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# Polish Notation

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**Summary.** This article is the first in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([12] and [13]) concerning a logic proposed by Prof. Andrzej Grzegorczyk ([14]).

We present some *mathematical folklore* about representing formulas in "Polish notation", that is, with operators of fixed arity prepended to their arguments. This notation, which was published by Jan Łukasiewicz in [15], eliminates the need for parentheses and is generally well suited for rigorous reasoning about syntactic properties of formulas.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [11], [7], [8], [3], [9], [16], [19], [17], [18], and [10].

## 1. Preliminaries

From now on $k$, $m$, $n$ denote natural numbers, $a$, $b$, $c$, $c_1$, $c_2$ denote objects, $x$, $y$, $z$, $X$, $Y$, $Z$ denote sets, $D$ denotes a non empty set, $p$, $q$, $r$, $s$, $t$, $u$, $v$ denote finite sequences, $P$, $Q$, $R$, $P_1$, $P_2$, $Q_1$, $Q_2$, $R_1$, $R_2$ denote finite sequence-membered sets, and $S$, $T$ denote non empty, finite sequence-membered sets.

Let $D$ be a non empty set and $P$, $Q$ be subsets of $D^*$. The functor $^\frown(D, P, Q)$ yielding a subset of $D^*$ is defined by the term

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(Def. 1) $\{p \frown q$, where $p$ is a finite sequence of elements of $D$, $q$ is a finite sequence of elements of $D : p \in P$ and $q \in Q\}$.

Let us consider $P$ and $Q$. The functor $P \frown Q$ yielding a finite sequence-membered set is defined by

(Def. 2) for every $a$, $a \in it$ iff there exists $p$ and there exists $q$ such that $a = p \frown q$ and $p \in P$ and $q \in Q$.

Let $\beta$ be an empty set. One can check that $\beta \frown P$ is empty and $P \frown \beta$ is empty.

Let us consider $S$ and $T$. One can check that $S \frown T$ is non empty.

Now we state the propositions:

(1) If $p \frown q = r \frown s$, then there exists $t$ such that $p \frown t = r$ or $p = r \frown t$.

(2) $(P \frown Q) \frown R = P \frown (Q \frown R)$.

PROOF: For every $a$, $a \in (P \frown Q) \frown R$ iff $a \in P \frown (Q \frown R)$ by [4, (32)]. $\square$

Note that $\{\emptyset\}$ is non empty and finite sequence-membered.

(3) (i) $P \frown \{\emptyset\} = P$, and

(ii) $\{\emptyset\} \frown P = P$.

PROOF: For every $a$, $a \in P \frown \{\emptyset\}$ iff $a \in P$ by [4, (34)]. For every $a$, $a \in \{\emptyset\} \frown P$ iff $a \in P$ by [4, (34)]. $\square$

Let us consider $P$. The functor $P^{\frown\frown}$ yielding a function is defined by

(Def. 3) $\operatorname{dom} it = \mathbb{N}$ and $it(0) = \{\emptyset\}$ and for every $n$, there exists $Q$ such that $Q = it(n)$ and $it(n+1) = Q \frown P$.

Let us consider $n$. The functor $P \frown n$ yielding a finite sequence-membered set is defined by the term

(Def. 4) $(P^{\frown\frown})(n)$.

Now we state the proposition:

(4) $\emptyset \in P \frown 0$.

Let us consider $P$. Let $n$ be a zero natural number. Note that $P \frown n$ is non empty.

Let $\beta$ be an empty set and $n$ be a non zero natural number. One can verify that $\beta \frown n$ is empty.

Let us consider $P$. The functor $P^*$ yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) $\bigcup$ the set of all $P \frown n$ where $n$ is a natural number.

(5) $a \in P^*$ if and only if there exists $n$ such that $a \in P \frown n$.

Let us consider $P$.

(6) (i) $P \frown 0 = \{\emptyset\}$, and

(ii) for every $n$, $P \frown (n+1) = (P \frown n) \frown P$.

(7) $P^\frown 1 = P$. The theorem is a consequence of (6) and (3).

(8) $P^\frown n \subseteq P^*$.

(9) (i) $\emptyset \in P^*$, and

(ii) $P \subseteq P^*$.

The theorem is a consequence of (4), (5), and (7).

(10) $P^\frown (m+n) = (P^\frown m)^\frown (P^\frown n)$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv P^\frown (m + \$_1) = (P^\frown m)^\frown (P^\frown \$_1)$. $\mathcal{X}[0]$. For every $k$ such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every $k$, $\mathcal{X}[k]$ from [2, Sch. 2]. □

(11) If $p \in P^\frown m$ and $q \in P^\frown n$, then $p^\frown q \in P^\frown (m+n)$. The theorem is a consequence of (10).

(12) If $p$, $q \in P^*$, then $p^\frown q \in P^*$. The theorem is a consequence of (5) and (11).

(13) If $P \subseteq R^*$ and $Q \subseteq R^*$, then $P^\frown Q \subseteq R^*$. The theorem is a consequence of (12).

(14) If $Q \subseteq P^*$, then $Q^\frown n \subseteq P^*$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv Q^\frown \$_1 \subseteq P^*$. $\mathcal{X}[0]$. For every $k$ such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every $k$, $\mathcal{X}[k]$ from [2, Sch. 2]. □

(15) If $Q \subseteq P^*$, then $Q^* \subseteq P^*$. The theorem is a consequence of (5) and (14).

(16) If $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$, then $P_1{}^\frown Q_1 \subseteq P_2{}^\frown Q_2$.

