) + (n^f, 2) + \ldots < 2 \cdot n^{f(\operatorname{len} f)}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequence f such that $n > 1$ and $f(\operatorname{len} f) \leqslant \$_1$ and $f \neq \emptyset$ holds $\sum n^f < 2 \cdot n^{f(\operatorname{len} f)}$. For every natural-valued finite sequence f such that $n > 1$ and $\operatorname{len} f = 1$ holds $\sum n^f < 2 \cdot n^{f(\operatorname{len} f)}$ by [26, (25)], [19, (83)], [6, (40)], [11, (73)]. $\mathcal{P}[0]$ by [26, (25)], [4, (25)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (8), (25), (13)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $\sum n^f = n^f(1) + (n^f, 2) + \ldots$. $\square$

(27) Let us consider increasing, natural-valued finite sequences f_1, f_2. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $f_1 = f_2$. PROOF: For every natural-valued finite sequence f such that $n > 1$ and $\sum n^f \leqslant 0$ holds $f = \emptyset$ by [11, (85)], [19, (83)]. Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequences f_1, f_2 such that $n > 1$ and $\sum n^{f_1} \leqslant \$_1$ and $\sum n^{f_1} = \sum n^{f_2}$ holds $f_1 = f_2$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by (21), (22), [4, (8)], [11, (72)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = \sum n^{f_1}$. $n^{f_2}(1) + (n^{f_2}, 2) + \ldots = \sum n^{f_2}$. $\square$

(28) Let us consider a natural-valued function f. If $n > 1$, then $\operatorname{Coim}(n^f, n^k) = \operatorname{Coim}(f, k)$. PROOF: $\operatorname{Coim}(n^f, n^k) \subseteq \operatorname{Coim}(f, k)$ by [17, (30)]. $\square$

(29) Let us consider natural-valued functions f_1, f_2. Suppose $n > 1$. Then f_1 and f_2 are fiberwise equipotent if and only if n^{f_1} and n^{f_2} are fiberwise equipotent. PROOF: If f_1 and f_2 are fiberwise equipotent, then n^{f_1} and n^{f_2} are fiberwise equipotent by [9, (72)], [17, (30)], (28). For every object x, $\overline{\overline{\mathrm{Coim}(f_1, x)}} = \overline{\overline{\mathrm{Coim}(f_2, x)}}$ by [9, (72)], [17, (30)], (28). □

(30) Let us consider one-to-one, natural-valued finite sequences f_1, f_2. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $\mathrm{rng}\, f_1 = \mathrm{rng}\, f_2$.

PROOF: Reconsider $F_1 = f_1$, $F_2 = f_2$ as a finite sequence of elements of $\mathbb{R}$. Set $s_1 = \mathrm{sort_a}\, F_1$. Set $s_2 = \mathrm{sort_a}\, F_2$. n^{F_1} and n^{s_1} are fiberwise equipotent. n^{F_2} and n^{s_2} are fiberwise equipotent. For every extended reals e_1, e_2 such that e_1, $e_2 \in \mathrm{dom}\, s_1$ and $e_1 < e_2$ holds $s_1(e_1) < s_1(e_2)$ by [16, (2)], [2, (77)]. For every extended reals e_1, e_2 such that e_1, $e_2 \in \mathrm{dom}\, s_2$ and $e_1 < e_2$ holds $s_2(e_1) < s_2(e_2)$ by [16, (2)], [2, (77)]. $\sum n^{s_1} = n^{s_1}(1) + (n^{s_1}, 2) + \ldots$. $\sum n^{f_1} = n^{f_1}(1) + (n^{f_1}, 2) + \ldots$. $\sum n^{s_1} = \sum n^{s_2}$. $n^{s_1}(1) + (n^{s_1}, 2) + \ldots = n^{s_2}(1) + (n^{s_2}, 2) + \ldots$ and s_1 is increasing and natural-valued. □

(31) There exists an increasing, natural-valued finite sequence f such that $n = 2^f(1) + (2^f, 2) + \ldots$.

PROOF: Set $D = \mathrm{digits}(n, 2)$. Consider d being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, d = \mathrm{dom}\, D$ and for every natural number i such that $i \in \mathrm{dom}\, d$ holds $d(i) = D(i) \cdot 2^i$ and $\mathrm{value}(D, 2) = \sum d$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \mathrm{len}\, d$, then there exists an increasing, natural-valued finite sequence f such that ($\mathrm{len}\, f = 0$ or $f(\mathrm{len}\, f) < \$_1$) and $\sum 2^f = \sum (d{\restriction}\$_1)$. $\mathcal{P}$[(0 **qua** natural number)] by [11, (72)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [29, (86)], [20, (65)], [4, (25), (23)]. $\mathcal{P}[i]$ from [4, Sch. 2]. Consider f being an increasing, natural-valued finite sequence such that $\mathrm{len}\, f = 0$ or $f(\mathrm{len}\, f) < \mathrm{len}\, d$ and $\sum 2^f = \sum (d{\restriction}\mathrm{len}\, d)$. $\sum 2^f = 2^f(1) + (2^f, 2) + \ldots$. □

4. VALUE-BASED FUNCTION (RE)ORGANIZATION

Let o be a function yielding function and x, y be objects. The functor $o_{x,y}$ yielding a set is defined by the term

(Def. 5) $o(x)(y)$.

Let F be a function yielding function. We say that F is double one-to-one if and only if

(Def. 6) for every objects x_1, x_2, y_1, y_2 such that $x_1 \in \mathrm{dom}\, F$ and $y_1 \in \mathrm{dom}(F(x_1))$ and $x_2 \in \mathrm{dom}\, F$ and $y_2 \in \mathrm{dom}(F(x_2))$ and $F_{x_1,y_1} = F_{x_2,y_2}$ holds $x_1 = x_2$ and $y_1 = y_2$.

Let D be a set. Observe that every finite sequence of elements of D^* which is empty is also double one-to-one and there exists a function yielding function which is double one-to-one and there exists a finite sequence of elements of D^* which is double one-to-one.

Let F be a double one-to-one, function yielding function and x be an object. One can check that $F(x)$ is one-to-one.

Let F be a one-to-one function. One can check that $\langle F\rangle$ is double one-to-one.

Now we state the propositions:

(32) Let us consider a function yielding function f. Then f is double one-to-one if and only if for every x, $f(x)$ is one-to-one and for every x and y such that $x \neq y$ holds $\mathrm{rng}(f(x))$ misses $\mathrm{rng}(f(y))$.

(33) Let us consider a set D, and double one-to-one finite sequences f_1, f_2 of elements of D^*. Suppose $\mathrm{Values}\, f_1$ misses $\mathrm{Values}\, f_2$. Then $f_1 \frown f_2$ is double one-to-one. The theorem is a consequence of (1).

Let D be a finite set.

A double reorganization of D is a double one-to-one finite sequence of elements of D^* and is defined by

(Def. 7) $\mathrm{Values}\, it = D$.

Now we state the propositions:

(34) (i) $\emptyset$ is a double reorganization of $\emptyset$, and

(ii) $\langle\emptyset\rangle$ is a double reorganization of $\emptyset$.

(35) Let us consider a finite set D, and a one-to-one, onto finite sequence F of elements of D. Then $\langle F\rangle$ is a double reorganization of D.

(36) Let us consider finite sets D_1, D_2. Suppose D_1 misses D_2. Let us consider a double reorganization o_1 of D_1, and a double reorganization o_2 of D_2. Then $o_1 \frown o_2$ is a double reorganization of $D_1 \cup D_2$. The theorem is a consequence of (33) and (2).

(37) Let us consider a finite set D, a double reorganization o of D, and a one-to-one finite sequence F. Suppose $i \in \mathrm{dom}\, o$ and $\mathrm{rng}\, F \cap D \subseteq \mathrm{rng}(o(i))$. Then $o +\cdot (i, F)$ is a double reorganization of $\mathrm{rng}\, F \cup (D \setminus \mathrm{rng}(o(i)))$.

Proof: Set $r_1 = \mathrm{rng}\, F$. Set $o_3 = o(i)$. Set $r_4 = \mathrm{rng}\, o_3$. Set $o_4 = o +\cdot (i, F)$. $\mathrm{rng}\, o_4 \subseteq (r_1 \cup (D \setminus r_4))^*$ by [7, (31), (32)]. o_4 is double one-to-one by [7, (32)], (1). $\mathrm{Values}\, o_4 \subseteq r_1 \cup (D \setminus r_4)$ by (1), [7, (31), (32)]. $D \setminus r_4 \subseteq \mathrm{Values}\, o_4$ by (1), [7, (32)]. $r_1 \subseteq \mathrm{Values}\, o_4$. □

Let D be a finite set and n be a non zero natural number. One can check that there exists a double reorganization of D which is n-element.

Let D be a finite, natural-membered set, o be a double reorganization of D, and x be an object. One can verify that $o(x)$ is natural-valued.

Now we state the propositions:

(38) Let us consider a non empty finite sequence F, and a finite function G. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$. Then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{dom} G$ such that for every n, $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\operatorname{len}(o(n))})$.

Proof: Set $D = \operatorname{dom} G$. Set $d =$ the one-to-one ,onto finite sequence of elements of D. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \overline{\overline{G}}$, then there exists a $(\operatorname{len} F)$-element double reorganization o of $d^{\circ}(\operatorname{Seg} \$_1)$ such that for every k, $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\operatorname{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(39) Let us consider a non empty finite sequence F, and a finite sequence G. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$. Then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{dom} G$ such that for every n, $o(n)$ is increasing and $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\operatorname{len}(o(n))})$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} G$, then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{Seg} \$_1$ such that for every k, $o(k)$ is increasing and $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\operatorname{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

Let f be a finite function, o be a double reorganization of $\operatorname{dom} f$, and x be an object. One can check that $f \cdot o(x)$ is finite sequence-like and there exists a finite sequence which is complex-functions-valued and finite sequence-yielding.

Let f be a function yielding function and g be a function. We introduce $g \odot f$ as a synonym of $[g, f]$.

One can check that $g \odot f$ is function yielding.

Let f be a $((\operatorname{dom} g)^*)$-valued finite sequence. One can check that $g \odot f$ is finite sequence-yielding.

Let x be an object. Let us note that $(g \odot f)(x)$ is $(\operatorname{len}(f(x)))$-element.

Let f be a function yielding finite sequence. One can verify that $g \odot f$ is finite sequence-like and $g \odot f$ is $(\operatorname{len} f)$-element.

Let f be a function yielding function and g be a complex-valued function. One can check that $g \odot f$ is complex-functions-valued.

Let g be a natural-valued function. One can check that $g \odot f$ is natural-functions-valued.

Let us consider a function yielding function f and a function g. Now we state the propositions:

(40) $\operatorname{Values} g \odot f = g^{\circ}(\operatorname{Values} f)$.

Proof: Set $g_3 = g \odot f$. $\operatorname{Values} g_3 \subseteq g^{\circ}(\operatorname{Values} f)$ by (1), [9, (11), (12)]. Consider b being an object such that $b \in \operatorname{dom} g$ and $b \in \operatorname{Values} f$ and

$g(b) = a$. Consider x, y being objects such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom}(f(x))$ and $b = f(x)(y)$. □

(41) $(g \odot f)(x) = g \cdot f(x)$.

Now we state the proposition:

(42) Let us consider a function yielding function f, a finite sequence g, and objects x, y. Then $(g \odot f)_{x,y} = g(f_{x,y})$. The theorem is a consequence of (41).

Let f be a complex-functions-valued, finite sequence-yielding function. The functor $\sum f$ yielding a complex-valued function is defined by

(Def. 8) $\operatorname{dom} \mathit{it} = \operatorname{dom} f$ and for every set x, $\mathit{it}(x) = \sum(f(x))$.

Let f be a complex-functions-valued, finite sequence-yielding finite sequence. One can verify that $\sum f$ is finite sequence-like and $\sum f$ is $(\operatorname{len} f)$-element.

Let f be a natural-functions-valued, finite sequence-yielding function. One can verify that $\sum f$ is natural-valued.

Let f, g be complex-functions-valued finite sequences. One can check that $f \frown g$ is complex-functions-valued.

Let f, g be extended real-valued finite sequences. One can verify that $f \frown g$ is extended real-valued.

Let f be a complex-functions-valued function and X be a set. One can check that $f{\restriction}X$ is complex-functions-valued.

Let f be a finite sequence-yielding function. One can check that $f{\restriction}X$ is finite sequence-yielding.

Let F be a complex-valued function. One can check that $\langle F \rangle$ is complex-functions-valued.

Let us consider finite sequences f, g. Now we state the propositions:

(43) If $f \frown g$ is finite sequence-yielding, then f is finite sequence-yielding and g is finite sequence-yielding.

(44) If $f \frown g$ is complex-functions-valued, then f is complex-functions-valued and g is complex-functions-valued.

Now we state the propositions:

(45) Let us consider a complex-valued finite sequence f. Then $\sum \langle f \rangle = \langle \sum f \rangle$.

(46) Let us consider complex-functions-valued, finite sequence-yielding finite sequences f, g. Then $\sum(f \frown g) = \sum f \frown \sum g$.
Proof: For every i such that $1 \leqslant i \leqslant \operatorname{len} f + \operatorname{len} g$ holds $(\sum(f \frown g))(i) = (\sum f \frown \sum g)(i)$ by [26, (25)], [6, (25)]. □

(47) Let us consider a complex-valued finite sequence f, and a double reorganization o of $\operatorname{dom} f$. Then $\sum f = \sum \sum(f \odot o)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every complex-valued finite sequence f for every double reorganization o of $\operatorname{dom} f$ such that $\operatorname{len} f = \$_1$ holds $\sum f = \sum\sum(f \odot o)$. $\mathcal{P}[0]$ by [26, (29)], [11, (72)], [23, (11)], [11, (81)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (11)], [26, (25)], (1), [12, (116)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

Let us note that $\mathbb{N}^*$ is natural-functions-membered and $\mathbb{C}^*$ is complex-functions-membered.

Now we state the proposition:

(48) Let us consider a finite sequence f of elements of $\mathbb{C}^*$. Then $\sum$(the concatenation of $\mathbb{C} \odot f) = \sum\sum f$.

Proof: Set C = the concatenation of $\mathbb{C}$. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f of elements of $\mathbb{C}^*$ such that $\operatorname{len} f = \$_1$ holds $\sum(C \odot f) = \sum\sum f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (46), (45). $\mathcal{P}[i]$ from [4, Sch. 2]. □

Let f be a finite function.

A valued reorganization of f is a double reorganization of $\operatorname{dom} f$ and is defined by

(Def. 9) for every n, there exists x such that $x = f(it_{n,1})$ and ... and $x = f(it_{n,\operatorname{len}(it(n))})$ and for every natural numbers n_1, n_2, i_1, i_2 such that $i_1 \in \operatorname{dom}(it(n_1))$ and $i_2 \in \operatorname{dom}(it(n_2))$ and $f(it_{n_1,i_1}) = f(it_{n_2,i_2})$ holds $n_1 = n_2$.

Now we state the propositions:

(49) Let us consider a finite function f, and a valued reorganization o of f. Then

(i) $\operatorname{rng}((f \odot o)(n)) = \emptyset$, or

(ii) $\operatorname{rng}((f \odot o)(n)) = \{f(o_{n,1})\}$ and $1 \in \operatorname{dom}(o(n))$.

Proof: Consider y such that $y \in \operatorname{rng}((f \odot o)(n))$. Consider x such that $x \in \operatorname{dom}((f \odot o)(n))$ and $(f \odot o)(n)(x) = y$. $n \in \operatorname{dom}(f \odot o)$. Consider w being an object such that $w = f(o_{n,1})$ and ... and $w = f(o_{n,\operatorname{len}(o(n))})$. $\operatorname{rng}((f \odot o)(n)) \subseteq \{f(o_{n,1})\}$ by [9, (11), (12)], [26, (25)]. □

(50) Let us consider a finite sequence f, and valued reorganizations o_1, o_2 of f. Suppose $\operatorname{rng}((f \odot o_1)(i)) = \operatorname{rng}((f \odot o_2)(i))$. Then $\operatorname{rng}(o_1(i)) = \operatorname{rng}(o_2(i))$.

(51) Let us consider a finite sequence f, a complex-valued finite sequence g, and double reorganizations o_1, o_2 of $\operatorname{dom} g$. Suppose o_1 is a valued reorganization of f and o_2 is a valued reorganization of f and $\operatorname{rng}((f \odot o_1)(i)) = \operatorname{rng}((f \odot o_2)(i))$. Then $(\sum(g \odot o_1))(i) = (\sum(g \odot o_2))(i)$. The theorem is a consequence of (41).

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Euler's Partition Theorem[1]

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Summary. In this article we prove the Euler's Partition Theorem which states that the number of integer partitions with odd parts equals the number of partitions with distinct parts. The formalization follows H.S. Wilf's lecture notes [28] (see also [1]).

Euler's Partition Theorem is listed as item `#45` from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` [27].

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The notation and terminology used in this paper have been introduced in the following articles: [22], [2], [3], [17], [7], [16], [19], [14], [15], [23], [9], [10], [24], [5], [18], [6], [11], [29], [12], [26], and [13].

1. Preliminaries

From now on x, y denote objects and i, j, k, m, n denote natural numbers.

Let r be an extended real number. One can verify that $\langle r \rangle$ is extended real-valued and $\langle r \rangle$ is decreasing, increasing, non-decreasing, and non-increasing.

Now we state the proposition:

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(1) Let us consider non-decreasing, extended real-valued finite sequences f, g. If $f(\operatorname{len} f) \leqslant g(1)$, then $f \frown g$ is non-decreasing.
PROOF: Set $f_3 = f \frown g$. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} f_3$ and $e_1 \leqslant e_2$ holds $f_3(e_1) \leqslant f_3(e_2)$ by [7, (25)], [25, (25)]. □

Let R be a binary relation. We say that R is odd-valued if and only if

(Def. 1) $\operatorname{rng} R \subseteq \mathbb{N}_{\mathrm{odd}}$.

(2) $n \in \mathbb{N}_{\mathrm{odd}}$ if and only if n is odd.

Let us note that every binary relation which is odd-valued is also non-zero and natural-valued.

Let F be a function. Observe that F is odd-valued if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every x such that $x \in \operatorname{dom} F$ holds $F(x)$ is an odd natural number.

One can check that every binary relation which is empty is also odd-valued.

Let i be an odd natural number. Let us observe that $\langle i \rangle$ is odd-valued.

Let f, g be odd-valued finite sequences. Note that $f \frown g$ is odd-valued and every binary relation which is $\mathbb{N}_{\mathrm{odd}}$-valued is also odd-valued.

Let n be a natural number. A partition of n is a non-zero, non-decreasing, natural-valued finite sequence and is defined by

(Def. 3) $\sum it = n$.

Now we state the proposition:

(3) $\emptyset$ is a partition of 0.

Let n be a natural number. Observe that there exists a partition of n which is odd-valued and there exists a partition of n which is one-to-one.

Let us observe that sethood property holds for partitions of n.

Let f be an odd-valued finite sequence.

An odd organization of f is a valued reorganization of f and is defined by

(Def. 4) $2 \cdot n - 1 = f(it_{n,1})$ and ... and $2 \cdot n - 1 = f(it_{n,\operatorname{len}(it(n))})$.