(17) If $P \subseteq Q$, then for every $n$, $P^\frown n \subseteq Q^\frown n$.

PROOF: Define $\mathcal{S}$[natural number] $\equiv P^\frown \$_1 \subseteq Q^\frown \$_1$. $P^\frown 0 = \{\emptyset\}$. For every $n$ such that $\mathcal{S}[n]$ holds $\mathcal{S}[n+1]$. For every $n$, $\mathcal{S}[n]$ from [2, Sch. 2]. □

Let us consider $S$ and $n$. Let us observe that $S^\frown n$ is non empty and finite sequence-membered.

## 2. THE LANGUAGE

In the sequel $\alpha$ denotes a function from $P$ into $\mathbb{N}$ and $U$, $V$, $W$ denote subsets of $P^*$.

Let us consider $P$, $\alpha$, and $U$. The Polish-expression layer($P$, $\alpha$, $U$) yielding a subset of $P^*$ is defined by

(Def. 6) for every $a$, $a \in it$ iff $a \in P^*$ and there exists $p$ and there exists $q$ and there exists $n$ such that $a = p^\frown q$ and $p \in P$ and $n = \alpha(p)$ and $q \in U^\frown n$.

Now we state the proposition:

(18) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in U^\frown n$. Then $p^\frown q \in$ the Polish-expression layer($P$, $\alpha$, $U$). The theorem is a consequence of (14), (9), and (12).

Let us consider $P$ and $\alpha$. The Polish atoms($P$, $\alpha$) yielding a subset of $P^*$ is defined by

(Def. 7) for every $a$, $a \in it$ iff $a \in P$ and $\alpha(a) = 0$.

The Polish operations($P$, $\alpha$) yielding a subset of $P$ is defined by the term

(Def. 8) $\{t$, where $t$ is an element of $P^* : t \in P$ and $\alpha(t) \neq 0\}$.

Now we state the propositions:

(19) The Polish atoms($P$, $\alpha$) $\subseteq$ the Polish-expression layer($P$, $\alpha$, $U$). The theorem is a consequence of (4) and (18).

(20) Suppose $U \subseteq V$. Then the Polish-expression layer($P$, $\alpha$, $U$) $\subseteq$ the Polish-expression layer($P$, $\alpha$, $V$). The theorem is a consequence of (17).

(21) Suppose $u \in$ the Polish-expression layer($P$, $\alpha$, $U$). Then there exists $p$ and there exists $q$ such that $p \in P$ and $u = p \frown q$.

Let us consider $P$ and $\alpha$. The Polish-expression hierarchy($P$, $\alpha$) yielding a function is defined by

(Def. 9) $\operatorname{dom} it = \mathbb{N}$ and $it(0) =$ the Polish atoms($P$, $\alpha$) and for every $n$, there exists $U$ such that $U = it(n)$ and $it(n+1) =$ the Polish-expression layer($P$, $\alpha$, $U$).

Let us consider $n$. The Polish-expression hierarchy($P$, $\alpha$, $n$) yielding a subset of $P^*$ is defined by the term

(Def. 10) (the Polish-expression hierarchy($P$, $\alpha$))($n$).

Now we state the proposition:

(22) The Polish-expression hierarchy($P$, $\alpha$, 0) = the Polish atoms($P$, $\alpha$).

Let us consider $P$, $\alpha$, and $n$. Now we state the propositions:

(23) The Polish-expression hierarchy($P$, $\alpha$, $n+1$) = the Polish-expression layer($P$, $\alpha$, the Polish-expression hierarchy($P$, $\alpha$, $n$)).

(24) The Polish-expression hierarchy($P$, $\alpha$, $n$) $\subseteq$ the Polish-expression hierarchy($P$, $\alpha$, $n+1$).
PROOF: Define $\mathcal{S}$[natural number] $\equiv$ the Polish-expression hierarchy($P$, $\alpha$, $\$_1$) $\subseteq$ the Polish-expression hierarchy($P$, $\alpha$, $\$_1 + 1$). $\mathcal{S}[0]$. For every $k$ such that $\mathcal{S}[k]$ holds $\mathcal{S}[k+1]$. For every $k$, $\mathcal{S}[k]$ from [2, Sch. 2]. □

Now we state the proposition:

(25) The Polish-expression hierarchy($P$, $\alpha$, $n$) $\subseteq$ the Polish-expression hierarchy($P$, $\alpha$, $n+m$).
PROOF: Define $\mathcal{S}$[natural number] $\equiv$ the Polish-expression hierarchy($P$, $\alpha$, $n$) $\subseteq$ the Polish-expression hierarchy($P$, $\alpha$, $n + \$_1$). For every $k$ such that $\mathcal{S}[k]$ holds $\mathcal{S}[k+1]$. For every $k$, $\mathcal{S}[k]$ from [2, Sch. 2]. □

Let us consider $P$ and $\alpha$. The Polish-expression set$(P, \alpha)$ yielding a subset of $P^*$ is defined by the term

(Def. 11) $\bigcup$the set of all the Polish-expression hierarchy$(P, \alpha, n)$ where $n$ is a natural number.

Now we state the propositions:

(26) The Polish-expression hierarchy$(P, \alpha, n) \subseteq$ the Polish-expression set$(P, \alpha)$.

(27) Suppose $q \in$ (the Polish-expression set$(P, \alpha)) ^\frown n$. Then there exists $m$ such that $q \in$ (the Polish-expression hierarchy$(P, \alpha, m)) ^\frown n$.
Proof: Define $\mathcal{S}$[natural number] $\equiv$ for every $q$ such that $q \in$ (the Polish-expression set$(P, \alpha))^\frown \$_1$ there exists $m$ such that $q \in$ (the Polish-expression hierarchy$(P, \alpha, m)) ^\frown \$_1$. $\mathcal{S}[0]$. For every $k$ such that $\mathcal{S}[k]$ holds $\mathcal{S}[k+1]$. For every $k$, $\mathcal{S}[k]$ from [2, Sch. 2]. □

(28) Suppose $a \in$ the Polish-expression set$(P, \alpha)$. Then there exists $n$ such that $a \in$ the Polish-expression hierarchy$(P, \alpha, n+1)$. The theorem is a consequence of (24).

Let us consider $P$ and $\alpha$.

A Polish expression of $P$ and $\alpha$ is an element of the Polish-expression set$(P, \alpha)$. Let us consider $n$ and $t$. Assume $t \in P$. The Polish operation$(P, \alpha, n, t)$ yielding a function from (the Polish-expression set$(P, \alpha)) ^\frown n$ into $P^*$ is defined by

(Def. 12) for every $q$ such that $q \in \operatorname{dom} it$ holds $it(q) = t ^\frown q$.

Let us consider $X$ and $Y$. Let $F$ be a partial function from $X$ to $2^Y$. One can check that $F$ is disjoint valued if and only if the condition (Def. 13) is satisfied.

(Def. 13) for every $a$ and $b$ such that $a, b \in \operatorname{dom} F$ and $a \neq b$ holds $F(a)$ misses $F(b)$.

Let $X$ be a set. One can check that there exists a finite sequence of elements of $2^X$ which is disjoint valued.

Now we state the proposition:

(29) Let us consider a set $X$, a disjoint valued finite sequence $B$ of elements of $2^X$, $a$, $b$, and $c$. If $a \in B(b)$ and $a \in B(c)$, then $b = c$ and $b \in \operatorname{dom} B$.

Let us consider $X$. Let $B$ be a disjoint valued finite sequence of elements of $2^X$. The arity from list $B$ yielding a function from $X$ into $\mathbb{N}$ is defined by

(Def. 14) for every $a$ such that $a \in X$ holds there exists $n$ such that $a \in B(n)$ and $a \in B(it(a))$ or there exists no $n$ such that $a \in B(n)$ and $it(a) = 0$.

Now we state the propositions:

(30) Let us consider a disjoint valued finite sequence $B$ of elements of $2^X$, and $a$. Suppose $a \in X$. Then (the arity from list $B)(a) \neq 0$ if and only if

there exists $n$ such that $a \in B(n)$. The theorem is a consequence of (29).

(31) Let us consider a disjoint valued finite sequence $B$ of elements of $2^X$, $a$, and $n$. Suppose $a \in B(n)$. Then (the arity from list $B)(a) = n$. The theorem is a consequence of (29).

(32) Suppose $r \in$ the Polish-expression set($P$, $\alpha$). Then there exists $n$ and there exists $p$ and there exists $q$ such that $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set($P$, $\alpha$)) $\frown n$ and $r = p \frown q$. The theorem is a consequence of (28), (23), (26), and (17).

Let us consider $P$, $\alpha$, and $Q$. We say that $Q$ is $\alpha$-closed if and only if

(Def. 15) for every $p$, $n$, and $q$ such that $p \in P$ and $n = \alpha(p)$ and $q \in Q \frown n$ holds $p \frown q \in Q$.

Now we state the propositions:

(33) The Polish-expression set($P$, $\alpha$) is $\alpha$-closed. The theorem is a consequence of (27), (18), (23), and (26).

(34) If $Q$ is $\alpha$-closed, then the Polish atoms($P$, $\alpha$) $\subseteq Q$. The theorem is a consequence of (4).

(35) If $Q$ is $\alpha$-closed, then the Polish-expression hierarchy($P$, $\alpha$, $n$) $\subseteq Q$.
PROOF: Define $\mathcal{X}$[natural number] $\equiv$ the Polish-expression hierarchy($P$, $\alpha$, $\$_1$) $\subseteq Q$. $\mathcal{X}[0]$. For every $k$ such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every $k$, $\mathcal{X}[k]$ from [2, Sch. 2]. $\square$

(36) The Polish atoms($P$, $\alpha$) $\subseteq$ the Polish-expression set($P$, $\alpha$). The theorem is a consequence of (33) and (34).

(37) If $Q$ is $\alpha$-closed, then the Polish-expression set($P$, $\alpha$) $\subseteq Q$. The theorem is a consequence of (28) and (35).

(38) Suppose $r \in$ the Polish-expression set($P$, $\alpha$). Then there exists $n$ and there exists $t$ and there exists $q$ such that $t \in P$ and $n = \alpha(t)$ and $r =$ (the Polish operation($P$, $\alpha$, $n$, $t$))($q$). The theorem is a consequence of (28), (23), (26), and (17).

(39) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set($P$, $\alpha$)) $\frown n$. Then (the Polish operation($P$, $\alpha$, $n$, $p$))($q$) $\in$ the Polish-expression set($P$, $\alpha$). The theorem is a consequence of (33).

The scheme *AInd* deals with a finite sequence-membered set $\mathcal{P}$ and a function $\alpha$ from $\mathcal{P}$ into $\mathbb{N}$ and a unary predicate $\mathcal{X}$ and states that

(Sch. 1) For every $a$ such that $a \in$ the Polish-expression set($\mathcal{P}$, $\alpha$) holds $\mathcal{X}[a]$

provided

- for every $p$, $q$, and $n$ such that $p \in \mathcal{P}$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set($\mathcal{P}$, $\alpha$)) $\frown n$ holds $\mathcal{X}[p \frown q]$.