(4) Let us consider an odd-valued finite sequence f, and a double reorganization o of $\operatorname{dom} f$. Suppose for every n, $2 \cdot n - 1 = f(o_{n,1})$ and ... and $2 \cdot n - 1 = f(o_{n,\operatorname{len}(o(n))})$. Then o is an odd organization of f.
PROOF: For every n, there exists x such that $x = f(o_{n,1})$ and ... and $x = f(o_{n,\operatorname{len}(o(n))})$. For every natural numbers n_1, n_2, i_1, i_2 such that $i_1 \in \operatorname{dom}(o(n_1))$ and $i_2 \in \operatorname{dom}(o(n_2))$ and $f(o_{n_1,i_1}) = f(o_{n_2,i_2})$ holds $n_1 = n_2$ by [25, (25)]. □

(5) Let us consider an odd-valued finite sequence f, a complex-valued finite sequence g, and double reorganizations o_1, o_2 of $\operatorname{dom} g$. Suppose o_1 is an odd organization of f and o_2 is an odd organization of f. Then $(\sum(g \odot o_1))(i) = (\sum(g \odot o_2))(i)$.

PROOF: For every double reorganizations o_1, o_2 of $\operatorname{dom} g$ such that o_1 is an odd organization of f and o_2 is an odd organization of f holds $\operatorname{rng}((f \odot o_1)(n)) \subseteq \operatorname{rng}((f \odot o_2)(n))$ by [19, (49), (1)], [25, (29), (25)]. □

(6) Let us consider a partition p of n. Then there exists an odd-valued finite sequence O and there exists a natural-valued finite sequence a such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ and $p(1) = O(1) \cdot 2^{a(1)}$ and ... and $p(\operatorname{len} p) = O(\operatorname{len} p) \cdot 2^{a(\operatorname{len} p)}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every i and j such that $p(\$_1) = 2^i \cdot (2 \cdot j + 1)$ holds $\$_2 = \langle 2 \cdot j + 1, i \rangle$. For every k such that $k \in \operatorname{Seg} \operatorname{len} p$ there exists x such that $\mathcal{P}[k, x]$ by [20, (1)], [4, (4)]. Consider O_3 being a finite sequence such that $\operatorname{dom} O_3 = \operatorname{Seg} \operatorname{len} p$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} p$ holds $\mathcal{P}[k, O_3(k)]$ from [7, Sch. 1]. Define $\mathcal{Q}(\text{object}) = O_3(\$_1)_{\mathbf{1}}$. Consider O being a finite sequence such that $\operatorname{len} O = \operatorname{len} p$ and for every k such that $k \in \operatorname{dom} O$ holds $O(k) = \mathcal{Q}(k)$ from [7, Sch. 2]. For every x such that $x \in \operatorname{dom} O$ holds $O(x)$ is an odd natural number by [20, (1)]. Define $\mathcal{T}(\text{object}) = O_3(\$_1)_{\mathbf{2}}$. Consider A being a finite sequence such that $\operatorname{len} A = \operatorname{len} p$ and for every k such that $k \in \operatorname{dom} A$ holds $A(k) = \mathcal{T}(k)$ from [7, Sch. 2]. For every x such that $x \in \operatorname{dom} A$ holds $A(x)$ is natural by [20, (1)]. Set $O_2 = O \cdot 2^A$. $p(1) = O(1) \cdot 2^{A(1)}$ and ... and $p(\operatorname{len} p) = O(\operatorname{len} p) \cdot 2^{A(\operatorname{len} p)}$ by [25, (25)], [20, (1)]. For every i such that $i \in \operatorname{dom} p$ holds $p(i) = O_2(i)$ by [25, (25)]. □

(7) Let us consider a finite set D, and a function f from D into $\mathbb{N}$. Then there exists a finite sequence K of elements of D such that for every element d of D, $\overline{\overline{\operatorname{Coim}(K, d)}} = f(d)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite set D such that $\overline{\overline{D}} = \$_1$ for every function f from D into $\mathbb{N}$, there exists a finite sequence K of elements of D such that for every element d of D, $\overline{\overline{\operatorname{Coim}(K, d)}} = f(d)$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [21, (55)], [8, (63)], [25, (57)], [13, (56)]. $\mathcal{P}[i]$ from [5, Sch. 2]. □

(8) Let us consider complex-valued finite sequences f_1, f_2, g_1, g_2. Suppose $\operatorname{len} f_1 = \operatorname{len} g_1$. Then $(f_1 \frown f_2) \cdot (g_1 \frown g_2) = (f_1 \cdot g_1) \frown (f_2 \cdot g_2)$.

(9) Let us consider natural-valued finite sequences f, K. Suppose for every i, $\overline{\overline{\operatorname{Coim}(K, i)}} = f(i)$. Then $\sum K = 1 \cdot f(1) + 2 \cdot f(2) + ((\mathrm{id}_{\operatorname{dom} f} \cdot f), 3) + \ldots$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural-valued finite sequences f, K such that $\operatorname{len} f = \$_1$ and for every i, $\overline{\overline{\operatorname{Coim}(K, i)}} = f(i)$ holds $\sum K = ((\mathrm{id}_{\operatorname{dom} f} \cdot f), 1) + \ldots$. $\mathcal{P}[0]$ by [25, (25)], [9, (72)], [19, (20), (22)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [25, (55)], [5, (13)], [7, (59)], [8, (51)]. $\mathcal{P}[i]$ from [5, Sch. 2]. □

(10) Let us consider a natural-valued finite sequence g, and a double reorgani-

zation s_1 of $\mathrm{dom}\, g$. Then there exists a $(2 \cdot \mathrm{len}\, s_1)$-element finite sequence K of elements of $\mathbb{N}$ such that for every j, $K(2 \cdot j) = 0$ and $K(2 \cdot j - 1) = g(s_{1_{j,1}}) + ((g \odot s_1)(j), 2) + \ldots$. Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ if $\$_1 = 2 \cdot j - 1$, then $\$_2 = g(s_{1_{j,1}}) + ((g \odot s_1)(j), 2) + \ldots$ and if $\$_1 = 2 \cdot j$, then $\$_2 = 0$. Set $S = \mathrm{Seg}(2 \cdot \mathrm{len}\, s_1)$. For every k such that $k \in S$ there exists x such that $\mathcal{P}[k, x]$ by [22, (9)]. Consider f being a finite sequence such that $\mathrm{dom}\, f = S$ and for every i such that $i \in S$ holds $\mathcal{P}[i, f(i)]$ from [7, Sch. 1]. $\mathrm{rng}\, f \subseteq \mathbb{N}$ by [22, (9)]. $f(2 \cdot i) = 0$. $f(2 \cdot i - 1) = g(s_{1_{i,1}}) + ((g \odot s_1)(i), 2) + \ldots$ by [25, (25)], [5, (13)], [19, (15)]. $\square$

2. Euler Transformation

Now we state the proposition:

(11) Let us consider a one-to-one partition d of n. Then there exists an odd-valued partition e of n such that for every natural number j for every odd-valued finite sequence O_1 for every natural-valued finite sequence a_1 such that $\mathrm{len}\, O_1 = \mathrm{len}\, d = \mathrm{len}\, a_1$ and $d = O_1 \cdot 2^{a_1}$ for every double reorganization s_1 of $\mathrm{dom}\, d$ such that $1 = O_1(s_{1_{1,1}})$ and ... and $1 = O_1(s_{1_{1,\mathrm{len}(s_1(1))}})$ and $3 = O_1(s_{1_{2,1}})$ and ... and $3 = O_1(s_{1_{2,\mathrm{len}(s_1(2))}})$ and $5 = O_1(s_{1_{3,1}})$ and ... and $5 = O_1(s_{1_{3,\mathrm{len}(s_1(3))}})$ and for every i, $2 \cdot i - 1 = O_1(s_{1_{i,1}})$ and ... and $2 \cdot i - 1 = O_1(s_{1_{i,\mathrm{len}(s_1(i))}})$ holds $\overline{\overline{\mathrm{Coim}(e, 1)}} = 2^{a_1}(s_{1_{1,1}}) + ((2^{a_1} \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, 3)}} = 2^{a_1}(s_{1_{2,1}}) + ((2^{a_1} \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, 5)}} = 2^{a_1}(s_{1_{3,1}}) + ((2^{a_1} \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, j \cdot 2 - 1)}} = 2^{a_1}(s_{1_{j,1}}) + ((2^{a_1} \odot s_1)(j), 2) + \ldots$.

Proof: Consider O being an odd-valued finite sequence, a being a natural-valued finite sequence such that $\mathrm{len}\, O = \mathrm{len}\, d = \mathrm{len}\, a$ and $d = O \cdot 2^a$ and $d(1) = O(1) \cdot 2^{a(1)}$ and ... and $d(\mathrm{len}\, d) = O(\mathrm{len}\, d) \cdot 2^{a(\mathrm{len}\, d)}$. $n = d(1) + ((d, 2) + \ldots + (d, \mathrm{len}\, d))$ by [19, (22)]. $n = 2^{a(1)} \cdot O(1) + 2^{a(2)} \cdot O(2) + ((O \cdot 2^a, 3) + \ldots + (O \cdot 2^a, \mathrm{len}\, d))$ by [19, (20)], [25, (25)]. Reconsider $s_1 =$ the odd organization of O as a double reorganization of $\mathrm{dom}\, 2^a$. Consider μ being a $(2 \cdot \mathrm{len}\, s_1)$-element finite sequence of elements of $\mathbb{N}$ such that for every j, $\mu(2 \cdot j) = 0$ and $\mu(2 \cdot j - 1) = 2^a(s_{1_{j,1}}) + ((2^a \odot s_1)(j), 2) + \ldots$. Set $\alpha = a \cdot s_1(1)$. Set $\beta = a \cdot s_1(2)$. Set $\gamma = a \cdot s_1(3)$. $n = (2^\alpha(1) + (2^\alpha, 2) + \ldots) \cdot 1 + (2^\beta(1) + (2^\beta, 2) + \ldots) \cdot 3 + (2^\gamma(1) + (2^\gamma, 2) + \ldots) \cdot 5 + ((\mathrm{id}_{\mathrm{dom}\, \mu} \cdot \mu), 7) + \ldots$ by [25, (29)], [19, (41)], [25, (25)], [9, (12)]. $n = \mu(1) \cdot 1 + \mu(3) \cdot 3 + \mu(5) \cdot 5 + ((\mathrm{id}_{\mathrm{dom}\, \mu} \cdot \mu), 7) + \ldots$ by [19, (42), (41), (25)]. Consider K being an odd-valued finite sequence such that K is non-decreasing and for every i, $\overline{\overline{\mathrm{Coim}(K, i)}} = \mu(i)$. $n = \overline{\overline{\mathrm{Coim}(K, 1)}} \cdot 1 + \overline{\overline{\mathrm{Coim}(K, 3)}} \cdot 3 + \overline{\overline{\mathrm{Coim}(K, 5)}} \cdot 5 + ((\mathrm{id}_{\mathrm{dom}\, \mu} \cdot \mu), 7) + \ldots$. $n = \sum K$ by [19, (20)], (9). For every j such that $1 \leqslant j \leqslant \operatorname{len} d$ holds $O(j) = O_1(j)$ and $a(j) = a_1(j)$ by [25, (25)], [22, (9)], [4, (4)]. For every j, $\overline{\overline{\operatorname{Coim}(K, j \cdot 2 - 1)}} = 2^{a_1}(sort1_{j,1}) + ((2^{a_1} \odot sort1)(j), 2) + \ldots$ by [19, (42)], [25, (29)], [9, (72)], [19, (22)]. □

Let n be a natural number and p be a one-to-one partition of n. The Euler transformation p yielding an odd-valued partition of n is defined by

(Def. 5) for every odd-valued finite sequence O and for every natural-valued finite sequence a such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every double reorganization s_1 of $\operatorname{dom} p$ such that $1 = O({s_1}_{1,1})$ and ... and $1 = O({s_1}_{1,\operatorname{len}(s_1(1))})$ and $3 = O({s_1}_{2,1})$ and ... and $3 = O({s_1}_{2,\operatorname{len}(s_1(2))})$ and $5 = O({s_1}_{3,1})$ and ... and $5 = O({s_1}_{3,\operatorname{len}(s_1(3))})$ and for every i, $2 \cdot i - 1 = O({s_1}_{i,1})$ and ... and $2 \cdot i - 1 = O({s_1}_{i,\operatorname{len}(s_1(i))})$ holds $\overline{\overline{\operatorname{Coim}(it, 1)}} = 2^a({s_1}_{1,1}) + ((2^a \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(it, 3)}} = 2^a({s_1}_{2,1}) + ((2^a \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(it, 5)}} = 2^a({s_1}_{3,1}) + ((2^a \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(it, j \cdot 2 - 1)}} = 2^a({s_1}_{j,1}) + ((2^a \odot s_1)(j), 2) + \ldots$.

Now we state the proposition:

(12) Let us consider a natural number n, a one-to-one partition p of n, and an odd-valued partition e of n. Then $e =$ the Euler transformation p if and only if for every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every j, $\overline{\overline{\operatorname{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$.

Proof: If $e =$ the Euler transformation p, then for every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every j, $\overline{\overline{\operatorname{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$ by [25, (29)], [19, (42), (20)]. For every j and for every odd-valued finite sequence O and for every natural-valued finite sequence a such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every double reorganization s_1 of $\operatorname{dom} p$ such that $1 = O({s_1}_{1,1})$ and ... and $1 = O({s_1}_{1,\operatorname{len}(s_1(1))})$ and $3 = O({s_1}_{2,1})$ and ... and $3 = O({s_1}_{2,\operatorname{len}(s_1(2))})$ and $5 = O({s_1}_{3,1})$ and ... and $5 = O({s_1}_{3,\operatorname{len}(s_1(3))})$ and for every i, $2 \cdot i - 1 = O({s_1}_{i,1})$ and ... and $2 \cdot i - 1 = O({s_1}_{i,\operatorname{len}(s_1(i))})$ holds $\overline{\overline{\operatorname{Coim}(e, 1)}} = 2^a({s_1}_{1,1}) + ((2^a \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e, 3)}} = 2^a({s_1}_{2,1}) + ((2^a \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e, 5)}} = 2^a({s_1}_{3,1}) + ((2^a \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e, j \cdot 2 - 1)}} = 2^a({s_1}_{j,1}) + ((2^a \odot s_1)(j), 2) + \ldots$ by [25, (29)], (4), [19, (42), (20)]. □

One can verify that every real-valued function which is one-to-one and non-decreasing is also increasing.

(13) Let us consider an odd-valued finite sequence O, a natural-valued finite sequence a, and an odd organization s of O. Suppose $\operatorname{len} O = \operatorname{len} a$ and $O \cdot 2^a$ is one-to-one. Then $(a \odot s)(i)$ is one-to-one.

Proof: $(a \odot s)(i)$ is one-to-one by [9, (11), (12)], [25, (25)]. □

(14) Let us consider one-to-one partitions p_1, p_2 of n. Suppose the Euler transformation $p_1 =$ the Euler transformation p_2. Then $p_1 = p_2$.

(15) Let us consider an odd-valued partition e of n. Then there exists a one-to-one partition p of n such that $e =$ the Euler transformation p.

Proof: Define $\mathcal{K}(\text{object}) = \overline{\overline{\mathrm{Coim}(e, \$_1)}}$. Consider H being a finite sequence such that $\operatorname{len} H = n$ and for every k such that $k \in \operatorname{dom} H$ holds $H(k) = \mathcal{K}(k)$ from [7, Sch. 2]. $\operatorname{rng} H \subseteq \mathbb{N}$. $\sum e = \sum(\mathrm{idseq}(n) \cdot H)$ by [25, (25)], [5, (14)], [9, (72)], [30, (5)]. Define $\mathcal{F}[\text{natural number}, \text{object}] \equiv$ there exists an increasing, natural-valued finite sequence f such that $H(\$_1) = 2^f(1) + (2^f, 2) + \ldots$ and $\$_2 = \$_1 \cdot 2^f$. There exists a finite sequence p of elements of $\mathbb{N}^*$ such that $\operatorname{dom} p = \operatorname{Seg} \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{F}[k, p(k)]$ by [19, (31)]. Consider p being a finite sequence of elements of $\mathbb{N}^*$ such that $\operatorname{dom} p = \operatorname{Seg} \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{F}[k, p(k)]$. For every k such that $p(k) \neq \emptyset$ holds k is odd by [18, (83)], [12, (85)], [19, (22)], [9, (72)]. Set $N =$ the concatenation of $\mathbb{N}$. Set $n_3 = N \odot p$. Set $s_2 = \mathrm{sort_a}\, n_3$. s_2 is a one-to-one partition of n by [19, (1)], [25, (25)], [12, (45)], [18, (83)]. For every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} s_2 = \operatorname{len} a$ and $s_2 = O \cdot 2^a$ for every j, $\overline{\overline{\mathrm{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$ by [25, (29)], [5, (14)], [9, (72)], [25, (25)]. □

3. Main Theorem

Now we state the proposition:

(16) Euler's partition theorem: $\overline{\overline{\text{the set of all } p \text{ where } p \text{ is an odd-valued partition of } n}} = \overline{\overline{\text{the set of all } p \text{ where } p \text{ is a one-to-one partition of } n}}$. The theorem is a consequence of (15) and (14).

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Introduction to Diophantine Approximation

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Summary. In this article we formalize some results of Diophantine approximation, i.e. the approximation of an irrational number by rationals. A typical example is finding an integer solution (x, y) of the inequality $|x\theta - y| \leqslant 1/x$, where θ is a real number. First, we formalize some lemmas about continued fractions. Then we prove that the inequality has infinitely many solutions by continued fractions. Finally, we formalize Dirichlet's proof (1842) of existence of the solution [12], [1].

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Keywords: irrational number; approximation; continued fraction; rational number; Dirichlet's proof

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The notation and terminology used in this paper have been introduced in the following articles: [24], [2], [6], [22], [14], [5], [11], [7], [8], [28], [20], [26], [3], [25], [19], [4], [9], [32], [15], [13], [21], [30], [31], [18], [23], [29], and [10].

1. Irrational Numbers and Continued Fractions

From now on i, j, k, m, n, m_1, n_1 denote natural numbers, a, r, r_1, r_2 denote real numbers, m_0, c_3, c_1 denote integers, and x_1, x_2, o denote objects.

Now we state the proposition:

(1) (i) $r = (\mathrm{rfs}\,r)(0)$, and

(ii) $r = (\mathrm{scf}\,r)(0) + (1/_{(\mathrm{rfs}\,r)(1)})$, and

(iii) $(\mathrm{rfs}\,r)(n) = (\mathrm{scf}\,r)(n) + (1/_{(\mathrm{rfs}\,r)(n+1)})$.