## 3. Parsing

In the sequel $k$, $l$, $m$, $n$, $i$, $j$ denote natural numbers, $a$, $b$, $c$, $c_1$, $c_2$ denote objects, $x$, $y$, $z$, $X$, $Y$, $Z$ denote sets, $D$, $D_1$, $D_2$ denote non empty sets, $p$, $q$, $r$, $s$, $t$, $u$, $v$ denote finite sequences, and $P$, $Q$, $R$ denote finite sequence-membered sets.

Let us consider $P$. We say that $P$ is antichain-like if and only if

(Def. 16) for every $p$ and $q$ such that $p$, $p \frown q \in P$ holds $q = \emptyset$.

Now we state the propositions:

(40) $P$ is antichain-like if and only if for every $p$ and $q$ such that $p$, $p \frown q \in P$ holds $p = p \frown q$.

Proof: If $P$ is antichain-like, then for every $p$ and $q$ such that $p$, $p \frown q \in P$ holds $p = p \frown q$ by [4, (34)]. □

(41) If $P \subseteq Q$ and $Q$ is antichain-like, then $P$ is antichain-like.

Note that every finite sequence-membered set which is trivial is also antichain-like.

Now we state the proposition:

(42) If $P = \{a\}$, then $P$ is antichain-like.

Note that there exists a non empty, finite sequence-membered set which is antichain-like and every finite sequence-membered set which is empty is also antichain-like.

An antichain is an antichain-like, finite sequence-membered set. In the sequel $B$, $C$ denote antichains.

Let us consider $B$. One can verify that every subset of $B$ is antichain-like and finite sequence-membered.

A Polish-language is a non empty antichain. From now on $S$, $T$ denote Polish-languages.

Let $D$ be a non empty set and $\psi$ be a subset of $D^*$. Note that $\psi$ is antichain-like if and only if the condition (Def. 17) is satisfied.

(Def. 17) for every finite sequence $g$ of elements of $D$ and for every finite sequence $h$ of elements of $D$ such that $g$, $g \frown h \in \psi$ holds $h = \varepsilon_D$.

Now we state the proposition:

(43) If $p \frown q = r \frown s$ and $p$, $r \in B$, then $p = r$ and $q = s$. The theorem is a consequence of (1) and (40).

Let us consider $B$ and $C$. Note that $B \frown C$ is antichain-like.

Now we state the propositions:

(44) If for every $p$ and $q$ such that $p$, $q \in P$ holds $\operatorname{dom} p = \operatorname{dom} q$, then $P$ is antichain-like.

PROOF: For every $p$ and $q$ such that $p$, $p \frown q \in P$ holds $p = p \frown q$ by [4, (21)]. □

(45) If for every $p$ such that $p \in P$ holds $\mathrm{dom}\, p = a$, then $P$ is antichain-like. The theorem is a consequence of (44).

(46) If $\emptyset \in B$, then $B = \{\emptyset\}$.

PROOF: For every $a$ such that $a \in B$ holds $a = \emptyset$ by [4, (34)]. □

Let us consider $B$ and $n$. Note that $B \frown n$ is antichain-like.

Let us consider $T$. Let us observe that there exists a subset of $T^*$ which is non empty and antichain-like and $T \frown n$ is non empty.

A Polish-language of $T$ is a non empty, antichain-like subset of $T^*$.

A Polish arity-function of $T$ is a function from $T$ into $\mathbb{N}$ and is defined by

(Def. 18) there exists $a$ such that $a \in T$ and $it(a) = 0$.

One can verify that every Polish-language of $T$ is non empty, antichain-like, and finite sequence-membered.

In the sequel $\alpha$ denotes a Polish arity-function of $T$ and $U$, $V$, $W$ denote Polish-languages of $T$.

Let us consider $T$ and $\alpha$. Let $t$ be an element of $T$. Let us observe that the functor $\alpha(t)$ yields a natural number. Let us consider $U$. Note that the Polish-expression layer($T$, $\alpha$, $U$) is defined by

(Def. 19) for every $a$, $a \in it$ iff there exists an element $t$ of $T$ and there exists an element $u$ of $T^*$ such that $a = t \frown u$ and $u \in U \frown \alpha(t)$.

Let us consider $B$ and $p$. We say that $p$ is $B$-headed if and only if

(Def. 20) there exists $q$ and there exists $r$ such that $q \in B$ and $p = q \frown r$.

Let us consider $P$. We say that $P$ is $B$-headed if and only if

(Def. 21) for every $p$ such that $p \in P$ holds $p$ is $B$-headed.

Now we state the propositions:

(47) If $p$ is $B$-headed and $B \subseteq C$, then $p$ is $C$-headed.

(48) If $P$ is $B$-headed and $B \subseteq C$, then $P$ is $C$-headed.

Let us consider $B$ and $P$. Observe that $B \frown P$ is $B$-headed.

Now we state the propositions:

(49) If $p$ is $(B \frown C)$-headed, then $p$ is $B$-headed.

(50) $B$ is $B$-headed. The theorem is a consequence of (3).

Let us consider $B$. Let us observe that there exists a finite sequence-membered set which is $B$-headed.

Let $P$ be a $B$-headed, finite sequence-membered set. Let us note that every subset of $P$ is $B$-headed.

Let us consider $S$. Let us observe that there exists a finite sequence-membered set which is non empty and $S$-headed.

Now we state the proposition:

(51) $S \frown (m+n)$ is $(S \frown m)$-headed. The theorem is a consequence of (10).

Let us consider $S$ and $p$. The functor $S$-head$(p)$ yielding a finite sequence is defined by

(Def. 22) (i) $it \in S$ and there exists $r$ such that $p = it \frown r$, **if** $p$ is $S$-headed,

(ii) $it = \emptyset$, **otherwise**.

The functor $S$-tail$(p)$ yielding a finite sequence is defined by

(Def. 23) $p = (S\text{-head}(p)) \frown it$.

Now we state the propositions:

(52) If $s \in S$, then $S$-head$(s \frown t) = s$ and $S$-tail$(s \frown t) = t$.

(53) If $s \in S$, then $S$-head$(s) = s$ and $S$-tail$(s) = \emptyset$. The theorem is a consequence of (52).

Let us consider $S$, $T$, and $u$. Now we state the propositions:

(54) If $u \in S \frown T$, then $S$-head$(u) \in S$ and $S$-tail$(u) \in T$. The theorem is a consequence of (52).

(55) If $S \subseteq T$ and $u$ is $S$-headed, then $S$-head$(u) = T$-head$(u)$ and $S$-tail$(u) = T$-tail$(u)$. The theorem is a consequence of (52).

Now we state the propositions:

(56) Suppose $s$ is $S$-headed. Then

(i) $s \frown t$ is $S$-headed, and

(ii) $S$-head$(s \frown t) = S$-head$(s)$, and

(iii) $S$-tail$(s \frown t) = (S\text{-tail}(s)) \frown t$.

The theorem is a consequence of (52).

(57) If $m+1 \leqslant n$ and $s \in S \frown n$, then $s$ is $(S \frown m)$-headed and $S \frown m$-tail$(s)$ is $S$-headed. The theorem is a consequence of (51), (10), (54), and (7).

(58) (i) $s$ is $(S \frown 0)$-headed, and

(ii) $S \frown 0$-head$(s) = \emptyset$, and

(iii) $S \frown 0$-tail$(s) = s$.

The theorem is a consequence of (4) and (52).

Let us consider $T$ and $\alpha$. One can verify that the Polish atoms$(T, \alpha)$ is non empty and antichain-like.

Let us consider $U$. Let us observe that the Polish-expression layer$(T, \alpha, U)$ is non empty and antichain-like.

One can verify that the Polish-expression layer$(T, \alpha, U)$ yields a Polish-language of $T$. The Polish operations$(T, \alpha)$ yielding a subset of $T$ is defined by the term

(Def. 24) $\{t$, where $t$ is an element of $T : \alpha(t) \neq 0\}$.

Let us consider $n$. Let us note that the Polish-expression hierarchy($T$, $\alpha$, $n$) is antichain-like and non empty.

One can check that the Polish-expression hierarchy($T$, $\alpha$, $n$) yields a Polish-language of $T$. The functor Polish-WFF-set($T, \alpha$) yielding a Polish-language of $T$ is defined by the term

(Def. 25) the Polish-expression set($T$, $\alpha$).

A Polish WFF of $T$ and $\alpha$ is an element of Polish-WFF-set($T, \alpha$). Let $t$ be an element of $T$. The Polish operation($T$, $\alpha$, $t$) yielding a function from Polish-WFF-set($T, \alpha$) $\frown$ $\alpha(t)$ into Polish-WFF-set($T, \alpha$) is defined by the term

(Def. 26) the Polish operation($T$, $\alpha$, $\alpha(t)$, $t$).

Assume $\alpha(t) = 1$. The functor Polish-unOp($T, \alpha, t$) yielding a unary operation on Polish-WFF-set($T, \alpha$) is defined by the term

(Def. 27) the Polish operation($T$, $\alpha$, $t$).

Assume $\alpha(t) = 2$. The functor Polish-binOp($T, \alpha, t$) yielding a binary operation on Polish-WFF-set($T, \alpha$) is defined by

(Def. 28) for every $u$ and $v$ such that $u$, $v \in$ Polish-WFF-set($T, \alpha$) holds $it(u, v) =$ (the Polish operation($T$, $\alpha$, $t$))($u \frown v$).

In the sequel $\varphi$, $\psi$ denote Polish WFFs of $T$ and $\alpha$.

Let us consider $X$ and $Y$. Let $F$ be a partial function from $X$ to $2^Y$. We say that $F$ is exhaustive if and only if

(Def. 29) for every $a$ such that $a \in Y$ there exists $b$ such that $b \in \operatorname{dom} F$ and $a \in F(b)$.

Let $X$ be a non empty set. Observe that there exists a finite sequence of elements of $2^X$ which is non exhaustive and disjoint valued.

Now we state the proposition:

(59) Let us consider a partial function $F$ from $X$ to $2^Y$. Then $F$ is not exhaustive if and only if there exists $a$ such that $a \in Y$ and for every $b$ such that $b \in \operatorname{dom} F$ holds $a \notin F(b)$.

Let us consider $T$. Let $B$ be a non exhaustive, disjoint valued finite sequence of elements of $2^T$. The Polish arity from list $B$ yielding a Polish arity-function of $T$ is defined by the term

(Def. 30) the arity from list $B$.

One can check that there exists an antichain-like, finite sequence-membered set which has non empty elements and there exists a Polish-language which is non trivial and every antichain-like, finite sequence-membered set which is non trivial has also non empty elements.

Let us consider $S$, $n$, and $m$. Let $p$ be an element of $S \frown (n + 1 + m)$. The functor $\mathrm{decomp}(S, n, m, p)$ yielding an element of $S$ is defined by the term

(Def. 31) $S$-head($S \frown n$-tail($p$)).