Let us assume that r is irrational. Now we state the propositions:

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(2) $(\mathrm{rfs}\,r)(n)$ is irrational.
PROOF: Reconsider $r_3 = (\mathrm{rfs}\,r)(n)$ as a real number. $(\mathrm{scf}\,r_3)(m) = (\mathrm{scf}\,r)(n+m)$ and $(\mathrm{rfs}\,r_3)(m) = (\mathrm{rfs}\,r)(n+m)$. Consider n_1 such that for every m_1 such that $m_1 \geqslant n_1$ holds $(\mathrm{scf}\,r_3)(m_1) = 0$. For every m_1 such that $m_1 \geqslant n_1$ holds $(\mathrm{scf}\,r)(n+m_1) = 0$. For every m such that $m \geqslant n_1 + n$ holds $(\mathrm{scf}\,r)(m) = 0$ by [28, (3)]. □

(3) (i) $(\mathrm{rfs}\,r)(n) \neq 0$, and
(ii) $(\mathrm{rfs}\,r)(1) \cdot (\mathrm{rfs}\,r)(2) \neq 0$, and
(iii) $(\mathrm{scf}\,r)(1) \cdot (\mathrm{rfs}\,r)(2) + 1 \neq 0$.
PROOF: $(\mathrm{rfs}\,r)(n) \neq 0$ by [21, (28), (42)]. $(\mathrm{rfs}\,r)(1) \neq 0$ and $(\mathrm{rfs}\,r)(2) \neq 0$. $(\mathrm{rfs}\,r)(1) = (\mathrm{scf}\,r)(1) + (1/_{(\mathrm{rfs}\,r)(1+1)})$. □

(4) (i) $(\mathrm{scf}\,r)(n) < (\mathrm{rfs}\,r)(n) < (\mathrm{scf}\,r)(n) + 1$, and
(ii) $1 < (\mathrm{rfs}\,r)(n+1)$.
The theorem is a consequence of (2) and (1).

(5) $0 < (\mathrm{scf}\,r)(n+1)$. The theorem is a consequence of (4).

Let us consider r and n. Observe that $(cn\,r)(n)$ is integer.

Let us assume that r is irrational. Now we state the propositions:

(6) $(cd\,r)(n+1) \geqslant (cd\,r)(n)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (cd\,r)(\$_1) \leqslant (cd\,r)(\$_1 + 1)$. $\mathcal{P}[0]$ by (4), [28, (7)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by (4), [28, (7)], [21, (51)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(7) $(cd\,r)(n) \geqslant 1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (cd\,r)(\$_1) \geqslant 1$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(8) $(cd\,r)(n+2) > (cd\,r)(n+1)$. The theorem is a consequence of (5) and (7).

(9) $(cd\,r)(n) \geqslant n$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (cd\,r)(\$_1) \geqslant \$_1$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by (7), (5), [21, (40)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

Now we state the proposition:

(10) If $c_3 = (cn\,r)(n)$ and $c_1 = (cd\,r)(n)$ and $c_3 \neq 0$, then c_3 and c_1 are relatively prime.

Let us assume that r is irrational. Now we state the propositions:

(11) (i) $(cd\,r)(n+1) \cdot (\mathrm{rfs}\,r)(n+2) + (cd\,r)(n) > 0$, and

(ii) $(cd\, r)(n+1) \cdot (\mathrm{rfs}\, r)(n+2) - (cd\, r)(n) > 0$.

The theorem is a consequence of (7), (4), and (6).

(12) $(cd\, r)(n+1) \cdot ((cd\, r)(n+1) \cdot (\mathrm{rfs}\, r)(n+2) + (cd\, r)(n)) > 0$. The theorem is a consequence of (7) and (11).

(13) $r = (cn\, r)(n+1) \cdot (\mathrm{rfs}\, r)(n+2) + (cn\, r)(n)/_{(cd\, r)(n+1) \cdot (\mathrm{rfs}\, r)(n+2) + (cd\, r)(n)}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv r = (cn\, r)(\$_1 + 1) \cdot (\mathrm{rfs}\, r)(\$_1 + 2) + (cn\, r)(\$_1)/_{(cd\, r)(\$_1+1) \cdot (\mathrm{rfs}\, r)(\$_1+2) + (cd\, r)(\$_1)}$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(14) $((cn\, r)(n+1)/_{(cd\, r)(n+1)}) - r = (-1)^n/_{(cd\, r)(n+1) \cdot ((cd\, r)(n+1) \cdot (\mathrm{rfs}\, r)(n+2) + (cd\, r)(n))}$. The theorem is a consequence of (7), (11), and (13).

Now we state the propositions:

(15) If r is irrational and n is even and $n > 0$, then $r > (cn\, r)(n)/_{(cd\, r)(n)}$. The theorem is a consequence of (12) and (14).

(16) If r is irrational and n is odd, then $r < (cn\, r)(n)/_{(cd\, r)(n)}$. The theorem is a consequence of (12) and (14).

(17) Suppose r is irrational and $n > 0$. Then $|r - ((cn\, r)(n)/_{(cd\, r)(n)})| + |r - ((cn\, r)(n+1)/_{(cd\, r)(n+1)})| = |((cn\, r)(n)/_{(cd\, r)(n)}) - ((cn\, r)(n+1)/_{(cd\, r)(n+1)})|$. The theorem is a consequence of (15) and (16).

Let us assume that r is irrational. Now we state the propositions:

(18) $|r - ((cn\, r)(n)/_{(cd\, r)(n)})| > 0$.

(19) $(cd\, r)(n+2) \geqslant 2 \cdot (cd\, r)(n)$. The theorem is a consequence of (5), (7), and (6).

(20) $|r - ((cn\, r)(n+1)/_{(cd\, r)(n+1)})| < 1/_{(cd\, r)(n+1) \cdot (cd\, r)(n+2)}$. The theorem is a consequence of (7), (4), and (14).

(21) (i) $|r \cdot (cd\, r)(n+1) - (cn\, r)(n+1)| < |r \cdot (cd\, r)(n) - (cn\, r)(n)|$, and

(ii) $|r - ((cn\, r)(n+1)/_{(cd\, r)(n+1)})| < |r - ((cn\, r)(n)/_{(cd\, r)(n)})|$.

The theorem is a consequence of (13), (11), (4), (7), (18), and (6).

Now we state the propositions:

(22) If r is irrational and $m > n$, then $|r - ((cn\, r)(n)/_{(cd\, r)(n)})| > |r - ((cn\, r)(m)/_{(cd\, r)(m)})|$.

Proof: Define $\mathcal{P}$[natural number] $\equiv |r - ((cn\, r)(n)/_{(cd\, r)(n)})| > |r - ((cn\, r)(n+1+\$_1)/_{(cd\, r)(n+1+\$_1)})|$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(23) If r is irrational, then $|r - ((cn\, r)(n)/_{(cd\, r)(n)})| < 1/_{(cd\, r)(n)^2}$.

Proof: $|r - ((cn\,r)(n)/_{(cd\,r)(n)})| < 1/_{(cd\,r)(n)^2}$ by [28, (43)], (7), [16, (1)], (6). □

(24) Let us consider a subset S of $\mathbb{Q}$, and r. Suppose r is irrational and $S = \{p$, where p is an element of $\mathbb{Q} : |r - p| < 1/_{(\mathrm{den}\,p)^2}\}$. Then S is infinite.

Proof: Define $\mathcal{F}$(natural number) $= (cn\,r)(\$_1 + 1)/_{(cd\,r)(\$_1+1)}$. Consider f being a sequence of real numbers such that for every natural number n, $f(n) = \mathcal{F}(n)$ from [17, Sch. 1]. For every real number o such that $o \in \mathrm{rng}\, f$ holds $o \in S$ by [21, (50)], (7), [15, (28)], [16, (1)]. f is one-to-one. □

(25) If r is irrational, then $\mathrm{cocf}\, r$ is convergent and $\lim \mathrm{cocf}\, r = r$.

Proof: For every real number p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|(\mathrm{cocf}\, r)(m) - r| < p$ by [27, (25)], [28, (3)], [17, (8)], [6, (52)]. □

2. Integer Solution of $|x\theta - y| \leqslant 1/x$

Let us observe that there exists a natural number which is greater than 1. From now on t denotes a greater than 1 natural number.

Let us consider t. The functor EDI(t) yielding a sequence of subsets of $\mathbb{R}$ is defined by

(Def. 1) for every natural number n, $it(n) = [n/_t, n + 1/_t[$.

Now we state the propositions:

(26) (The partial unions of EDI(t))(i) $= [0, i + 1/_t[$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial unions of EDI(t))($\$_1$) $= [0, \$_1 + 1/_t[$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(27) Let us consider a real number r, and a natural number i. If $\lfloor r \cdot t \rfloor = i$, then $r \in (\mathrm{EDI}(t))(i)$.

(28) If r_1, $r_2 \in (\mathrm{EDI}(t))(i)$, then $|r_1 - r_2| < t^{-1}$.

(29) (The partial unions of EDI(t))($t - 1$) $= [0, 1[$. The theorem is a consequence of (26).

(30) Let us consider a real number r. Suppose $r \in [0, 1[$. Then there exists a natural number i such that

(i) $i \leqslant t - 1$, and

(ii) $r \in (\mathrm{EDI}(t))(i)$.

The theorem is a consequence of (29).

(31) Let us consider a real number r, and a natural number i. If $r \in (\mathrm{EDI}(t))(i)$, then $\lfloor r \cdot t \rfloor = i$.

(32) Let us consider a real number r. Suppose $r \in [0, 1[$. Then there exists a natural number i such that

(i) $i \leqslant t - 1$, and

(ii) $\lfloor r \cdot t \rfloor = i$.

The theorem is a consequence of (30) and (31).

Let us consider t and a. The functor $\mathrm{FDP}(t, a)$ yielding a finite sequence of elements of $\mathbb{Z}_t$ is defined by

(Def. 2) $\mathrm{len}\, it = t + 1$ and for every i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \lfloor \mathrm{frac}((i - 1) \cdot a) \cdot t \rfloor$.

Let us note that $\mathrm{rng}\, \mathrm{FDP}(t, a)$ is non empty.

Now we state the proposition:

(33) $\overline{\overline{\mathrm{rng}\, \mathrm{FDP}(t, a)}} \in \overline{\overline{\mathrm{dom}\, \mathrm{FDP}(t, a)}}$.

Let us consider t and a. One can verify that $\mathrm{FDP}(t, a)$ is non one-to-one.

3. PROOF OF DIRICHLET'S THEOREM

Now we state the proposition:

(34) DIRICHLET'S APPROXIMATION THEOREM:
There exist integers x, y such that

(i) $|x \cdot a - y| < 1/t$, and

(ii) $0 < x \leqslant t$.

The theorem is a consequence of (27) and (28).

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Finite Product of Semiring of Sets

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Summary. We formalize that the image of a semiring of sets [17] by an injective function is a semiring of sets. We offer a non-trivial example of a semiring of sets in a topological space [21]. Finally, we show that the finite product of a semiring of sets is also a semiring of sets [21] and that the finite product of a classical semiring of sets [8] is a classical semiring of sets. In this case, we use here the notation from the book of Aliprantis and Border [1].

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The notation and terminology used in this paper have been introduced in the following articles: [9], [2], [3], [4], [22], [7], [15], [23], [10], [11], [6], [12], [20], [26], [27], [19], [14], [16], [25], [18], and [13].

1. Preliminaries

From now on X_1, X_2, X_3, X_4 denote sets.

Now we state the propositions:

(1) (i) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$, and

(ii) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$, and

(iii) $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$.

(2) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = (X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2)$.

(3) $(X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$.

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(4) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$. The theorem is a consequence of (2) and (3).

(5) $\bigcup\{X_1, X_2, X_3\} = (X_1 \cup X_2) \cup X_3$.

2. The Direct Image of a Semiring of Sets by an Injective Function

Now we state the proposition:

(6) Let us consider sets T, S, a function f from T into S, and a family G of subsets of T. Then $f^\circ G = \{f^\circ A$, where A is a subset of $T : A \in G\}$.

Let T, S be sets, f be a function from T into S, and G be a finite family of subsets of T. Let us note that $f^\circ G$ is finite.

Let f be a function and A be a countable set. Let us note that $f^\circ A$ is countable.

The scheme *FraenkelCountable* deals with a set $\mathcal{A}$ and a set $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) $\{\mathcal{F}(w)$, where w is an element of $\mathcal{A} : w \in \mathcal{X}\}$ is countable

provided

- $\mathcal{X}$ is countable.

Let T, S be sets, f be a function from T into S, and G be a countable family of subsets of T. Let us note that $f^\circ G$ is countable.

Let X, Y be sets, S be a family of subsets of X with the empty element, and f be a function from X into Y. One can verify that $f^\circ S$ has the empty element.

Now we state the propositions:

(7) Let us consider sets X, Y, a function f from X into Y, and families S_1, S_2 of subsets of X. If $S_1 \subseteq S_2$, then $f^\circ S_1 \subseteq f^\circ S_2$. The theorem is a consequence of (6).

(8) Let us consider sets X, Y, a $\cap$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^\circ S$ is a $\cap$-closed family of subsets of Y.

(9) Let us consider non empty sets X, Y, a $\cap_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^\circ S$ is a $\cap_{fp}$-closed family of subsets of Y.

(10) Let us consider non empty sets X, Y, a $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one and $f^\circ S$ is not empty. Then $f^\circ S$ is a $\setminus^{\subseteq}_{fp}$-closed family of subsets of Y.

Proof: Reconsider $f_1 = f^\circ S$ as a family of subsets of Y. f_1 is $\setminus^{\subseteq}_{fp}$-closed by [10, (64), (87)], [11, (103)], [26, (123)]. $\square$

(11) Let us consider non empty sets X, Y, a $\setminus_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^\circ S$ is a $\setminus_{fp}$-closed family of subsets of Y.

(12) Let us consider non empty sets X, Y, a semiring S of sets of X, and a function f from X into Y. If f is one-to-one, then $f^\circ S$ is a semiring of sets of Y.

3. The Set of Set Differences of All Elements of a Semiring of Sets

Now we state the proposition:

(13) Let us consider a 1-element finite sequence X. Suppose $X(1)$ is not empty. Then there exists a function I from $X(1)$ into $\prod X$ such that

(i) I is one-to-one and onto, and

(ii) for every object x such that $x \in X(1)$ holds $I(x) = \langle x \rangle$.

Let X be a set. Observe that 2_*^X is $\cap$-closed and there exists a $\cap$-closed family of subsets of X which has the empty element and there exists a $\cap$-closed family of subsets of X with the empty element which is $\cup$-closed.

Let X, Y be non empty sets. Let us observe that $X \setminus\!\!\setminus Y$ is non empty.

Now we state the proposition:

(14) Let us consider a set X, and a family S of subsets of X with the empty element. Then $S \setminus\!\!\setminus S =$ the set of all $A \setminus B$ where A, B are elements of S.

Let X be a set and S be a family of subsets of X with the empty element. The functor semidiff S yielding a family of subsets of X is defined by the term

(Def. 1) $S \setminus\!\!\setminus S$.

Now we state the proposition:

(15) Let us consider a set X, a family S of subsets of X with the empty element, and an object x. Suppose $x \in$ semidiff S. Then there exist elements A, B of S such that $x = A \setminus B$. The theorem is a consequence of (14).

Let X be a set and S be a family of subsets of X with the empty element. Observe that semidiff S has the empty element.

Let S be a $\cap$-closed, $\cup$-closed family of subsets of X with the empty element. Note that semidiff S is $\cap$-closed and $\setminus_{fp}$-closed.

Now we state the proposition:

(16) Let us consider a set X, and a $\cap$-closed, $\cup$-closed family S of subsets of X with the empty element. Then semidiff S is a semiring of sets of X.

4. The Collection of All Locally Closed Sets $LC(X, \tau)$ of a Topological Space (X, τ)

Let T be a non empty topological space. The functor $\mathrm{LC}(T)$ yielding a family of subsets of Ω_T is defined by the term

(Def. 2) $\{A \cap B$, where A, B are subsets of $T : A$ is open and B is closed$\}$.

Let us note that $\mathrm{LC}(T)$ is $\cap$-closed and $\setminus_{fp}$-closed and has the empty element.

(17) Let us consider a non empty topological space T. Then $\mathrm{LC}(T)$ is a semiring of sets of Ω_T.

5. The Finite Product of Semirings of Sets

Let n be a natural number. Note that there exists an n-element finite sequence which is non-empty.

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A semiring family of X is an n-element finite sequence and is defined by

(Def. 3) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $it(i)$ is a semiring of sets of $X(i)$.

In the sequel n denotes a non zero natural number and X denotes a non-empty, n-element finite sequence. Now we state the propositions:

(18) Let us consider a semiring family S of X. Then $\mathrm{dom}\, S = \mathrm{dom}\, X$.

(19) Let us consider a semiring family S of X, and a natural number i. If $i \in \mathrm{Seg}\, n$, then $\bigcup(S(i)) \subseteq X(i)$.

(20) Let us consider a function f, and an n-element finite sequence X. If $f \in \prod X$, then f is an n-element finite sequence.

Let n be a non zero natural number and X be an n-element finite sequence. The functor SemiringProduct X yielding a set is defined by

(Def. 4) for every object f, $f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod X$.

Now we state the propositions:

(21) Let us consider an n-element finite sequence X. Then $\mathrm{SemiringProduct}\, X \subseteq 2^{(\bigcup \bigcup X)^{\mathrm{dom}\, X}}$.

(22) Let us consider a semiring family S of X. Then SemiringProduct S is a family of subsets of $\prod X$.
Proof: Reconsider $S_1 = \mathrm{SemiringProduct}\, S$ as a subset of $2^{(\bigcup \bigcup S)^{\mathrm{dom}\, S}}$. $S_1 \subseteq 2^{\prod X}$ by [3, (9)], (18), [7, (89)], (19). □

(23) Let us consider a non-empty, 1-element finite sequence X. Then $\prod X =$ the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (13).

One can check that $\prod \langle \emptyset \rangle$ is empty. Now we state the propositions:

(24) Let us consider a non empty set x. Then $\prod \langle x \rangle =$ the set of all $\langle y \rangle$ where y is an element of x. The theorem is a consequence of (23).

(25) Let us consider a non-empty, 1-element finite sequence X, and a semiring family S of X. Then SemiringProduct $S =$ the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$. Proof: S is non-empty by (18), [7, (3)]. $\prod S =$ the set of all $\langle s \rangle$ where s is an element of $S(1)$. $\square$

Let us consider sets x, y. Now we state the propositions:

(26) $\prod \langle x \rangle \cap \prod \langle y \rangle = \prod \langle x \cap y \rangle$. The theorem is a consequence of (24).

(27) $\prod \langle x \rangle \setminus \prod \langle y \rangle = \prod \langle x \setminus y \rangle$. The theorem is a consequence of (24).

Let us consider a non-empty, 1-element finite sequence X and a semiring family S of X. Now we state the propositions:

(28) the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$ is a semiring of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (24), (26), and (27).

(29) SemiringProduct S is a semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (28).

(30) Let us consider sets X_1, X_2, a semiring S_1 of sets of X_1, and a semiring S_2 of sets of X_2. Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1, s_2 is an element of S_2 is a semiring of sets of $X_1 \times X_2$.