Let $p$ be an element of $S \frown n$. The functor $\mathrm{decomp}(S, n, p)$ yielding a finite sequence of elements of $S$ is defined by

(Def. 32) $\mathrm{dom}\, it = \mathrm{Seg}\, n$ and for every $m$ such that $m \in \mathrm{Seg}\, n$ there exists $k$ such that $m = k + 1$ and $it(m) = S$-head($S \frown k$-tail($p$)).

Now we state the propositions:

(60) Let us consider an element $s$ of $S \frown n$, and an element $t$ of $T \frown n$. If $S \subseteq T$ and $s = t$, then $\mathrm{decomp}(S, n, s) = \mathrm{decomp}(T, n, t)$.
PROOF: Set $p = \mathrm{decomp}(S, n, s)$. Set $q = \mathrm{decomp}(T, n, t)$. For every $a$ such that $a \in \mathrm{Seg}\, n$ holds $p(a) = q(a)$ by (17), [4, (1)], (57), (55). □

(61) Let us consider an element $q$ of $S \frown 0$. Then $\mathrm{decomp}(S, 0, q) = \emptyset$.

(62) Let us consider an element $q$ of $S \frown n$. Then $\mathrm{len}\, \mathrm{decomp}(S, n, q) = n$.

(63) Let us consider an element $q$ of $S \frown 1$. Then $\mathrm{decomp}(S, 1, q) = \langle q \rangle$. The theorem is a consequence of (7), (58), (53), and (62).

(64) Let us consider elements $p$, $q$ of $S$, and an element $r$ of $S \frown 2$. Suppose $r = p \frown q$. Then $\mathrm{decomp}(S, 2, r) = \langle p, q \rangle$. The theorem is a consequence of (58), (52), (7), (53), and (62).

(65) Polish-WFF-set($T, \alpha$) is $T$-headed. The theorem is a consequence of (28), (23), and (21).

(66) The Polish-expression hierarchy($T$, $\alpha$, $n$) is $T$-headed. The theorem is a consequence of (26) and (65).

Let us consider $T$, $\alpha$, and $\varphi$. The functor Polish-WFF-head $\varphi$ yielding an element of $T$ is defined by the term

(Def. 33) $T$-head($\varphi$).

Let us consider $n$. Let $\varphi$ be an element of the Polish-expression hierarchy($T$, $\alpha$, $n$). The functor Polish-WFF-head $\varphi$ yielding an element of $T$ is defined by the term

(Def. 34) $T$-head($\varphi$).

Let us consider $\varphi$. The Polish arity $\varphi$ yielding a natural number is defined by the term

(Def. 35) $\alpha$(Polish-WFF-head $\varphi$).

Let us consider $n$. Let $\varphi$ be an element of the Polish-expression hierarchy($T$, $\alpha$, $n$). The Polish arity $\varphi$ yielding a natural number is defined by the term

(Def. 36) $\alpha$(Polish-WFF-head $\varphi$).

Now we state the propositions:

(67) $T$-tail($\varphi$) $\in$ Polish-WFF-set$(T, \alpha) \frown$ (the Polish arity $\varphi$). The theorem is a consequence of (32) and (52).

(68) Let us consider an element $\varphi$ of the Polish-expression hierarchy($T$, $\alpha$, $n+1$). Then $T$-tail($\varphi$) $\in$ (the Polish-expression hierarchy($T$, $\alpha$, $n$)) $\frown$ (the Polish arity $\varphi$). The theorem is a consequence of (23) and (52).

Let us consider $T$, $\alpha$, and $\varphi$. The functor $(T, \alpha)$-tail $\varphi$ yielding an element of Polish-WFF-set$(T, \alpha) \frown$ (the Polish arity $\varphi$) is defined by the term

(Def. 37) $T$-tail($\varphi$).

Now we state the proposition:

(69) If $T$-head($\varphi$) $\in$ the Polish atoms($T$, $\alpha$), then $\varphi = T$-head($\varphi$). The theorem is a consequence of (67) and (6).

Let us consider $T$, $\alpha$, and $n$. Let $\varphi$ be an element of the Polish-expression hierarchy($T, \alpha, n+1$). The functor $(T, \alpha)$-tail $\varphi$ yielding an element of (the Polish-expression hierarchy($T$, $\alpha$, $n$)) $\frown$ (the Polish arity $\varphi$) is defined by the term

(Def. 38) $T$-tail($\varphi$).

Let us consider $\varphi$. The functor Polish-WFF-args $\varphi$ yielding a finite sequence of elements of Polish-WFF-set$(T, \alpha)$ is defined by the term

(Def. 39) decomp(Polish-WFF-set$(T, \alpha)$, the Polish arity $\varphi$, $(T, \alpha)$-tail $\varphi$).

Let us consider $n$. Let $\varphi$ be an element of the Polish-expression hierarchy($T$, $\alpha$, $n+1$). The functor Polish-WFF-args $\varphi$ yielding a finite sequence of elements of the Polish-expression hierarchy($T$, $\alpha$, $n$) is defined by the term

(Def. 40) decomp(the Polish-expression hierarchy($T$, $\alpha$, $n$), the Polish arity $\varphi$, $(T, \alpha)$-tail $\varphi$).

Now we state the propositions:

(70) Let us consider an element $t$ of $T$, and $u$.
Suppose $u \in$ Polish-WFF-set$(T, \alpha) \frown \alpha(t)$.
Then $T$-tail((the Polish operation($T$, $\alpha$, $t$))($u$)) $= u$. The theorem is a consequence of (52).