(31) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a semiring family S_3 of X_3, and a semiring family S_1 of X_1. Suppose SemiringProduct S_3 is a semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a semiring of sets of $\prod X_1$. Let us consider a family S_4 of subsets of $\prod X_3 \times \prod X_1$. Suppose $S_4 =$ the set of all $s_1 \times s_2$ where s_1 is an element of SemiringProduct S_3, s_2 is an element of SemiringProduct S_1. Then there exists a function I from $\prod X_3 \times \prod X_1$ into $\prod (X_3 \frown X_1)$ such that

(i) I is one-to-one and onto, and

(ii) for every finite sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \frown y$, and

(iii) $I^{\circ} S_4 =$ SemiringProduct$(S_3 \frown S_1)$.

Proof: $\bigcup (S_1(1)) \subseteq X_1(1)$. Consider I being a function from $\prod X_3 \times \prod X_1$ into $\prod (X_3 \frown X_1)$ such that I is one-to-one and I is onto and for every finite sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x,y) = x \mathbin{^\frown} y$. $I^\circ S_4 = \mathrm{SemiringProduct}(S_3 \mathbin{^\frown} S_1)$ by (25), (20), [7, (89)], [24, (153)]. □

(32) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a semiring family S_3 of X_3, and a semiring family S_1 of X_1. Suppose SemiringProduct S_3 is a semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a semiring of sets of $\prod X_1$. Then $\mathrm{SemiringProduct}(S_3 \mathbin{^\frown} S_1)$ is a semiring of sets of $\prod(X_3 \mathbin{^\frown} X_1)$. The theorem is a consequence of (30), (31), (9), and (10).

(33) Let us consider a semiring family S of X. Then SemiringProduct S is a semiring of sets of $\prod X$. PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every non-empty, $\$_1$-element finite sequence X for every semiring family S of X, SemiringProduct S is a semiring of sets of $\prod X$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$ from [5, Sch. 10]. □

Let n be a non zero natural number, X be a non-empty, n-element finite sequence, and S be a semiring family of X. We say that S is $\cap$-closed yielding if and only if

(Def. 5) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $S(i)$ is $\cap$-closed.

Note that there exists a semiring family of X which is $\cap$-closed yielding.

6. THE FINITE PRODUCT OF CLASSICAL SEMIRINGS OF SETS

Let X be a set. Note that there exists a semiring of sets of X which is $\cap$-closed.

Let us consider a non-empty, 1-element finite sequence X and a $\cap$-closed yielding semiring family S of X. Now we state the propositions:

(34) the set of all $\prod\langle s\rangle$ where s is an element of $S(1)$ is a $\cap$-closed semiring of sets of the set of all $\langle x\rangle$ where x is an element of $X(1)$. The theorem is a consequence of (26) and (28).

(35) SemiringProduct S is a $\cap$-closed semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (34).

Now we state the propositions:

(36) Let us consider sets X_1, X_2, a $\cap$-closed semiring S_1 of sets of X_1, and a $\cap$-closed semiring S_2 of sets of X_2. Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1, s_2 is an element of S_2 is a $\cap$-closed semiring of sets of $X_1 \times X_2$.

(37) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a $\cap$-closed yielding semiring family S_3 of X_3, and a $\cap$-closed yielding semiring family S_1 of X_1. Suppose SemiringProduct

S_3 is a $\cap$-closed semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a $\cap$-closed semiring of sets of $\prod X_1$. Then SemiringProduct$(S_3 \frown S_1)$ is a $\cap$-closed semiring of sets of $\prod(X_3 \frown X_1)$. The theorem is a consequence of (30), (31), (36), (8), and (10).

Let us consider n and X. Let S be a $\cap$-closed yielding semiring family of X. One can check that SemiringProduct S is $\cap$-closed.

(38) Let us consider a $\cap$-closed yielding semiring family S of X.
Then SemiringProduct S is a $\cap$-closed semiring of sets of $\prod X$.

7. MEASURABLE RECTANGLE

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A classical semiring family of X is an n-element finite sequence and is defined by

(Def. 6) for every natural number i such that $i \in \operatorname{Seg} n$ holds $it(i)$ is a semi-diff-closed, $\cap$-closed family of subsets of $X(i)$ with the empty element.

Let X be an n-element finite sequence. We introduce MeasurableRectangle X as a synonym of SemiringProduct X. Now we state the propositions:

(39) Every classical semiring family of X is a $\cap$-closed yielding semiring family of X.

(40) Let us consider a classical semiring family S of X.
Then MeasurableRectangle S is a semi-diff-closed, $\cap$-closed family of subsets of $\prod X$ with the empty element. The theorem is a consequence of (39) and (33).

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Two Axiomatizations of Nelson Algebras

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Summary. Nelson algebras were first studied by Rasiowa and Białynicki-Birula [1] under the name N-lattices or quasi-pseudo-Boolean algebras. Later, in investigations by Monteiro and Brignole [3, 4], and [2] the name "Nelson algebras" was adopted – which is now commonly used to show the correspondence with Nelson's paper [14] on constructive logic with strong negation.

By a Nelson algebra we mean an abstract algebra

$$\langle L, \top, -, \neg, \rightarrow, \Rightarrow, \sqcup, \sqcap \rangle$$

where L is the carrier, $-$ is a quasi-complementation (Rasiowa used the sign $\sim$, but in Mizar "$-$" should be used to follow the approach described in [12] and [10]), $\neg$ is a weak pseudo-complementation, $\rightarrow$ is weak relative pseudo-complementation and $\Rightarrow$ is implicative operation. $\sqcup$ and $\sqcap$ are ordinary lattice binary operations of supremum and infimum.

In this article we give the definition and basic properties of these algebras according to [16] and [15]. We start with preliminary section on quasi-Boolean algebras (i.e. de Morgan bounded lattices). Later we give the axioms in the form of Mizar adjectives with names corresponding with those in [15]. As our main result we give two axiomatizations (non-equational and equational) and the full formal proof of their equivalence.

The second set of equations is rather long but it shows the logical essence of Nelson lattices. This formalization aims at the construction of algebraic model of rough sets [9] in our future submissions. Section 4 contains all items from Th. 1.2 and 1.3 (and the itemization is given in the text). In the fifth section we provide full formal proof of Th. 2.1 p. 75 [16].

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The notation and terminology used in this paper have been introduced in the following articles: [5], [6], [7], [18], [11], [13], [17], and [8].

1. De Morgan and Quasi-Boolean Lattices

Let L be a non empty ortholattice structure. We say that L is de Morgan if and only if

(Def. 1) for every elements x, y of L, $(x \sqcap y)^{\rm c} = x^{\rm c} \sqcup y^{\rm c}$.

One can verify that every non empty ortholattice structure which is de Morgan and involutive is also de Morgan and every non empty ortholattice structure which is de Morgan and involutive is also de Morgan.

Every non empty ortholattice structure which is trivial is also de Morgan and there exists a non empty ortholattice structure which is de Morgan, involutive, bounded, distributive, and lattice-like.

A de Morgan algebra is a de Morgan, involutive, distributive, lattice-like, non empty ortholattice structure.

A quasi-Boolean algebra is a bounded de Morgan algebra. From now on L denotes a quasi-Boolean algebra and x, y, z denote elements of L.

Now we state the propositions:

(1) $(x \sqcup y)^{\rm c} = x^{\rm c} \sqcap y^{\rm c}$.

(2) $(\top_L)^{\rm c} = \bot_L$.

(3) $(\bot_L)^{\rm c} = \top_L$.

(4) $x \sqcap (x \sqcap y) = x \sqcap y$.

(5) $x \sqcup (x \sqcup y) = x \sqcup y$.

2. The Structure and Operators in Nelson Algebras

We consider Nelson structures which extend ortholattice structures and are systems

$$\langle \text{a carrier, a unity, a complement operation, a weak pseudo-complementation,}$$

$$\text{a weak relative pseudo-complementation, an implicative operation,}$$

$$\text{a join operation, a meet operation} \rangle$$

where the carrier is a set, the unity is an element of the carrier, the complement operation and the weak pseudo-complementation are unary operations on the carrier, the weak relative pseudo-complementation and the implicative operation and the join operation and the meet operation are binary operations on the carrier.

Note that there exists a Nelson structure which is strict and non empty and there exists a non empty Nelson structure which is trivial, de Morgan, involutive, bounded, distributive, and lattice-like.

Let L be a non empty Nelson structure and a, b be elements of L. The functor $a \to b$ yielding an element of L is defined by the term

(Def. 2) (the weak relative pseudo-complementation of $L)(a, b)$.

We say that $a < b$ if and only if

(Def. 3) $a \to b = \top_L$.

We say that $a \leqslant b$ if and only if

(Def. 4) $a = a \sqcap b$.

Let a be an element of L. The functor $\neg\, a$ yielding an element of L is defined by the term

(Def. 5) (the weak pseudo-complementation of $L)(a)$.

Let a, b be elements of L. The functor $a \Rightarrow b$ yielding an element of L is defined by the term

(Def. 6) (the implicative operation of $L)(a, b)$.

3. The Non-Equational Axiomatization

Let L be a non empty Nelson structure. We say that L has reflexive $<$ if and only if

(Def. 7) for every element a of L, $a < a$.

We say that L has transitive $<$ if and only if

(Def. 8) for every elements a, b, c of L such that $a < b < c$ holds $a < c$.

Let L be a bounded, lattice-like, non empty Nelson structure. We say that L is quasi-Boolean if and only if

(Def. 9) L is de Morgan, involutive, and distributive.

Let us note that every bounded, lattice-like, non empty Nelson structure which is quasi-Boolean is also de Morgan, involutive, and distributive.

Every bounded, lattice-like, non empty Nelson structure which is de Morgan, involutive, and distributive is also quasi-Boolean.

Let L be a non empty Nelson structure. We say that L satisfies (qpB_3) if and only if

(Def. 10) for every elements x, a, b of L, $a \sqcap x < b$ iff $x < a \to b$.

We say that L satisfies (qpB_4) if and only if

(Def. 11) for every elements a, b of L, $a \Rightarrow b = (a \rightarrow b) \sqcap (-b \rightarrow -a)$.

We say that L satisfies (qpB$_5$) if and only if

(Def. 12) for every elements a, b of L, $a \Rightarrow b = \top_L$ iff $a \sqcap b = a$.

We say that L satisfies (qpB$_6$) if and only if

(Def. 13) for every elements a, b, c of L such that $a < c$ and $b < c$ holds $a \sqcup b < c$.

We say that L satisfies (qpB$_7$) if and only if

(Def. 14) for every elements a, b, c of L such that $a < b$ and $a < c$ holds $a < b \sqcap c$.

We say that L satisfies (qpB$_8$) if and only if

(Def. 15) for every elements a, b of L, $a \sqcap -b < -(a \rightarrow b)$.

We say that L satisfies (qpB$_9$) if and only if

(Def. 16) for every elements a, b of L, $-(a \rightarrow b) < a \sqcap -b$.

We say that L satisfies (qpB$_{10}$) if and only if

(Def. 17) for every element a of L, $a < -\neg a$.

We say that L satisfies (qpB$_{11}$) if and only if

(Def. 18) for every element a of L, $-\neg a < a$.

We say that L satisfies (qpB$_{12}$) if and only if

(Def. 19) for every elements a, b of L, $a \sqcap -a < b$.

We say that L satisfies (qpB$_{13}$) if and only if

(Def. 20) for every element a of L, $\neg a = a \rightarrow -\top_L$.

Let us observe that there exists a bounded, lattice-like, non empty Nelson structure which is quasi-Boolean and has reflexive $<$ and transitive $<$ and satisfies (qpB$_3$), (qpB$_4$), (qpB$_5$), (qpB$_6$), (qpB$_7$), (qpB$_8$), (qpB$_9$), (qpB$_{10}$), (qpB$_{11}$), (qpB$_{12}$), and (qpB$_{13}$).

A Nelson algebra is a quasi-Boolean, bounded, lattice-like, non empty Nelson structure with reflexive $<$ and transitive $<$. Let L be a bounded, non empty Nelson structure and a, b be elements of L. Let us observe that the functor $a \Rightarrow b$ is defined by the term

(Def. 21) $(a \rightarrow b) \sqcap (-b \rightarrow -a)$.

From now on L denotes a Nelson algebra and a, b, c, d, x, y, z denote elements of L.

Now we state the propositions:

(6) $a \sqsubseteq b$ if and only if $a \leqslant b$.

(7) $a \leqslant b \leqslant a$ if and only if $a = b$.

PROOF: If $a \leqslant b \leqslant a$, then $a = b$ by [18, (4), (8)]. □

(8) $a \sqcap b = \top_L$ if and only if $a = \top_L$ and $b = \top_L$.

(9) $a \leqslant b$ if and only if $a < b$ and $-b < -a$. The theorem is a consequence of (8).

(10) $a \sqcap b < a$. The theorem is a consequence of (9).

(11) $a < a \sqcup b$. The theorem is a consequence of (9).

(12) $a \leqslant b$ if and only if $a \Rightarrow b = \top_L$.

(13) $-(a \sqcap b) = -a \sqcup -b$. The theorem is a consequence of (1).

(14) $(a \sqcap -a) \sqcap (b \sqcup -b) = a \sqcap -a$. The theorem is a consequence of (1), (13), and (9).

(15) If $a \leqslant b \leqslant c$, then $a \leqslant c$.

(16) If $b \leqslant c$, then $a \sqcup b \leqslant a \sqcup c$ and $a \sqcap b \leqslant a \sqcap c$. The theorem is a consequence of (9), (1), and (13).

(17) $-a \sqcup b \leqslant a \rightarrow b$. The theorem is a consequence of (1), (2), (9), (10), (16), and (15).

(18) $(a \rightarrow b) \sqcap (-a \sqcup b) = -a \sqcup b$. The theorem is a consequence of (1), (13), (17), (10), (9), and (7).

(19) $-a \sqcup b < a \rightarrow b$. The theorem is a consequence of (18) and (9).

(20) $a \sqcap (a \rightarrow b) = a \sqcap (-a \sqcup b)$. The theorem is a consequence of (11), (10), (13), (1), (19), (9), and (7).

(21) If $-x < -y$, then $-(z \rightarrow x) < -(z \rightarrow y)$.

Let us assume that $x < y$. Now we state the propositions:

(22) $a \sqcap (a \rightarrow x) < y$. The theorem is a consequence of (20) and (10).

(23) $a \rightarrow x < a \rightarrow y$. The theorem is a consequence of (22).

(24) $a \rightarrow (b \sqcap c) = (a \rightarrow b) \sqcap (a \rightarrow c)$. The theorem is a consequence of (11), (13), (10), (23), (9), and (7).

4. Properties of Nelson Algebras

Now we state the propositions:

(25) [see also [16] p. 69, Th. 1.2 (5)]:
$a \Rightarrow a = \top_L$.

(26) [see also [16] p. 69, Th. 1.2 (6)]:
If $a \Rightarrow b = \top_L$ and $b \Rightarrow c = \top_L$, then $a \Rightarrow c = \top_L$.

(27) [see also [16] p. 69, Th. 1.2 (7)]:
If $a \Rightarrow b = \top_L$ and $b \Rightarrow a = \top_L$, then $a = b$.

(28) [see also [16] p. 69, Th. 1.2 (8)]:
$a \Rightarrow \top_L = \top_L$.

(29) [see also [16] p. 69, Th. 1.3 (9)]:
$a \to a = \top_L$.

(30) [see also [16] p. 69, Th. 1.3 (10)]:
If $a \to b = \top_L$ and $b \to c = \top_L$, then $a \to c = \top_L$.

(31) [see also [16] p. 69, Th. 1.3 (11)]:
If $b < c$, then $a \sqcup b < a \sqcup c$ and $a \sqcap b < a \sqcap c$.

(32) [see also [16] p. 69, Th. 1.3 (12)]:
If $a < b$ and $c < d$, then $a \sqcup c < b \sqcup d$ and $a \sqcap c < b \sqcap d$.

(33) [see also [16] p. 69, Th. 1.3 (13)]:
$a \sqcap (a \to b) < b$.

(34) [see also [16] p. 69, Th. 1.3 (14)]:
$a \to (b \to c) = (a \sqcap b) \to c$.

(35) [see also [16] p. 69, Th. 1.3 (15)]:
$a \to (b \to c) = b \to (a \to c)$.

(36) [see also [16] p. 69, Th. 1.3 (16)]:
$a < (a \to b) \to b$. The theorem is a consequence of (33).

(37) [see also [16] p. 71, Th. 1.3 (50)]:
$a < b \to (a \sqcap b)$. The theorem is a consequence of (9).

(38) [see also [16] p. 69, Th. 1.3 (17)]:
$a \sqcap -a \leqslant b \sqcup -b$. The theorem is a consequence of (1) and (9).

(39) [see also [16] p. 70, Th. 1.3 (18)]:
$a \leqslant b \Rightarrow a \sqcap b$. The theorem is a consequence of (37) and (9).

(40) [see also [16] p. 70, Th. 1.3 (19)]:
$a \to \neg b = b \to \neg a$. The theorem is a consequence of (35).

(41) [see also [16] p. 70, Th. 1.3 (20)]:
$a \to \top_L = \top_L$. The theorem is a consequence of (9).

(42) [see also [16] p. 70, Th. 1.3 (21)]:
$\bot_L \to a = \top_L$. The theorem is a consequence of (9).

(43) [see also [16] p. 70, Th. 1.3 (22)]:
$\top_L \to b = b$. The theorem is a consequence of (9), (33), and (7).

(44) [see also [16] p. 70, Th. 1.3 (23)]:
If $a = \top_L$ and $a \to b = \top_L$, then $b = \top_L$.

(45) [see also [16] p. 70, Th. 1.3 (24)]:
$a \to (b \to a) = \top_L$. The theorem is a consequence of (9).

(46) [see also [16] p. 70, Th. 1.3 (25)]:
$(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top_L$. The theorem is a consequence of (33) and (35).

(47) [see also [16] p. 70, Th. 1.3 (26)]:
$a \rightarrow (a \sqcup b) = \top_L$. The theorem is a consequence of (11).

(48) [see also [16] p. 70, Th. 1.3 (27)]:
$b \rightarrow (a \sqcup b) = \top_L$. The theorem is a consequence of (11).

(49) [see also [16] p. 70, Th. 1.3 (28)]:
$(a \rightarrow c) \rightarrow ((b \rightarrow c) \rightarrow ((a \sqcup b) \rightarrow c)) = \top_L$. The theorem is a consequence of (33) and (10).

(50) [see also [16] p. 70, Th. 1.3 (29)]:
$(a \sqcap b) \rightarrow a = \top_L$. The theorem is a consequence of (10).

(51) [see also [16] p. 70, Th. 1.3 (30)]:
$(a \sqcap b) \rightarrow b = \top_L$. The theorem is a consequence of (10).

(52) [see also [16] p. 70, Th. 1.3 (31)]:
$(a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow (b \sqcap c))) = \top_L$. The theorem is a consequence of (33).

(53) [see also [16] p. 70, Th. 1.3 (32)]:
$(a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a) = \top_L$. The theorem is a consequence of (40) and (29).

(54) [see also [16] p. 70, Th. 1.3 (33)]:
$\neg(a \rightarrow a) \rightarrow b = \top_L$. The theorem is a consequence of (29), (2), (43), and (42).

(55) [see also [16] p. 70, Th. 1.3 (34)]:
$-a \rightarrow (a \rightarrow b) = \top_L$.

(56) [see also [16] p. 70, Th. 1.3 (35)]:
$(-(a \rightarrow b) \rightarrow (a \sqcap -b)) \sqcap ((a \sqcap -b) \rightarrow -(a \rightarrow b)) = \top_L$.

(57) [see also [16] p. 70, Th. 1.3 (36)]:
$(-\neg a \rightarrow a) \sqcap (a \rightarrow -\neg a) = \top_L$.

(58) [see also [16] p. 70, Th. 1.3 (37)]:
$--a = a$.

(59) [see also [16] p. 70, Th. 1.3 (38)]:
$-(a \sqcup b) = -a \sqcap -b$.

(60) [see also [16] p. 70, Th. 1.3 (39)]:
$-(a \sqcap b) = -a \sqcup -b$. The theorem is a consequence of (1).

(61) [see also [16] p. 70, Th. 1.3 (40)]:
If $a < b$, then $b \rightarrow c < a \rightarrow c$ and $c \rightarrow a < c \rightarrow b$. The theorem is a consequence of (43), (46), (10), and (41).

(62) [see also [16] p. 70, Th. 1.3 (41)]:
$(a \rightarrow b) \rightarrow ((c \rightarrow d) \rightarrow ((a \sqcap c) \rightarrow (b \sqcap d))) = \top_L$. The theorem is a consequence of (33).

(63) [SEE ALSO [16] P. 70, TH. 1.3 (42)]:
$(a \to b) \to ((c \to d) \to ((a \sqcup c) \to (b \sqcup d))) = \top_L$. The theorem is a consequence of (10).

(64) [SEE ALSO [16] P. 70, TH. 1.3 (43)]:
$(b \to a) \to ((c \to d) \to ((a \to c) \to (b \to d))) = \top_L$. The theorem is a consequence of (33).

5. ALTERNATIVE EQUATIONAL AXIOMATICS BY RASIOWA

Let L be a non empty Nelson structure. We say that L satisfies (qpB$_0^*$) if and only if

(Def. 22) for every elements a, b of L, $a \leqslant b$ iff $a \Rightarrow b = \top_L$.

We say that L satisfies (qpB$_1^*$) if and only if

(Def. 23) for every elements a, b of L, $a \to (b \to a) = \top_L$.

We say that L satisfies (qpB$_2^*$) if and only if

(Def. 24) for every elements a, b, c of L, $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top_L$.

We say that L satisfies (qpB$_3^*$) if and only if

(Def. 25) for every element a of L, $\top_L \to a = a$.

We say that L satisfies (qpB$_5^*$) if and only if

(Def. 26) for every elements a, b of L, $(a \Rightarrow b) \to ((b \Rightarrow a) \to b) = (b \Rightarrow a) \to ((a \Rightarrow b) \to a)$.

We say that L satisfies (qpB$_6^*$) if and only if

(Def. 27) for every elements a, b of L, $a \to (a \sqcup b) = \top_L$.

We say that L satisfies (qpB$_7^*$) if and only if

(Def. 28) for every elements a, b of L, $b \to (a \sqcup b) = \top_L$.

We say that L satisfies (qpB$_8^*$) if and only if

(Def. 29) for every elements a, b, c of L, $(a \to c) \to ((b \to c) \to ((a \sqcup b) \to c)) = \top_L$.

We say that L satisfies (qpB$_9^*$) if and only if

(Def. 30) for every elements a, b of L, $(a \sqcap b) \to a = \top_L$.

We say that L satisfies (qpB$_{10}^*$) if and only if

(Def. 31) for every elements a, b of L, $(a \sqcap b) \to b = \top_L$.

We say that L satisfies (qpB$_{11}^*$) if and only if

(Def. 32) for every elements a, b, c of L, $(a \to b) \to ((a \to c) \to (a \to (b \sqcap c))) = \top_L$.

We say that L satisfies (qpB$_{12}^*$) if and only if

(Def. 33) for every elements a, b of L, $(a \to \neg b) \to (b \to \neg a) = \top_L$.

We say that L satisfies (qpB$_{13}^*$) if and only if

(Def. 34) for every elements a, b of L, $\neg(a \to a) \to b = \top_L$.

We say that L satisfies (qpB$^*_{14}$) if and only if

(Def. 35) for every elements a, b of L, $-a \to (a \to b) = \top_L$.

We say that L satisfies (qpB$^*_{15}$) if and only if

(Def. 36) for every elements a, b of L, $(-(a \to b) \to (a \sqcap -b)) \sqcap ((a \sqcap -b) \to -(a \to b)) = \top_L$.

We say that L satisfies (qpB$^*_{17}$) if and only if

(Def. 37) for every elements a, b of L, $-(a \sqcup b) = -a \sqcap -b$.

We say that L satisfies (qpB$^*_{19}$) if and only if

(Def. 38) for every element a of L, $(-\neg a \to a) \sqcap (a \to -\neg a) = \top_L$.

We introduce L satisfies (qpB$^*_4$) as a synonym of L satisfies (qpB$_4$) and L satisfies (qpB$^*_{16}$) as a synonym of L is de Morgan and L satisfies (qpB$^*_{18}$) as a synonym of L is involutive.

Note that every Nelson algebra satisfies (qpB$^*_1$), (qpB$^*_2$), (qpB$^*_3$), (qpB$^*_4$), (qpB$^*_5$), (qpB$^*_6$), (qpB$^*_7$), (qpB$^*_8$), (qpB$^*_9$), (qpB$^*_{10}$), (qpB$^*_{11}$), (qpB$^*_{12}$), (qpB$^*_{13}$), (qpB$^*_{14}$), (qpB$^*_{15}$), (qpB$^*_{16}$), (qpB$^*_{17}$), (qpB$^*_{18}$), and (qpB$^*_{19}$).

Now we state the proposition:

(65) Let us consider a non empty Nelson structure L. Suppose L satisfies (qpB$^*_0$). Then L is a Nelson algebra if and only if L satisfies (qpB$^*_1$), (qpB$^*_2$), (qpB$^*_3$), (qpB$^*_4$), (qpB$^*_5$), (qpB$^*_6$), (qpB$^*_7$), (qpB$^*_8$), (qpB$^*_9$), (qpB$^*_{10}$), (qpB$^*_{11}$), (qpB$^*_{12}$), (qpB$^*_{13}$), (qpB$^*_{14}$), (qpB$^*_{15}$), (qpB$^*_{16}$), (qpB$^*_{17}$), (qpB$^*_{18}$), and (qpB$^*_{19}$).

Proof: Reconsider $L_1 = L$ as a de Morgan, non empty Nelson structure. For every elements a, b of L_1 such that $a \sqcap b = \top_{L_1}$ holds $a = \top_{L_1}$ and $b = \top_{L_1}$. For every elements a, b of L_1, $a \leqslant b$ iff $a < b$ and $-b < -a$. Set $d = (\top_L)^{\mathrm{c}}$. For every element a of L, $a \leqslant \top_L$. For every element a of L, $d \leqslant a$. For every element a of L, $d \sqcap a = d$. For every element a of L_1, $a \to \top_{L_1} = \top_{L_1}$. For every elements a, b, c of L_1 such that $a \to b = \top_{L_1}$ and $b \to c = \top_{L_1}$ holds $a \to c = \top_{L_1}$. L_1 has transitive $<$. L_1 satisfies (qpB$_6$). For every element a of L_1, $a \to a = \top_{L_1}$. L_1 satisfies (qpB$_7$). For every elements a, b of L_1, $a \sqcap b \leqslant a$. For every elements a, b of L_1, $a \leqslant a \sqcup b$. For every elements a, b of L_1, $b \leqslant a \sqcup b$. For every elements a, b of L_1, $a \sqcap b \leqslant b$. For every element a of L_1, $a \Rightarrow a = \top_{L_1}$. For every elements a, b of L_1, $a = b$ iff $a \Rightarrow b = \top_{L_1}$ and $b \Rightarrow a = \top_{L_1}$. For every elements a, b of L_1, $a \leqslant b \leqslant a$ iff $a = b$. L_1 has reflexive $<$. For every elements a, b, c of L_1 such that $a < b$ holds $b \to c < a \to c$ and $c \to a < c \to b$. For every elements a, b of L_1, $a \to (b \to (a \sqcap b)) = \top_{L_1}$. For every elements a, b, c of L_1 such that $a < b \to c$ holds $b < a \to c$. For every elements a, c of L_1, $a \to (a \to c) < a \to c$. L_1 satisfies (qpB$_3$). For every elements a, b, c of L_1

such that $b < c$ holds $a \sqcap b < a \sqcap c$. For every elements a, b, c of L_1 such that $b < c$ holds $a \sqcup b < a \sqcup c$. For every elements a, b, c of L_1 such that $a \leqslant c$ and $b \leqslant c$ holds $a \sqcup b \leqslant c$. For every elements a, b, c of L_1 such that $c \leqslant a$ and $c \leqslant b$ holds $c \leqslant a \sqcap b$. For every elements a, b of L_1, $b \sqcup a \leqslant a \sqcup b$. For every elements a, b of L_1, $a \sqcup b = b \sqcup a$. For every elements a, b of L_1, $a \sqcap b \leqslant b \sqcap a$. For every elements a, b of L_1, $a \sqcap b = b \sqcap a$. For every elements a, b, c of L_1 such that $a \leqslant b$ holds $a \sqcup c \leqslant b \sqcup c$. For every elements a, b of L_1, $b = (a \sqcap b) \sqcup b$. For every elements a, b of L_1, $a \sqcap (a \sqcup b) = a$. For every elements a, b, c of L_1 such that $b \leqslant c$ holds $a \sqcap b \leqslant a \sqcap c$. For every elements a, b, c of L_1 such that $a \leqslant b \leqslant c$ holds $a \leqslant c$. For every elements a, b, c of L_1, $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$. For every elements a, b, c of L_1, $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$. Set $c = \top_{L_1}$. For every element a of L_1, $c \sqcup a = c$ and $a \sqcup c = c$ by [18, (4)]. L_1 is distributive. L_1 satisfies (qpB_5). L_1 satisfies (qpB_8). L_1 satisfies (qpB_9). L_1 satisfies (qpB_{10}). L_1 satisfies (qpB_{11}). L_1 satisfies (qpB_{12}). For every elements a, b, c of L_1, $\neg \top_{L_1} = -\top_{L_1}$. For every elements a, b of L_1, $a \rightarrow \neg b = b \rightarrow \neg a$. L_1 satisfies (qpB_{13}). □

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Groups – Additive Notation

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Summary. We translate the articles covering group theory already available in the Mizar Mathematical Library from multiplicative into additive notation. We adapt the works of Wojciech A. Trybulec [41, 42, 43] and Artur Korniłowicz [25].

In particular, these authors have defined the notions of group, abelian group, power of an element of a group, order of a group and order of an element, subgroup, coset of a subgroup, index of a subgroup, conjugation, normal subgroup, topological group, dense subset and basis of a topological group. Lagrange's theorem and some other theorems concerning these notions [9, 24, 22] are presented.

Note that "The term $\mathbb{Z}$-module is simply another name for an additive abelian group" [27]. We take an approach different than that used by Futa et al. [21] to use in a future article the results obtained by Artur Korniłowicz [25]. Indeed, Hölzl et al. showed that it was possible to build "a generic theory of limits based on filters" in Isabelle/HOL [23, 10]. Our goal is to define the convergence of a sequence and the convergence of a series in an abelian topological group [11] using the notion of filters.

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Keywords: additive group; subgroup; Lagrange theorem; conjugation; normal subgroup; index; additive topological group; basis; neighborhood; additive abelian group; Z-module

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The notation and terminology used in this paper have been introduced in the following articles: [12], [32], [31], [2], [18], [28], [33], [13], [19], [39], [14], [15], [1], [40], [26], [35], [36], [5], [6], [16], [30], [8], [46], [47], [44], [29], [37], [45], [25], [48], [20], [7], [38], and [17].

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1. Additive Notation for Groups – `GROUP_1`

From now on m, n denote natural numbers, i, j denote integers, S denotes a non empty additive magma, and r, r_1, r_2, s, s_1, s_2, t, t_1, t_2 denote elements of S.

The scheme *SeqEx2Dbis* deals with non empty sets $\mathcal{X}$, $\mathcal{Z}$ and a ternary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists a function f from $\mathbb{N} \times \mathcal{X}$ into $\mathcal{Z}$ such that for every natural number x for every element y of $\mathcal{X}$, $\mathcal{P}[x, y, f(x, y)]$

provided

- for every natural number x and for every element y of $\mathcal{X}$, there exists an element z of $\mathcal{Z}$ such that $\mathcal{P}[x, y, z]$.

Let I_1 be an additive magma. We say that I_1 is add-unital if and only if

(Def. 1) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$.

We say that I_1 is additive group-like if and only if

(Def. 2) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$ and there exists an element g of I_1 such that $h + g = e$ and $g + h = e$.

Let us note that every additive magma which is additive group-like is also add-unital and there exists an additive magma which is strict, additive group-like, add-associative, and non empty.

An additive group is an additive group-like, add-associative, non empty additive magma. Now we state the propositions:

(1) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and there exists t such that for every s_1, $s_1 + t = s_1$ and $t + s_1 = s_1$ and there exists s_2 such that $s_1 + s_2 = t$ and $s_2 + s_1 = t$. Then S is an additive group.

(2) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and for every r and s, there exists t such that $r + t = s$ and there exists t such that $t + r = s$. Then S is add-associative and additive group-like.

(3) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is add-associative and additive group-like.

From now on G denotes an additive group-like, non empty additive magma and e, h denote elements of G.

Let G be an additive magma. Assume G is add-unital. The functor 0_G yielding an element of G is defined by

(Def. 3) for every element h of G, $h + it = h$ and $it + h = h$.

Now we state the proposition:

(4) If for every h, $h + e = h$ and $e + h = h$, then $e = 0_G$.

From now on G denotes an additive group and f, g, h denote elements of G.

Let us consider G and h. The functor $-h$ yielding an element of G is defined by

(Def. 4) $h + it = 0_G$ and $it + h = 0_G$.

Let us note that the functor is involutive.

Now we state the propositions:

(5) If $h + g = 0_G$ and $g + h = 0_G$, then $g = -h$.

(6) If $h + g = h + f$ or $g + h = f + h$, then $g = f$.

(7) If $h + g = h$ or $g + h = h$, then $g = 0_G$. The theorem is a consequence of (6).

(8) $-0_G = 0_G$.

(9) If $-h = -g$, then $h = g$. The theorem is a consequence of (6).

(10) If $-h = 0_G$, then $h = 0_G$. The theorem is a consequence of (8).

(11) If $h + g = 0_G$, then $h = -g$ and $g = -h$. The theorem is a consequence of (6).

(12) $h + f = g$ if and only if $f = -h + g$. The theorem is a consequence of (6).

(13) $f + h = g$ if and only if $f = g + -h$. The theorem is a consequence of (6).

(14) There exists f such that $g + f = h$. The theorem is a consequence of (12).

(15) There exists f such that $f + g = h$. The theorem is a consequence of (13).

(16) $-(h + g) = -g + -h$. The theorem is a consequence of (11).

(17) $g + h = h + g$ if and only if $-(g + h) = -g + -h$. The theorem is a consequence of (16) and (6).

(18) $g + h = h + g$ if and only if $-g + -h = -h + -g$. The theorem is a consequence of (16) and (17).

(19) $g+h = h+g$ if and only if $g+-h = -h+g$. The theorem is a consequence of (18), (11), and (16).

From now on u denotes a unary operation on G.

Let us consider G. The functor add inverse G yielding a unary operation on G is defined by

(Def. 5) $it(h) = -h$.

Let G be an add-associative, non empty additive magma. Let us note that the addition of G is associative.

Let us consider an add-unital, non empty additive magma G. Now we state the propositions:

(20) 0_G is a unity w.r.t. the addition of G.

(21) $\mathbf{1}_\alpha = 0_G$, where α is the addition of G. The theorem is a consequence of (20).

Let G be an add-unital, non empty additive magma. Let us note that the addition of G is unital.

Now we state the proposition:

(22) add inverse G is an inverse operation w.r.t. the addition of G. The theorem is a consequence of (21).

Let us consider G. One can verify that the addition of G has inverse operation.

Now we state the proposition:

(23) The inverse operation w.r.t. the addition of G = add inverse G. The theorem is a consequence of (22).

Let G be a non empty additive magma. The functor mult G yielding a function from $\mathbb{N} \times$ (the carrier of G) into the carrier of G is defined by

(Def. 6) for every element h of G, $it(0, h) = 0_G$ and for every natural number n, $it(n + 1, h) = it(n, h) + h$.

Let us consider G, i, and h. The functor $i \cdot h$ yielding an element of G is defined by the term

(Def. 7) $$\begin{cases} (\mathrm{mult}\, G)(|i|, h), & \textbf{if } 0 \leqslant i, \\ -(\mathrm{mult}\, G)(|i|, h), & \textbf{otherwise}. \end{cases}$$

Let us consider n. One can check that the functor $n \cdot h$ is defined by the term

(Def. 8) $(\mathrm{mult}\, G)(n, h)$.

Now we state the propositions:

(24) $0 \cdot h = 0_G$.

(25) $1 \cdot h = h$.

(26) $2 \cdot h = h + h$. The theorem is a consequence of (25).

(27) $3 \cdot h = h + h + h$. The theorem is a consequence of (26).

(28) $2 \cdot h = 0_G$ if and only if $-h = h$. The theorem is a consequence of (26) and (11).

(29) If $i \leqslant 0$, then $i \cdot h = -|i| \cdot h$. The theorem is a consequence of (8).

(30) $i \cdot 0_G = 0_G$. The theorem is a consequence of (8).

(31) $(-1) \cdot h = -h$. The theorem is a consequence of (25).

(32) $(i + j) \cdot h = i \cdot h + j \cdot h$.

PROOF: Define $\mathcal{P}[\text{integer}] \equiv$ for every i, $(i + \$_1) \cdot h = i \cdot h + \$_1 \cdot h$. For every j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j-1]$ and $\mathcal{P}[j+1]$. $\mathcal{P}[0]$. For every j, $\mathcal{P}[j]$ from [40, Sch. 4]. □

(33) (i) $(i+1) \cdot h = i \cdot h + h$, and

(ii) $(i+1) \cdot h = h + i \cdot h$.

The theorem is a consequence of (25) and (32).

(34) $(-i) \cdot h = -i \cdot h$.

Let us assume that $g + h = h + g$. Now we state the propositions:

(35) $i \cdot (g + h) = i \cdot g + i \cdot h$. The theorem is a consequence of (16).

(36) $i \cdot g + j \cdot h = j \cdot h + i \cdot g$. The theorem is a consequence of (19) and (16).

(37) $g + i \cdot h = i \cdot h + g$. The theorem is a consequence of (25) and (36).

Let us consider G and h. We say that h is of order 0 if and only if

(Def. 9) if $n \cdot h = 0_G$, then $n = 0$.

One can check that 0_G is non of order 0.