(71) Suppose $\varphi \in$ the Polish-expression hierarchy($T$, $\alpha$, $n+1$).
Then rng Polish-WFF-args $\varphi \subseteq$ the Polish-expression hierarchy($T$, $\alpha$, $n$).
The theorem is a consequence of (60) and (26).

(72) Let us consider a finite sequence $p$, a function $f$ from $Y$ into $D$, and a function $g$ from $Z$ into $D$. Suppose $\operatorname{rng} p \subseteq Y$ and $\operatorname{rng} p \subseteq Z$ and for every $a$ such that $a \in \operatorname{rng} p$ holds $f(a) = g(a)$. Then $f \cdot p = g \cdot p$.
PROOF: Reconsider $p_1 = p$ as a finite sequence of elements of $Y$. Reconsider $q = f \cdot p_1$ as a finite sequence. Reconsider $p_2 = p$ as a finite sequence of elements of $Z$. Reconsider $r = g \cdot p_2$ as a finite sequence. $q = r$ by [6, (33)], [4, (1)], [7, (13), (3)]. $\square$

Let us consider $T$, $\alpha$, and $D$. The Polish recursion-domain($\alpha$, $D$) yielding a subset of $T \times D^*$ is defined by the term

(Def. 41) $\{\langle t, p \rangle$, where $t$ is an element of $T$, $p$ is a finite sequence of elements of $D : \operatorname{len} p = \alpha(t)\}$.

A Polish recursion-function of $\alpha$ and $D$ is a function from the Polish recursion-domain($\alpha$, $D$) into $D$. From now on $f$ denotes a Polish recursion-function of $\alpha$ and $D$ and $\gamma$, $\gamma_1$, $\gamma_2$ denote functions from Polish-WFF-set$(T, \alpha)$ into $D$.

Let us consider $T$, $\alpha$, $D$, $f$, and $\gamma$. We say that $\gamma$ is $f$-recursive if and only if

(Def. 42) for every $\varphi$, $\gamma(\varphi) = f(\langle T\text{-head}(\varphi), \gamma \cdot \text{Polish-WFF-args}\, \varphi \rangle)$.

Now we state the proposition:

(73) If $\gamma_1$ is $f$-recursive and $\gamma_2$ is $f$-recursive, then $\gamma_1 = \gamma_2$. The theorem is a consequence of (36), (17), (33), (52), (60), (72), and (37).

From now on $L$ denotes a non trivial Polish-language, $\beta$ denotes a Polish arity-function of $L$, $g$ denotes a Polish recursion-function of $\beta$ and $D$, $J$, $J_1$ denote subsets of Polish-WFF-set$(L, \beta)$, $H$ denotes a function from $J$ into $D$, $H_1$ denotes a function from $J_1$ into $D$.

Let us consider $L$, $\beta$, $D$, $g$, $J$, and $H$. We say that $H$ is $g$-recursive if and only if

(Def. 43) for every Polish WFF $\varphi$ of $L$ and $\beta$ such that $\varphi \in J$ and rng Polish-WFF-args $\varphi \subseteq J$ holds
$H(\varphi) = g(\langle L\text{-head}(\varphi), H \cdot \text{Polish-WFF-args}\, \varphi \rangle)$.

Now we state the propositions:

(74) There exists $J$ and there exists $H$ such that $J =$ the Polish-expression hierarchy($L$, $\beta$, $n$) and $H$ is $g$-recursive.
PROOF: Define $\mathcal{X}$[natural number] $\equiv$ there exists $J$ and there exists $H$ such that $J =$ the Polish-expression hierarchy($L$, $\beta$, $\$_1$) and $H$ is $g$-recursive. For every $n$, $\mathcal{X}[n]$ from [2, Sch. 2]. □

(75) There exists a function $\gamma$ from Polish-WFF-set$(L, \beta)$ into $D$ such that $\gamma$ is $g$-recursive.
PROOF: Set $W =$ Polish-WFF-set$(L, \beta)$. Define $\mathcal{X}$[object, object] $\equiv$ there exists $n$ and there exists $J_1$ and there exists $H_1$ such that $J_1 =$ the Polish-expression hierarchy($L$, $\beta$, $n$) and $H_1$ is $g$-recursive and $\$_1 \in J_1$ and $\$_2 = H_1(\$_1)$. For every $a$ such that $a \in W$ there exists $b$ such that $b \in D$ and $\mathcal{X}[a, b]$ by (28), (74), [8, (5)]. Consider $\gamma$ being a function from $W$ into $D$ such that for every $a$ such that $a \in W$ holds $\mathcal{X}[a, \gamma(a)]$ from [8, Sch. 1]. □

(76) Let us consider an element $t$ of $L$. Then the Polish operation($L$, $\beta$, $t$) is one-to-one.

PROOF: Set $f$ = the Polish operation($L$, $\beta$, $t$). For every $a$ and $b$ such that $a$, $b \in \operatorname{dom} f$ and $f(a) = f(b)$ holds $a = b$ by [4, (33)]. □

(77) Let us consider elements $t$, $u$ of $L$. Suppose rng(the Polish operation($L$, $\beta$, $t$)) meets rng(the Polish operation($L$, $\beta$, $u$)). Then $t = u$. The theorem is a consequence of (43).

(78) Let us consider an element $t$ of $L$, and $a$. Suppose $a \in$ dom(the Polish operation($L$, $\beta$, $t$)). Then there exists $p$ such that

(i) $p$ = (the Polish operation($L$, $\beta$, $t$))($a$), and

(ii) $L$-head($p$) = $t$.