Let us consider h. The functor $\operatorname{ord}(h)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 10) (i) $it = 0$, **if** h is of order 0,

(ii) $it \cdot h = 0_G$ and $it \neq 0$ and for every m such that $m \cdot h = 0_G$ and $m \neq 0$ holds $it \leqslant m$, **otherwise**.

Now we state the propositions:

(38) $\operatorname{ord}(h) \cdot h = 0_G$.

(39) $\operatorname{ord}(0_G) = 1$.

(40) If $\operatorname{ord}(h) = 1$, then $h = 0_G$. The theorem is a consequence of (25).

Observe that there exists an additive group which is strict and Abelian.

Now we state the proposition:

(41) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is an Abelian additive group. The theorem is a consequence of (3).

In the sequel A denotes an Abelian additive group and a, b denote elements of A.

Now we state the propositions:

(42) $-(a + b) = -a + -b$.

(43) $i \cdot (a + b) = i \cdot a + i \cdot b$.

(44) $\langle$the carrier of A, the addition of $A, 0_A\rangle$ is Abelian, add-associative, right zeroed, and right complementable.

Let us consider an add-unital, non empty additive magma L and an element x of L. Now we state the propositions:

(45) $(\operatorname{mult} L)(1, x) = x$.

(46) $(\operatorname{mult} L)(2,x) = x + x$. The theorem is a consequence of (45).

Now we state the proposition:

(47) Let us consider an add-associative, Abelian, add-unital, non empty additive magma L, elements x, y of L, and a natural number n. Then $(\operatorname{mult} L)(n, x+y) = (\operatorname{mult} L)(n,x) + (\operatorname{mult} L)(n,y)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\operatorname{mult} L)(\$_1, x+y) = (\operatorname{mult} L)(\$_1, x) + (\operatorname{mult} L)(\$_1, y)$. For every natural number n, $\mathcal{P}[n]$ from [5, Sch. 2]. □

Let G, H be additive magmas and I_1 be a function from G into H. We say that I_1 preserves zero if and only if

(Def. 11) $I_1(0_G) = 0_H$.

2. Subgroups and Lagrange Theorem – GROUP_2

In the sequel x denotes an object, y, y_1, y_2, Y, Z denote sets, k denotes a natural number, G denotes an additive group, a, g, h denote elements of G, and A denotes a subset of G.

Let us consider G and A. The functor $-A$ yielding a subset of G is defined by the term

(Def. 12) $\{-g : g \in A\}$.

One can check that the functor is involutive.

Now we state the propositions:

(48) $x \in -A$ if and only if there exists g such that $x = -g$ and $g \in A$.

(49) $-\{g\} = \{-g\}$.

(50) $-\{g, h\} = \{-g, -h\}$.

(51) $-\emptyset_\alpha = \emptyset$, where α is the carrier of G.

(52) $-\Omega_\alpha =$ the carrier of G, where α is the carrier of G.

(53) $A \neq \emptyset$ if and only if $-A \neq \emptyset$. The theorem is a consequence of (48).

Let us consider G. Let A be an empty subset of G. Observe that $-A$ is empty.

Let A be a non empty subset of G. One can check that $-A$ is non empty.

In the sequel G denotes a non empty additive magma, A, B, C denote subsets of G, and a, b, g, g_1, g_2, h, h_1, h_2 denote elements of G.

Let G be an Abelian, non empty additive magma and A, B be subsets of G. One can check that the functor $A + B$ is commutative.

(54) $x \in A + B$ if and only if there exists g and there exists h such that $x = g + h$ and $g \in A$ and $h \in B$.

(55) $A \neq \emptyset$ and $B \neq \emptyset$ if and only if $A + B \neq \emptyset$. The theorem is a consequence of (54).

(56) If G is add-associative, then $(A+B)+C = A+(B+C)$.

(57) Let us consider an additive group G, and subsets A, B of G. Then $-(A+B) = -B + -A$. The theorem is a consequence of (16).

(58) $A+(B \cup C) = A+B \cup (A+C)$.

(59) $(A \cup B)+C = A+C \cup (B+C)$.

(60) $A+B \cap C \subseteq (A+B) \cap (A+C)$.

(61) $A \cap B+C \subseteq (A+C) \cap (B+C)$.

(62) (i) $\emptyset_{\alpha} + A = \emptyset$, and

(ii) $A + \emptyset_{\alpha} = \emptyset$,

where α is the carrier of G. The theorem is a consequence of (54).

(63) Let us consider an additive group G, and a subset A of G. Suppose $A \neq \emptyset$. Then

(i) $\Omega_{\alpha} + A =$ the carrier of G, and

(ii) $A + \Omega_{\alpha} =$ the carrier of G,

where α is the carrier of G.

(64) $\{g\} + \{h\} = \{g+h\}$.

(65) $\{g\} + \{g_1, g_2\} = \{g+g_1, g+g_2\}$.

(66) $\{g_1, g_2\} + \{g\} = \{g_1+g, g_2+g\}$.

(67) $\{g, h\} + \{g_1, g_2\} = \{g+g_1, g+g_2, h+g_1, h+g_2\}$.

Let us consider an additive group G and a subset A of G. Now we state the propositions:

(68) Suppose for every elements g_1, g_2 of G such that $g_1, g_2 \in A$ holds $g_1+g_2 \in A$ and for every element g of G such that $g \in A$ holds $-g \in A$. Then $A+A = A$.

(69) If for every element g of G such that $g \in A$ holds $-g \in A$, then $-A = A$.

(70) If for every a and b such that $a \in A$ and $b \in B$ holds $a+b = b+a$, then $A+B = B+A$.

(71) If G is an Abelian additive group, then $A+B = B+A$.

(72) Let us consider an Abelian additive group G, and subsets A, B of G. Then $-(A+B) = -A + -B$. The theorem is a consequence of (42).

Let us consider G, g, and A. The functors: $g+A$ and $A+g$ yielding subsets of G are defined by terms,

(Def. 13) $\{g\} + A$,

(Def. 14) $A + \{g\}$,

respectively. Now we state the propositions:

(73) $x \in g+A$ if and only if there exists h such that $x = g+h$ and $h \in A$.

(74) $x \in A + g$ if and only if there exists h such that $x = h + g$ and $h \in A$.

Let us assume that G is add-associative. Now we state the propositions:

(75) $(g + A) + B = g + (A + B)$.

(76) $(A + g) + B = A + (g + B)$.

(77) $(A + B) + g = A + (B + g)$.

(78) $(g + h) + A = g + (h + A)$. The theorem is a consequence of (64) and (56).

(79) $(g + A) + h = g + (A + h)$.

(80) $(A + g) + h = A + (g + h)$. The theorem is a consequence of (56) and (64).

(81) (i) $\emptyset_\alpha + a = \emptyset$, and

(ii) $a + \emptyset_\alpha = \emptyset$,

where α is the carrier of G.

From now on G denotes an additive group-like, non empty additive magma, h, g, g_1, g_2 denote elements of G, and A denotes a subset of G.

(82) Let us consider an additive group G, and an element a of G. Then

(i) $\Omega_\alpha + a =$ the carrier of G, and

(ii) $a + \Omega_\alpha =$ the carrier of G,

where α is the carrier of G.

(83) (i) $0_G + A = A$, and

(ii) $A + 0_G = A$.

The theorem is a consequence of (73) and (74).

(84) If G is an Abelian additive group, then $g + A = A + g$.

Let G be an additive group-like, non empty additive magma.

A subgroup of G is an additive group-like, non empty additive magma and is defined by

(Def. 15) the carrier of *it* $\subseteq$ the carrier of G and the addition of *it* $=$ (the addition of G) $\restriction$ (the carrier of *it*).

In the sequel H denotes a subgroup of G and h, h_1, h_2 denote elements of H.

Now we state the propositions:

(85) If G is finite, then H is finite.

(86) If $x \in H$, then $x \in G$.

(87) $h \in G$.

(88) h is an element of G.

(89) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 + h_2 = g_1 + g_2$.

Let G be an additive group. Let us observe that every subgroup of G is add-associative.

In the sequel G, G_1, G_2, G_3 denote additive groups, a, a_1, a_2, b, b_1, b_2, g, g_1, g_2 denote elements of G, A, B denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, and h, h_1, h_2 denote elements of H.

(90) $0_H = 0_G$. The theorem is a consequence of (87), (89), and (7).

(91) $0_{H_1} = 0_{H_2}$. The theorem is a consequence of (90).

(92) $0_G \in H$. The theorem is a consequence of (90).

(93) $0_{H_1} \in H_2$. The theorem is a consequence of (90) and (92).

(94) If $h = g$, then $-h = -g$. The theorem is a consequence of (87), (89), (90), and (11).

(95) add inverse H = add inverse $G \restriction$(the carrier of H). The theorem is a consequence of (87) and (94).

(96) If $g_1, g_2 \in H$, then $g_1 + g_2 \in H$. The theorem is a consequence of (89).

(97) If $g \in H$, then $-g \in H$. The theorem is a consequence of (94).

Let us consider G. Observe that there exists a subgroup of G which is strict.

(98) Suppose $A \neq \emptyset$ and for every g_1 and g_2 such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ and for every g such that $g \in A$ holds $-g \in A$. Then there exists a strict subgroup H of G such that the carrier of $H = A$.

Proof: Reconsider $D = A$ as a non empty set. Set $o =$ (the addition of G) $\restriction A$. $\operatorname{rng} o \subseteq A$ by [17, (87)], [14, (47)]. Set $H = \langle D, o \rangle$. H is additive group-like. $\square$

(99) If G is an Abelian additive group, then H is Abelian. The theorem is a consequence of (87) and (89).

Let G be an Abelian additive group. One can check that every subgroup of G is Abelian.

(100) G is a subgroup of G.

(101) Suppose G_1 is a subgroup of G_2 and G_2 is a subgroup of G_1. Then the additive magma of G_1 = the additive magma of G_2.

(102) If G_1 is a subgroup of G_2 and G_2 is a subgroup of G_3, then G_1 is a subgroup of G_3.

(103) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a subgroup of H_2.

(104) If for every g such that $g \in H_1$ holds $g \in H_2$, then H_1 is a subgroup of H_2. The theorem is a consequence of (87) and (103).

(105) Suppose the carrier of H_1 = the carrier of H_2. Then the additive magma of H_1 = the additive magma of H_2. The theorem is a consequence of (103) and (101).

(106) Suppose for every g, $g \in H_1$ iff $g \in H_2$. Then the additive magma of H_1 = the additive magma of H_2. The theorem is a consequence of (104) and (101).

Let us consider G. Let H_1, H_2 be strict subgroups of G. One can check that $H_1 = H_2$ if and only if the condition (Def. 16) is satisfied.

(Def. 16) for every g, $g \in H_1$ iff $g \in H_2$.

Now we state the propositions:

(107) Let us consider an additive group G, and a subgroup H of G. Suppose the carrier of $G \subseteq$ the carrier of H. Then the additive magma of H = the additive magma of G. The theorem is a consequence of (100) and (105).

(108) Suppose for every element g of G, $g \in H$. Then the additive magma of H = the additive magma of G. The theorem is a consequence of (100) and (106).

Let us consider G. The functor $\mathbf{0}_G$ yielding a strict subgroup of G is defined by

(Def. 17) the carrier of *it* $= \{0_G\}$.

The functor Ω_G yielding a strict subgroup of G is defined by the term

(Def. 18) the additive magma of G.

Note that the functor is projective.

Now we state the propositions:

(109) $\mathbf{0}_H = \mathbf{0}_G$. The theorem is a consequence of (90) and (102).

(110) $\mathbf{0}_{H_1} = \mathbf{0}_{H_2}$. The theorem is a consequence of (109).

(111) $\mathbf{0}_G$ is a subgroup of H. The theorem is a consequence of (109).

(112) Let us consider a strict additive group G. Then every subgroup of G is a subgroup of Ω_G.

(113) Every strict additive group is a subgroup of Ω_G.

(114) $\mathbf{0}_G$ is finite.

Let us consider G. Note that $\mathbf{0}_G$ is finite and there exists a subgroup of G which is strict and finite and there exists an additive group which is strict and finite.

Let G be a finite additive group. One can verify that every subgroup of G is finite.

Now we state the propositions:

(115) $\overline{\overline{\mathbf{0}_G}} = 1$.

(116) Let us consider a strict, finite subgroup H of G. If $\overline{\overline{H}} = 1$, then $H = \mathbf{0}_G$. The theorem is a consequence of (92).

(117) $\overline{\overline{H}} \subseteq \overline{\overline{G}}$.

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(118) $\overline{\overline{H}} \leqslant \overline{\overline{G}}$.

(119) Suppose $\overline{\overline{G}} = \overline{\overline{H}}$. Then the additive magma of $H =$ the additive magma of G.

PROOF: The carrier of $H =$ the carrier of G by [3, (48)]. $\square$

Let us consider G and H. The functor $\overline{H}$ yielding a subset of G is defined by the term

(Def. 19) the carrier of H.

Now we state the propositions:

(120) If $g_1, g_2 \in \overline{H}$, then $g_1 + g_2 \in \overline{H}$. The theorem is a consequence of (96).

(121) If $g \in \overline{H}$, then $-g \in \overline{H}$. The theorem is a consequence of (97).

(122) $\overline{H} + \overline{H} = \overline{H}$. The theorem is a consequence of (121), (120), and (68).

(123) $-\overline{H} = \overline{H}$. The theorem is a consequence of (121) and (69).

(124) (i) if $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$, then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$, and

(ii) if there exists H such that the carrier of $H = \overline{H_1} + \overline{H_2}$, then $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$.

The theorem is a consequence of (121), (16), (120), (55), and (98).

(125) Suppose G is an Abelian additive group. Then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$. The theorem is a consequence of (71) and (124).

Let us consider G, H_1, and H_2. The functor $H_1 \cap H_2$ yielding a strict subgroup of G is defined by

(Def. 20) the carrier of *it* $= \overline{H_1} \cap \overline{H_2}$.

Now we state the propositions:

(126) (i) for every subgroup H of G such that $H = H_1 \cap H_2$ holds the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2), and

(ii) for every strict subgroup H of G such that the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2) holds $H = H_1 \cap H_2$.

(127) $\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$.

(128) $x \in H_1 \cap H_2$ if and only if $x \in H_1$ and $x \in H_2$.

(129) Let us consider a strict subgroup H of G. Then $H \cap H = H$. The theorem is a consequence of (105).

Let us consider G, H_1, and H_2. Note that the functor $H_1 \cap H_2$ is commutative.

(130) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$. The theorem is a consequence of (105).

(131) (i) $\mathbf{0}_G \cap H = \mathbf{0}_G$, and

(ii) $H \cap \mathbf{0}_G = \mathbf{0}_G$.

The theorem is a consequence of (111).

(132) Let us consider a strict additive group G, and a strict subgroup H of G. Then

(i) $H \cap \Omega_G = H$, and

(ii) $\Omega_G \cap H = H$.

(133) Let us consider a strict additive group G. Then $\Omega_G \cap \Omega_G = G$.

(134) $H_1 \cap H_2$ is subgroup of H_1 and subgroup of H_2.

(135) Let us consider a subgroup H_1 of G. Then H_1 is a subgroup of H_2 if and only if the additive magma of $H_1 \cap H_2 =$ the additive magma of H_1.

(136) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of H_2. The theorem is a consequence of (102).

(137) If H_1 is subgroup of H_2 and subgroup of H_3, then H_1 is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (86), (128), and (104).

(138) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (126) and (103).

(139) If H_1 is finite or H_2 is finite, then $H_1 \cap H_2$ is finite.

Let us consider G, H, and A. The functors: $A + H$ and $H + A$ yielding subsets of G are defined by terms,

(Def. 21) $A + \overline{H}$,

(Def. 22) $\overline{H} + A$,

respectively. Now we state the propositions:

(140) $x \in A + H$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in A$ and $g_2 \in H$.

(141) $x \in H + A$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in H$ and $g_2 \in A$.

(142) $(A + B) + H = A + (B + H)$.

(143) $(A + H) + B = A + (H + B)$.

(144) $(H + A) + B = H + (A + B)$.

(145) $(A + H_1) + H_2 = A + (H_1 + \overline{H_2})$.

(146) $(H_1 + A) + H_2 = H_1 + (A + H_2)$.

(147) $(H_1 + \overline{H_2}) + A = H_1 + (H_2 + A)$.

(148) If G is an Abelian additive group, then $A + H = H + A$.

Let us consider G, H, and a. The functors: $a+H$ and $H+a$ yielding subsets of G are defined by terms,

(Def. 23) $a+\overline{H}$,

(Def. 24) $\overline{H}+a$,

respectively. Now we state the propositions:

(149) $x \in a+H$ if and only if there exists g such that $x = a+g$ and $g \in H$. The theorem is a consequence of (73).

(150) $x \in H+a$ if and only if there exists g such that $x = g+a$ and $g \in H$. The theorem is a consequence of (74).

(151) $(a+b)+H = a+(b+H)$.

(152) $(a+H)+b = a+(H+b)$.

(153) $(H+a)+b = H+(a+b)$.

(154) (i) $a \in a+H$, and

(ii) $a \in H+a$.

The theorem is a consequence of (92), (149), and (150).

(155) (i) $0_G+H = \overline{H}$, and

(ii) $H+0_G = \overline{H}$.

(156) (i) $\mathbf{0}_G+a = \{a\}$, and

(ii) $a+\mathbf{0}_G = \{a\}$.

The theorem is a consequence of (64).

(157) (i) $a+\Omega_G =$ the carrier of G, and

(ii) $\Omega_G+a =$ the carrier of G.

The theorem is a consequence of (63).

(158) If G is an Abelian additive group, then $a+H = H+a$.

(159) $a \in H$ if and only if $a+H = \overline{H}$. The theorem is a consequence of (149), (96), (97), and (92).

(160) $a+H = b+H$ if and only if $-b+a \in H$. The theorem is a consequence of (78), (83), and (159).

(161) $a+H = b+H$ if and only if $a+H$ meets $b+H$. The theorem is a consequence of (154), (149), (97), (13), (12), (96), and (160).

(162) $(a+b)+H \subseteq a+H+(b+H)$. The theorem is a consequence of (149) and (92).

(163) (i) $\overline{H} \subseteq a+H+(-a+H)$, and

(ii) $\overline{H} \subseteq -a+H+(a+H)$.

The theorem is a consequence of (83) and (162).

(164) $2 \cdot a + H \subseteq a + H + (a + H)$. The theorem is a consequence of (26) and (162).

(165) $a \in H$ if and only if $H + a = \overline{H}$. The theorem is a consequence of (150), (96), (97), and (92).

(166) $H + a = H + b$ if and only if $b + -a \in H$. The theorem is a consequence of (83), (80), and (165).

(167) $H + a = H + b$ if and only if $H + a$ meets $H + b$. The theorem is a consequence of (154), (150), (97), (12), (13), (96), and (166).

(168) $(H + a) + b \subseteq H + a + (H + b)$. The theorem is a consequence of (92), (150), and (80).

(169) (i) $\overline{H} \subseteq H + a + (H + -a)$, and

(ii) $\overline{H} \subseteq H + -a + (H + a)$.

The theorem is a consequence of (80), (83), and (168).

(170) $H + 2 \cdot a \subseteq H + a + (H + a)$. The theorem is a consequence of (80), (26), and (168).

(171) $a + H_1 \cap H_2 = (a + H_1) \cap (a + H_2)$. The theorem is a consequence of (149), (128), and (6).

(172) $H_1 \cap H_2 + a = (H_1 + a) \cap (H_2 + a)$. The theorem is a consequence of (150), (128), and (6).

(173) There exists a strict subgroup H_1 of G such that the carrier of $H_1 = a + H_2 + -a$. The theorem is a consequence of (154), (74), (149), (97), (150), (16), (73), (56), (96), and (98).

(174) $a + H \approx b + H$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = b + -a + g_1$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = b + H$. f is one-to-one. □

(175) $a + H \approx H + b$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = -a + g_1 + b$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = H + b$. f is one-to-one. □

(176) $H + a \approx H + b$. The theorem is a consequence of (175).

(177) (i) $\overline{H} \approx a + H$, and

(ii) $\overline{H} \approx H + a$.

The theorem is a consequence of (83), (174), and (176).

(178) (i) $\overline{\overline{H}} = \overline{\overline{a+H}}$, and

(ii) $\overline{\overline{H}} = \overline{\overline{H+a}}$.

(179) Let us consider a finite subgroup H of G. Then there exist finite sets B, C such that

(i) $B = a + H$, and

(ii) $C = H + a$, and

(iii) $\overline{\overline{H}} = \overline{\overline{B}}$, and

(iv) $\overline{\overline{H}} = \overline{\overline{C}}$.

The theorem is a consequence of (177).

Let us consider G and H. The functors: the left cosets of H and the right cosets of H yielding families of subsets of G are defined by conditions,

(Def. 25) $A \in$ the left cosets of H iff there exists a such that $A = a + H$,

(Def. 26) $A \in$ the right cosets of H iff there exists a such that $A = H + a$,

respectively. Now we state the propositions:

(180) If G is finite, then the right cosets of H is finite and the left cosets of H is finite.

(181) (i) $\overline{H} \in$ the left cosets of H, and

(ii) $\overline{H} \in$ the right cosets of H.

The theorem is a consequence of (83).

(182) The left cosets of $H \approx$ the right cosets of H.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g such that $\$_1 = g + H$ and $\$_2 = H + -g$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f =$ the right cosets of H. f is one-to-one. $\square$

(183) (i) $\bigcup$(the left cosets of H) = the carrier of G, and

(ii) $\bigcup$(the right cosets of H) = the carrier of G.

The theorem is a consequence of (87), (149), and (150).

(184) The left cosets of $\mathbf{0}_G =$ the set of all $\{a\}$. The theorem is a consequence of (156).

(185) The right cosets of $\mathbf{0}_G =$ the set of all $\{a\}$. The theorem is a consequence of (156).

Let us consider a strict subgroup H of G. Now we state the propositions:

(186) If the left cosets of $H =$ the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (149), (92), and (6).

(187) If the right cosets of $H =$ the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (150), (92), and (6).

(188) (i) the left cosets of $\Omega_G = \{\text{the carrier of } G\}$, and

(ii) the right cosets of $\Omega_G = \{\text{the carrier of } G\}$.

The theorem is a consequence of (157).

Let us consider a strict additive group G and a strict subgroup H of G. Now we state the propositions:

(189) If the left cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (149), (6), and (108).

(190) If the right cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (150), (6), and (108).

Let us consider G and H. The functor $|\bullet : H|$ yielding a cardinal number is defined by the term

(Def. 27) $\overline{\overline{\alpha}}$, where α is the left cosets of H.

Now we state the proposition:

(191) (i) $|\bullet : H| = \overline{\overline{\alpha}}$, and

(ii) $|\bullet : H| = \overline{\overline{\beta}}$,

where α is the left cosets of H and β is the right cosets of H.

Let us consider G and H. Assume the left cosets of H is finite. The functor $|\bullet : H|_{\mathbb{N}}$ yielding an element of $\mathbb{N}$ is defined by

(Def. 28) there exists a finite set B such that $B =$ the left cosets of H and $it = \overline{\overline{B}}$.

Now we state the proposition:

(192) Suppose the left cosets of H is finite. Then

(i) there exists a finite set B such that $B =$ the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$, and

(ii) there exists a finite set C such that $C =$ the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$.

The theorem is a consequence of (182).

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(193) LAGRANGE THEOREM FOR ADDITIVE GROUPS:
$\overline{\overline{G}} = \overline{\overline{H}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (179), (174), (161), and (183).

(194) $\overline{\overline{H}} \mid \overline{\overline{G}}$. The theorem is a consequence of (193).

(195) Let us consider a finite additive group G, subgroups I, H of G, and a subgroup J of H. Suppose $I = J$. Then $|\bullet : I|_{\mathbb{N}} = |\bullet : J|_{\mathbb{N}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (193).

(196) $|\bullet : \Omega_G|_{\mathbb{N}} = 1$. The theorem is a consequence of (188).

(197) Let us consider a strict additive group G, and a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 1$. Then $H = G$. The theorem is a consequence of (183) and (189).

(198) $|\bullet : \mathbf{0}_G| = \overline{\overline{G}}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\operatorname{dom} f =$ the carrier of G and for every object x such that $x \in$ the carrier of G holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\operatorname{rng} f =$ the left cosets of $\mathbf{0}_G$. f is one-to-one by [17, (3)]. □

(199) Let us consider a finite additive group G. Then $|\bullet : \mathbf{0}_G|_{\mathbb{N}} = \overline{\overline{G}}$. The theorem is a consequence of (193) and (115).

(200) Let us consider a finite additive group G, and a strict subgroup H of G. Suppose $|\bullet : H|_{\mathbb{N}} = \overline{\overline{G}}$. Then $H = \mathbf{0}_G$. The theorem is a consequence of (193) and (116).

(201) Let us consider a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H| = \overline{\overline{G}}$. Then

(i) G is finite, and

(ii) $H = \mathbf{0}_G$.

The theorem is a consequence of (200).

3. CLASSES OF CONJUGATION AND NORMAL SUBGROUPS – `GROUP_3`

From now on x, y, y_1, y_2 denote sets, G denotes an additive group, a, b, c, d, g, h denote elements of G, A, B, C, D denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, n denotes a natural number, and i denotes an integer.

Now we state the propositions:

(202) (i) $a + b + -b = a$, and

(ii) $a + -b + b = a$, and

(iii) $-b + b + a = a$, and

(iv) $b + -b + a = a$, and

(v) $a + (b + -b) = a$, and

(vi) $a + (-b + b) = a$, and

(vii) $-b + (b + a) = a$, and

(viii) $b + (-b + a) = a$.

(203) G is an Abelian additive group if and only if the addition of G is commutative.

(204) $\mathbf{0}_G$ is Abelian.

(205) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(206) If $A \subseteq B$, then $a + A \subseteq a + B$ and $A + a \subseteq B + a$.

(207) If H_1 is a subgroup of H_2, then $a + H_1 \subseteq a + H_2$ and $H_1 + a \subseteq H_2 + a$. The theorem is a consequence of (205).

(208) $a + H = \{a\} + H$.

(209) $H + a = H + \{a\}$.

(210) $(A + a) + H = A + (a + H)$. The theorem is a consequence of (142).

(211) $(a + H) + A = a + (H + A)$. The theorem is a consequence of (143).

(212) $(A + H) + a = A + (H + a)$. The theorem is a consequence of (143).

(213) $(H + a) + A = H + (a + A)$. The theorem is a consequence of (144).

(214) $(H_1 + a) + H_2 = H_1 + (a + H_2)$.

Let us consider G. The functor $\mathrm{SubGr}\, G$ yielding a set is defined by

(Def. 29) for every object x, $x \in$ *it* iff x is a strict subgroup of G.

Note that $\mathrm{SubGr}\, G$ is non empty.

Now we state the propositions:

(215) Let us consider a strict additive group G. Then $G \in \mathrm{SubGr}\, G$. The theorem is a consequence of (100).

(216) If G is finite, then $\mathrm{SubGr}\, G$ is finite.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H of G such that $\$_1 = H$ and $\$_2 =$ the carrier of H. For every object x such that $x \in \mathrm{SubGr}\, G$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = \mathrm{SubGr}\, G$ and for every object x such that $x \in \mathrm{SubGr}\, G$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\mathrm{rng}\, f \subseteq 2^{\alpha}$, where α is the carrier of G. f is one-to-one. $\square$

Let us consider G, a, and b. The functor $a \cdot b$ yielding an element of G is defined by the term

(Def. 30) $-b + a + b$.

Now we state the propositions:

(217) If $a \cdot g = b \cdot g$, then $a = b$. The theorem is a consequence of (6).

(218) $0_G \cdot a = 0_G$.

(219) If $a \cdot b = 0_G$, then $a = 0_G$. The theorem is a consequence of (11) and (7).

(220) $a \cdot 0_G = a$. The theorem is a consequence of (8).

(221) $a \cdot a = a$.

(222) (i) $a \cdot (-a) = a$, and

(ii) $(-a) \cdot a = -a$.

(223) $a \cdot b = a$ if and only if $a + b = b + a$. The theorem is a consequence of (12).

(224) $(a + b) \cdot g = a \cdot g + b \cdot g$.

(225) $a \cdot g \cdot h = a \cdot (g + h)$. The theorem is a consequence of (16).

(226) (i) $a \cdot b \cdot (-b) = a$, and

(ii) $a \cdot (-b) \cdot b = a$.

The theorem is a consequence of (225) and (220).

(227) $(-a) \cdot b = -a \cdot b$. The theorem is a consequence of (16).

(228) $(n \cdot a) \cdot b = n \cdot (a \cdot b)$.

(229) $(i \cdot a) \cdot b = i \cdot (a \cdot b)$. The theorem is a consequence of (29) and (227).

(230) If G is an Abelian additive group, then $a \cdot b = a$. The theorem is a consequence of (202).

(231) If for every a and b, $a \cdot b = a$, then G is Abelian. The theorem is a consequence of (223).

Let us consider G, A, and B. The functor $A \cdot B$ yielding a subset of G is defined by the term

(Def. 31) $\{g \cdot h : g \in A \text{ and } h \in B\}$.

Now we state the propositions:

(232) $x \in A \cdot B$ if and only if there exists g and there exists h such that $x = g \cdot h$ and $g \in A$ and $h \in B$.

(233) $A \cdot B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. The theorem is a consequence of (232).

(234) $A \cdot B \subseteq -B + A + B$.

(235) $(A + B) \cdot C \subseteq A \cdot C + B \cdot C$. The theorem is a consequence of (224).

(236) $A \cdot B \cdot C = A \cdot (B + C)$. The theorem is a consequence of (225).

(237) $(-A) \cdot B = -A \cdot B$. The theorem is a consequence of (227).

(238) $\{a\} \cdot \{b\} = \{a \cdot b\}$. The theorem is a consequence of (49), (64), (233), and (234).

(239) $\{a\} \cdot \{b, c\} = \{a \cdot b, a \cdot c\}$.

(240) $\{a, b\} \cdot \{c\} = \{a \cdot c, b \cdot c\}$.

(241) $\{a, b\} \cdot \{c, d\} = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$.

Let us consider G, A, and g. The functors: $A \cdot g$ and $g \cdot A$ yielding subsets of G are defined by terms, (Def. 32) $A \cdot \{g\}$,

(Def. 33) $\{g\} \cdot A$,

respectively. Now we state the propositions:

(242) $x \in A \cdot g$ if and only if there exists h such that $x = h \cdot g$ and $h \in A$.

(243) $x \in g \cdot A$ if and only if there exists h such that $x = g \cdot h$ and $h \in A$.

(244) $g \cdot A \subseteq -A + g + A$. The theorem is a consequence of (243) and (74).

(245) $A \cdot B \cdot g = A \cdot (B + g)$.

(246) $A \cdot g \cdot B = A \cdot (g + B)$.

(247) $g \cdot A \cdot B = g \cdot (A + B)$.

(248) $A \cdot a \cdot b = A \cdot (a + b)$. The theorem is a consequence of (236) and (64).

(249) $a \cdot A \cdot b = a \cdot (A + b)$.

(250) $a \cdot b \cdot A = a \cdot (b + A)$. The theorem is a consequence of (238) and (236).

(251) $A \cdot g = -g + A + g$. The theorem is a consequence of (234), (49), (74), (73), and (242).

(252) $(A + B) \cdot a \subseteq A \cdot a + B \cdot a$.

(253) $A \cdot 0_G = A$. The theorem is a consequence of (251), (83), and (8).

(254) If $A \neq \emptyset$, then $0_G \cdot A = \{0_G\}$. The theorem is a consequence of (243) and (218).

(255) (i) $A \cdot a \cdot (-a) = A$, and

(ii) $A \cdot (-a) \cdot a = A$.

The theorem is a consequence of (248) and (253).

(256) G is an Abelian additive group if and only if for every A and B such that $B \neq \emptyset$ holds $A \cdot B = A$. The theorem is a consequence of (230), (238), and (231).

(257) G is an Abelian additive group if and only if for every A and g, $A \cdot g = A$. The theorem is a consequence of (256), (238), and (231).

(258) G is an Abelian additive group if and only if for every A and g such that $A \neq \emptyset$ holds $g \cdot A = \{g\}$. The theorem is a consequence of (256), (238), and (231).

Let us consider G, H, and a. The functor $H \cdot a$ yielding a strict subgroup of G is defined by

(Def. 34) the carrier of *it* $= \overline{H} \cdot a$.

Now we state the propositions:

(259) $x \in H \cdot a$ if and only if there exists g such that $x = g \cdot a$ and $g \in H$. The theorem is a consequence of (242).

(260) The carrier of $H \cdot a = -a + H + a$. The theorem is a consequence of (251).

(261) $H \cdot a \cdot b = H \cdot (a + b)$. The theorem is a consequence of (248) and (105).

Let us consider a strict subgroup H of G. Now we state the propositions:

(262) $H \cdot 0_G = H$. The theorem is a consequence of (253) and (105).

(263) (i) $H \cdot a \cdot (-a) = H$, and

(ii) $H \cdot (-a) \cdot a = H$.

The theorem is a consequence of (261) and (262).

Now we state the propositions:

(264) $(H_1 \cap H_2) \cdot a = H_1 \cdot a \cap (H_2 \cdot a)$. The theorem is a consequence of (259), (128), and (217).

(265) $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

PROOF: Define $\mathcal{F}(\text{element of } G) = \$_1 \cdot a$. Consider f being a function from the carrier of G into the carrier of G such that for every g, $f(g) = \mathcal{F}(g)$ from [15, Sch. 4]. Set $g = f{\restriction}(\text{the carrier of } H)$. $\operatorname{rng} g =$ the carrier of $H \cdot a$ by [46, (62)], (88), (242), [14, (47)]. g is one-to-one by [46, (62)], (88), [14, (47)], (217). □

(266) H is finite if and only if $H \cdot a$ is finite. The theorem is a consequence of (265).

Let us consider G and a. Let H be a finite subgroup of G. Observe that $H \cdot a$ is finite.

Now we state the propositions:

(267) Let us consider a finite subgroup H of G. Then $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

(268) $\mathbf{0}_G \cdot a = \mathbf{0}_G$. The theorem is a consequence of (238) and (218).

(269) Let us consider a strict subgroup H of G. If $H \cdot a = \mathbf{0}_G$, then $H = \mathbf{0}_G$. The theorem is a consequence of (266), (115), (265), and (116).

(270) Let us consider an additive group G, and an element a of G. Then $\Omega_G \cdot a = \Omega_G$. The theorem is a consequence of (225), (220), and (259).

(271) Let us consider a strict subgroup H of G. If $H \cdot a = G$, then $H = G$. The theorem is a consequence of (259), (217), and (108).

(272) $|\bullet : H| = |\bullet : H \cdot a|$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists b such that $\$_1 = b + H$ and $\$_2 = b \cdot a + H \cdot a$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in$ the left cosets of H and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the left cosets of $H \cdot a$. f is one-to-one. □

(273) If the left cosets of H is finite, then $|\bullet : H|_{\mathbb{N}} = |\bullet : H \cdot a|_{\mathbb{N}}$. The theorem is a consequence of (272).

(274) If G is an Abelian additive group, then for every strict subgroup H of G and for every a, $H \cdot a = H$. The theorem is a consequence of (260), (158), (153), (155), and (105).

Let us consider G, a, and b. We say that a and b are conjugated if and only if

(Def. 35) there exists g such that $a = b \cdot g$.

Now we state the proposition:

(275) a and b are conjugated if and only if there exists g such that $b = a \cdot g$. The theorem is a consequence of (226).

Let us consider G, a, and b. Observe that a and b are conjugated is reflexive and symmetric.

Now we state the propositions:

(276) If a and b are conjugated and b and c are conjugated, then a and c are conjugated. The theorem is a consequence of (225).

(277) If a and 0_G are conjugated or 0_G and a are conjugated, then $a = 0_G$. The theorem is a consequence of (275) and (219).

(278) $a \cdot \overline{\Omega_G} = \{b : a \text{ and } b \text{ are conjugated}\}$. The theorem is a consequence of (243).

Let us consider G and a. The functor $a^\bullet$ yielding a subset of G is defined by the term

(Def. 36) $a \cdot \overline{\Omega_G}$.

Now we state the propositions:

(279) $x \in a^\bullet$ if and only if there exists b such that $b = x$ and a and b are conjugated. The theorem is a consequence of (278).

(280) $a \in b^\bullet$ if and only if a and b are conjugated. The theorem is a consequence of (279).

(281) $a \cdot g \in a^\bullet$.

(282) $a \in a^\bullet$.

(283) If $a \in b^\bullet$, then $b \in a^\bullet$. The theorem is a consequence of (280).

(284) $a^\bullet = b^\bullet$ if and only if $a^\bullet$ meets $b^\bullet$. The theorem is a consequence of (280), (279), and (276).

(285) $a^\bullet = \{0_G\}$ if and only if $a = 0_G$. The theorem is a consequence of (280), (279), and (277).

(286) $a^\bullet + A = A + a^\bullet$. The theorem is a consequence of (280), (202), (226), (224), (221), (225), (279), and (275).

Let us consider G, A, and B. We say that A and B are conjugated if and only if

(Def. 37) there exists g such that $A = B \cdot g$.

Now we state the propositions:

(287) A and B are conjugated if and only if there exists g such that $B = A \cdot g$. The theorem is a consequence of (255).

(288) A and A are conjugated. The theorem is a consequence of (253).

(289) If A and B are conjugated, then B and A are conjugated. The theorem is a consequence of (255).

Let us consider G, A, and B. Let us observe that A and B are conjugated is reflexive and symmetric.

Now we state the propositions:

(290) If A and B are conjugated and B and C are conjugated, then A and C are conjugated. The theorem is a consequence of (248).

(291) $\{a\}$ and $\{b\}$ are conjugated if and only if a and b are conjugated.
PROOF: If $\{a\}$ and $\{b\}$ are conjugated, then a and b are conjugated by (287), (238), (275), [17, (3)]. Consider g such that $a \cdot g = b$. $\{b\} = \{a\} \cdot g$. □

(292) If A and $\overline{H_1}$ are conjugated, then there exists a strict subgroup H_2 of G such that the carrier of $H_2 = A$.

Let us consider G and A. The functor $A^\bullet$ yielding a family of subsets of G is defined by the term

(Def. 38) $\{B : A$ and B are conjugated$\}$.

Now we state the propositions:

(293) $x \in A^\bullet$ if and only if there exists B such that $x = B$ and A and B are conjugated.

(294) $A \in B^\bullet$ if and only if A and B are conjugated.

(295) $A \cdot g \in A^\bullet$. The theorem is a consequence of (287).

(296) $A \in A^\bullet$.

(297) If $A \in B^\bullet$, then $B \in A^\bullet$. The theorem is a consequence of (294).

(298) $A^\bullet = B^\bullet$ if and only if $A^\bullet$ meets $B^\bullet$. The theorem is a consequence of (294) and (290).

(299) $\{a\}^\bullet = \{\{b\} : b \in a^\bullet\}$. The theorem is a consequence of (287), (275), (280), (238), and (291).

(300) If G is finite, then $A^\bullet$ is finite.

Let us consider G, H_1, and H_2. We say that H_1 and H_2 are conjugated if and only if

(Def. 39) there exists g such that the additive magma of $H_1 = H_2 \cdot g$.

Now we state the propositions:

(301) Let us consider strict subgroups H_1, H_2 of G. Then H_1 and H_2 are conjugated if and only if there exists g such that $H_2 = H_1 \cdot g$. The theorem is a consequence of (263).

(302) Let us consider a strict subgroup H_1 of G. Then H_1 and H_1 are conjugated. The theorem is a consequence of (262).

(303) Let us consider strict subgroups H_1, H_2 of G. If H_1 and H_2 are conjugated, then H_2 and H_1 are conjugated. The theorem is a consequence of (263).

Let us consider G. Let H_1, H_2 be strict subgroups of G. Observe that H_1 and H_2 are conjugated is reflexive and symmetric.

Now we state the proposition:

(304) Let us consider strict subgroups H_1, H_2 of G. Suppose H_1 and H_2 are conjugated and H_2 and H_3 are conjugated. Then H_1 and H_3 are conjugated. The theorem is a consequence of (261).

In the sequel L denotes a subset of $\mathrm{SubGr}\, G$.

Let us consider G and H. The functor $H^\bullet$ yielding a subset of $\mathrm{SubGr}\, G$ is defined by

(Def. 40) for every object x, $x \in it$ iff there exists a strict subgroup H_1 of G such that $x = H_1$ and H and H_1 are conjugated.

Now we state the propositions:

(305) If $x \in H^\bullet$, then x is a strict subgroup of G.

(306) Let us consider strict subgroups H_1, H_2 of G. Then $H_1 \in {H_2}^\bullet$ if and only if H_1 and H_2 are conjugated.

Let us consider a strict subgroup H of G. Now we state the propositions:

(307) $H \cdot g \in H^\bullet$. The theorem is a consequence of (301).

(308) $H \in H^\bullet$.

Let us consider strict subgroups H_1, H_2 of G. Now we state the propositions:

(309) If $H_1 \in {H_2}^\bullet$, then $H_2 \in {H_1}^\bullet$. The theorem is a consequence of (306).

(310) ${H_1}^\bullet = {H_2}^\bullet$ if and only if ${H_1}^\bullet$ meets ${H_2}^\bullet$. The theorem is a consequence of (308), (305), (306), and (304).

Now we state the propositions:

(311) If G is finite, then $H^\bullet$ is finite.

(312) Let us consider a strict subgroup H_1 of G. Then H_1 and H_2 are conjugated if and only if $\overline{H_1}$ and $\overline{H_2}$ are conjugated.

Let us consider G. Let I_1 be a subgroup of G. We say that I_1 is normal if and only if

(Def. 41) for every a, $I_1 \cdot a =$ the additive magma of I_1.

Let us note that there exists a subgroup of G which is strict and normal.

From now on N_2 denotes a normal subgroup of G.

Now we state the propositions:

(313) (i) $\mathbf{0}_G$ is normal, and

(ii) Ω_G is normal.

(314) Let us consider strict, normal subgroups N_1, N_2 of G. Then $N_1 \cap N_2$ is normal. The theorem is a consequence of (264).

(315) Let us consider a strict subgroup H of G. If G is an Abelian additive group, then H is normal.

(316) H is a normal subgroup of G if and only if for every a, $a + H = H + a$. The theorem is a consequence of (260), (79), (151), (83), (153), (155), and (105).

Let us consider a subgroup H of G. Now we state the propositions:

(317) H is a normal subgroup of G if and only if for every a, $a + H \subseteq H + a$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(318) H is a normal subgroup of G if and only if for every a, $H + a \subseteq a + H$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(319) H is a normal subgroup of G if and only if for every A, $A + H = H + A$. The theorem is a consequence of (140), (149), (316), (150), and (141).

Let us consider a strict subgroup H of G. Now we state the propositions:

(320) H is a normal subgroup of G if and only if for every a, H is a subgroup of $H \cdot a$. The theorem is a consequence of (100), (260), (80), (83), (207), and (318).

(321) H is a normal subgroup of G if and only if for every a, $H \cdot a$ is a subgroup of H. The theorem is a consequence of (100), (260), (80), (83), (207), and (317).

(322) H is a normal subgroup of G if and only if $H^\bullet = \{H\}$.

Proof: If H is a normal subgroup of G, then $H^\bullet = \{H\}$ by (301), (308), [17, (31)]. H is normal. □

(323) H is a normal subgroup of G if and only if for every a such that $a \in H$ holds $a^\bullet \subseteq \overline{H}$. The theorem is a consequence of (279), (275), (259), and (226).

Let us consider strict, normal subgroups N_1, N_2 of G. Now we state the propositions:

(324) $\overline{N_1} + \overline{N_2} = \overline{N_2} + \overline{N_1}$.

(325) There exists a strict, normal subgroup N of G such that the carrier of $N = \overline{N_1} + \overline{N_2}$. The theorem is a consequence of (124), (75), (316), (76), and (77).

Now we state the propositions:

(326) Let us consider a normal subgroup N of G. Then the left cosets of $N =$ the right cosets of N. The theorem is a consequence of (316).

(327) Let us consider a subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 2$. Then H is a normal subgroup of G.

Proof: There exists a finite set B such that $B =$ the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$. Consider x, y being objects such that $x \neq y$ and the left cosets of $H = \{x, y\}$. $\overline{H} \in$ the left cosets of H. Consider z_3 being an object such that $\{x, y\} = \{\overline{H}, z_3\}$. $\overline{H}$ misses z_3 by (155), (161), [34, (29)], [17, (4)]. $\bigcup$(the left cosets of H) = the carrier of G and $\bigcup$(the left cosets of H) $= \overline{H} \cup z_3$. $\bigcup$(the right cosets of H) = the carrier of G and $z_3 =$ (the carrier of G) $\setminus \overline{H}$. There exists a finite set C such that $C =$ the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$. Consider z_1, z_2 being objects such that $z_1 \neq z_2$ and the right cosets of $H = \{z_1, z_2\}$. $\overline{H} \in$ the right cosets of H. Consider z_4 being an object such that $\{z_1, z_2\} = \{\overline{H}, z_4\}$. $\overline{H}$ misses z_4 by (155), (167), [34, (29)], [17, (4)]. $\square$

Let us consider G and A. The functor $\mathrm{N}(A)$ yielding a strict subgroup of G is defined by

(Def. 42) the carrier of *it* $= \{h : A \cdot h = A\}$.

Now we state the propositions:

(328) $x \in \mathrm{N}(A)$ if and only if there exists h such that $x = h$ and $A \cdot h = A$.

(329) $\overline{\overline{A^\bullet}} = |\bullet : \mathrm{N}(A)|$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a such that $\$_1 = A \cdot a$ and $\$_2 = \mathrm{N}(A) + a$. For every object x such that $x \in A^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A^\bullet$ and for every object x such that $x \in A^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in A^\bullet$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the right cosets of $\mathrm{N}(A)$. f is one-to-one. $\square$

(330) Suppose $A^\bullet$ is finite or the left cosets of $\mathrm{N}(A)$ is finite. Then there exists a finite set C such that

(i) $C = A^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(A)|_{\mathbb{N}}$.

The theorem is a consequence of (329).

(331) $\overline{\overline{a^\bullet}} = |\bullet : \mathrm{N}(\{a\})|$.

Proof: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\mathrm{dom}\, f = a^\bullet$ and for every object x such that $x \in a^\bullet$ holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\mathrm{rng}\, f = \{a\}^\bullet$. f is one-to-one by [17, (3)]. □

(332) Suppose $a^\bullet$ is finite or the left cosets of $\mathrm{N}(\{a\})$ is finite. Then there exists a finite set C such that

(i) $C = a^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(\{a\})|_{\mathbb{N}}$.

The theorem is a consequence of (331).

Let us consider G and H. The functor $\mathrm{N}(H)$ yielding a strict subgroup of G is defined by the term

(Def. 43) $\mathrm{N}(\overline{H})$.

Let us consider a strict subgroup H of G. Now we state the propositions:

(333) $x \in \mathrm{N}(H)$ if and only if there exists h such that $x = h$ and $H \cdot h = H$. The theorem is a consequence of (328).

(334) $\overline{\overline{H^\bullet}} = |\bullet : \mathrm{N}(H)|$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H_1 of G such that $\$_1 = H_1$ and $\$_2 = \overline{H_1}$. For every object x such that $x \in H^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = H^\bullet$ and for every object x such that $x \in H^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\mathrm{rng}\, f = \overline{H}^\bullet$. f is one-to-one. □

(335) Suppose $H^\bullet$ is finite or the left cosets of $\mathrm{N}(H)$ is finite. Then there exists a finite set C such that

(i) $C = H^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(H)|_{\mathbb{N}}$.

The theorem is a consequence of (334).

Now we state the proposition:

(336) Let us consider a strict additive group G, and a strict subgroup H of G. Then H is a normal subgroup of G if and only if $\mathrm{N}(H) = G$. The theorem is a consequence of (333) and (108).

Let us consider a strict additive group G. Now we state the propositions:

(337) $\mathrm{N}(\mathbf{0}_G) = G$. The theorem is a consequence of (313) and (336).

(338) $\mathrm{N}(\Omega_G) = G$. The theorem is a consequence of (313) and (336).

4. Topological Groups – `TOPGRP_1`

In the sequel S, R denote 1-sorted structures, X denotes a subset of R, T denotes a topological structure, x denotes a set, H denotes a non empty additive magma, P, Q, P_1, Q_1 denote subsets of H, and h denotes an element of H.

Now we state the proposition:

(339) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $P + Q \subseteq P_1 + Q_1$.

Let us assume that $P \subseteq Q$. Now we state the propositions:

(340) $P + h \subseteq Q + h$. The theorem is a consequence of (74).

(341) $h + P \subseteq h + Q$. The theorem is a consequence of (73).

From now on a denotes an element of G.

Now we state the propositions:

(342) $a \in -A$ if and only if $-a \in A$.

(343) $A \subseteq B$ if and only if $-A \subseteq -B$.

(344) $(\text{add inverse}\, G)^\circ A = -A$.

(345) $(\text{add inverse}\, G)^{-1}(A) = -A$.

(346) add inverse G is one-to-one. The theorem is a consequence of (9).

(347) $\text{rng}\, \text{add inverse}\, G =$ the carrier of G.

Let G be an additive group. One can verify that add inverse G is one-to-one and onto.

Now we state the propositions:

(348) $(\text{add inverse}\, G)^{-1} = \text{add inverse}\, G$.

(349) $(\text{The addition of } H)^\circ(P \times Q) = P + Q$.

Let G be a non empty additive magma and a be an element of G. The functors: a^+ and ${}^+a$ yielding functions from G into G are defined by conditions,

(Def. 44) for every element x of G, $a^+(x) = a + x$,

(Def. 45) for every element x of G, ${}^+a(x) = x + a$,

respectively. Let G be an additive group. One can verify that a^+ is one-to-one and onto and ${}^+a$ is one-to-one and onto.

Now we state the propositions:

(350) $(h^+)^\circ P = h + P$. The theorem is a consequence of (73).

(351) $({}^+h)^\circ P = P + h$. The theorem is a consequence of (74).

(352) $(a^+)^{-1} = (-a)^+$.

(353) $({}^+a)^{-1} = {}^+(-a)$.

We consider topological additive group structures which extend additive magmas and topological structures and are systems

$$\langle \text{a carrier}, \text{an addition}, \text{a topology} \rangle$$

where the carrier is a set, the addition is a binary operation on the carrier, the topology is a family of subsets of the carrier.

Let A be a non empty set, R be a binary operation on A, and T be a family of subsets of A. Let us observe that $\langle A, R, T \rangle$ is non empty.

Let x be a set, R be a binary operation on $\{x\}$, and T be a family of subsets of $\{x\}$. Observe that $\langle \{x\}, R, T \rangle$ is trivial and every 1-element additive magma is additive group-like, add-associative, and Abelian and there exists a topological additive group structure which is strict and non empty and there exists a topological additive group structure which is strict, topological space-like, and 1-element.

Let G be an additive group-like, add-associative, non empty topological additive group structure. We say that G is inverse-continuous if and only if

(Def. 46) add inverse G is continuous.

Let G be a topological space-like topological additive group structure. We say that G is continuous if and only if

(Def. 47) for every function f from $G \times G$ into G such that $f =$ the addition of G holds f is continuous.

One can check that there exists a topological space-like, additive group-like, add-associative, 1-element topological additive group structure which is strict, Abelian, inverse-continuous, and continuous.

A semi additive topological group is a topological space-like, additive group-like, add-associative, non empty topological additive group structure.

A topological additive group is an inverse-continuous, continuous semi additive topological group. Now we state the propositions:

(354) Let us consider a continuous, non empty, topological space-like topological additive group structure T, elements a, b of T, and a neighbourhood W of $a + b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A + B \subseteq W$.

(355) Let us consider a topological space-like, non empty topological additive group structure T. Suppose for every elements a, b of T for every neighbourhood W of $a + b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A + B \subseteq W$. Then T is continuous.
PROOF: For every point W of $T \times T$ and for every neighbourhood G of $f(W)$, there exists a neighbourhood H of W such that $f^{\circ}H \subseteq G$ by [32, (10)], (349). □

(356) Let us consider an inverse-continuous semi additive topological group T, an element a of T, and a neighbourhood W of $-a$. Then there exists an open neighbourhood A of a such that $-A \subseteq W$.

(357) Let us consider a semi additive topological group T. Suppose for every

element a of T for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$. Then T is inverse-continuous. The theorem is a consequence of (344).

(358) Let us consider a topological additive group T, elements a, b of T, and a neighbourhood W of $a+-b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A+-B \subseteq W$. The theorem is a consequence of (354) and (356).

(359) Let us consider a semi additive topological group T. Suppose for every elements a, b of T for every neighbourhood W of $a+-b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+-B \subseteq W$. Then T is a topological additive group.

PROOF: For every element a of T and for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$ by [28, (4)]. For every elements a, b of T and for every neighbourhood W of $a+b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+B \subseteq W$. $\square$

Let G be a continuous, non empty, topological space-like topological additive group structure and a be an element of G. One can check that a^+ is continuous and ${}^+a$ is continuous.

Let us consider a continuous semi additive topological group G and an element a of G. Now we state the propositions:

(360) a^+ is a homeomorphism of G. The theorem is a consequence of (352).

(361) ${}^+a$ is a homeomorphism of G. The theorem is a consequence of (353).

Let G be a continuous semi additive topological group and a be an element of G. The functors: a^+ and ${}^+a$ yield homeomorphisms of G. Now we state the proposition:

(362) Let us consider an inverse-continuous semi additive topological group G. Then add inverse G is a homeomorphism of G. The theorem is a consequence of (348).

Let G be an inverse-continuous semi additive topological group. Let us note that the functor add inverse G yields a homeomorphism of G. Let us note that every semi additive topological group which is continuous is also homogeneous.

Let us consider a continuous semi additive topological group G, a closed subset F of G, and an element a of G. Now we state the propositions:

(363) $F+a$ is closed. The theorem is a consequence of (351).

(364) $a+F$ is closed. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, F be a closed subset of G, and a be an element of G. Let us note that $F+a$ is closed and $a+F$ is closed.

Now we state the proposition:

(365) Let us consider an inverse-continuous semi additive topological group G, and a closed subset F of G. Then $-F$ is closed. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and F be a closed subset of G. One can verify that $-F$ is closed.

Let us consider a continuous semi additive topological group G, an open subset O of G, and an element a of G. Now we state the propositions:

(366) $O + a$ is open. The theorem is a consequence of (351).

(367) $a + O$ is open. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, A be an open subset of G, and a be an element of G. One can check that $A + a$ is open and $a + A$ is open.

Now we state the proposition:

(368) Let us consider an inverse-continuous semi additive topological group G, and an open subset O of G. Then $-O$ is open. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and A be an open subset of G. Observe that $-A$ is open.

Let us consider a continuous semi additive topological group G and subsets A, O of G.

Let us assume that O is open. Now we state the propositions:

(369) $O + A$ is open.

PROOF: $\mathrm{Int}(O + A) = O + A$ by [48, (16)], (74), [48, (22)]. □

(370) $A + O$ is open.

PROOF: $\mathrm{Int}(A + O) = A + O$ by [48, (16)], (73), [48, (22)]. □

Let G be a continuous semi additive topological group, A be an open subset of G, and B be a subset of G. Note that $A + B$ is open and $B + A$ is open.

Now we state the propositions:

(371) Let us consider an inverse-continuous semi additive topological group G, a point a of G, and a neighbourhood A of a. Then $-A$ is a neighbourhood of $-a$. The theorem is a consequence of (343).

(372) Let us consider a topological additive group G, a point a of G, and a neighbourhood A of $a + -a$. Then there exists an open neighbourhood B of a such that $B + -B \subseteq A$. The theorem is a consequence of (358) and (342).

(373) Let us consider an inverse-continuous semi additive topological group G, and a dense subset A of G. Then $-A$ is dense. The theorem is a consequence of (345).

Let G be an inverse-continuous semi additive topological group and A be a dense subset of G. Observe that $-A$ is dense.

Let us consider a continuous semi additive topological group G, a dense subset A of G, and a point a of G. Now we state the propositions:

(374) $a + A$ is dense. The theorem is a consequence of (350).

(375) $A + a$ is dense. The theorem is a consequence of (351).

Let G be a continuous semi additive topological group, A be a dense subset of G, and a be a point of G. Let us observe that $A + a$ is dense and $a + A$ is dense.

Now we state the proposition:

(376) Let us consider a topological additive group G, a basis B of 0_G, and a dense subset M of G. Then $\{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$ is a basis of G.

PROOF: Set $Z = \{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$. $Z \subseteq$ the topology of G by [38, (12)]. For every subset W of G such that W is open for every point a of G such that $a \in W$ there exists a subset V of G such that $V \in Z$ and $a \in V$ and $V \subseteq W$ by (8), [28, (3)], (74), (372). $Z \subseteq 2^{\alpha}$, where α is the carrier of G. $\square$

One can check that every topological additive group is regular.

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