The theorem is a consequence of (52).

Let us consider $L$, $\beta$, an element $t$ of $L$, and a Polish WFF $\varphi$ of $L$ and $\beta$. Now we state the proposition:

(79) Polish-WFF-head $\varphi = t$ if and only if there exists an element $u$ of Polish-WFF-set$(L, \beta) \frown \beta(t)$ such that $\varphi$ = (the Polish operation($L$, $\beta$, $t$))($u$). The theorem is a consequence of (52).

Let us assume that $\beta(t) = 1$. Now we state the propositions:

(80) If Polish-WFF-head $\varphi = t$, then there exists a Polish WFF $\psi$ of $L$ and $\beta$ such that $\varphi$ = (Polish-unOp$(L, \beta, t)$)($\psi$). The theorem is a consequence of (79) and (7).

(81) (i) Polish-WFF-head((Polish-unOp$(L, \beta, t)$)($\varphi$)) = $t$, and

(ii) Polish-WFF-args((Polish-unOp$(L, \beta, t)$)($\varphi$)) = $\langle \varphi \rangle$.

The theorem is a consequence of (7), (79), (70), and (63).

Now we state the proposition:

(82) Suppose $\beta(t) = 2$. Then suppose Polish-WFF-head $\varphi = t$. Then there exist Polish WFFs $\psi$, $H$ of $L$ and $\beta$ such that $\varphi$ = (Polish-binOp$(L, \beta, t)$) $(\psi, H)$. The theorem is a consequence of (79), (6), and (7).

Now we state the propositions:

(83) Let us consider an element $t$ of $L$. Suppose $\beta(t) = 2$. Let us consider Polish WFFs $\varphi$, $\psi$ of $L$ and $\beta$. Then

(i) Polish-WFF-head(Polish-binOp$(L, \beta, t)$)$(\varphi, \psi) = t$, and

(ii) Polish-WFF-args(Polish-binOp$(L, \beta, t)$)$(\varphi, \psi) = \langle \varphi, \psi \rangle$.

The theorem is a consequence of (7), (11), (79), (64), and (70).

(84) Let us consider a Polish WFF $\varphi$ of $L$ and $\beta$. Then $\varphi \in$ the Polish atoms($L$, $\beta$) if and only if the Polish arity $\varphi = 0$. The theorem is a consequence of (53), (67), and (6).

(85) Let us consider a function $\gamma$ from Polish-WFF-set$(L, \beta)$ into $D$, an element $t$ of $L$, and a Polish WFF $\varphi$ of $L$ and $\beta$. Suppose $\gamma$ is $g$-recursive and $\beta(t) = 1$. Then $\gamma((\text{Polish-unOp}(L, \beta, t))(\varphi)) = g(t, \langle\gamma(\varphi)\rangle)$. The theorem is a consequence of (81).

Let us consider $S$. Let $p$ be a finite sequence of elements of $S$. The functor Flat$(p)$ yielding an element of $S \frown \operatorname{len} p$ is defined by

(Def. 44) $\text{decomp}(S, \operatorname{len} p, it) = p$.

Let us consider $L$ and $\beta$.

A substitution of $L$ and $\beta$ is a partial function from the Polish atoms$(L, \beta)$ to Polish-WFF-set$(L, \beta)$. Let $s$ be a substitution of $L$ and $\beta$. The functor Subst $s$ yielding a Polish recursion-function of $\beta$ and Polish-WFF-set$(L, \beta)$ is defined by

(Def. 45) for every element $t$ of $L$ and for every finite sequence $p$ of elements of Polish-WFF-set$(L, \beta)$ such that $\operatorname{len} p = \beta(t)$ holds if $t \in \operatorname{dom} s$, then $it(t, p) = s(t)$ and if $t \notin \operatorname{dom} s$, then $it(t, p) = t \frown \text{Flat}(p)$.

Let $\varphi$ be a Polish WFF of $L$ and $\beta$. The functor $s[\varphi]$ yielding a Polish WFF of $L$ and $\beta$ is defined by

(Def. 46) there exists a function $H$ from Polish-WFF-set$(L, \beta)$ into Polish-WFF-set$(L, \beta)$ such that $H$ is (Subst $s$)-recursive and $it = H(\varphi)$.

Now we state the proposition:

(86) Let us consider a substitution $s$ of $L$ and $\beta$, and a Polish WFF $\varphi$ of $L$ and $\beta$. If $s = \emptyset$, then $s[\varphi] = \varphi$.
Proof: Set $W$ = Polish-WFF-set$(L, \beta)$. Set $g$ = Subst $s$. Set $\gamma = \text{id}_W$. $\gamma$ is $g$-recursive by (62), [6, (32), (33)], [7, (3), (17), (13)]. □

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# Grzegorczyk's Logics. Part I

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**Summary.** This article is the second in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([9] and [10]) concerning a logic proposed by Prof. Andrzej Grzegorczyk ([11]).

This part presents the syntax and axioms of Grzegorczyk's *Logic of Descriptions* (LD) as originally proposed by him, as well as some theorems not depending on any semantic constructions. There are both some clear similarities and fundamental differences between LD and the non-Fregean logics introduced by Roman Suszko in [14]. In particular, we were inspired by Suszko's semantics for his non-Fregean logic SCI, presented in [15].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [16], [13], [2], [8], [4], [5], [1], [6], [? ], [18], [20], [19], [12], [17], and [7].

## 1. The Construction of Grzegorczyk's LD Language

From now on $k$, $m$, $n$ denote elements of $\mathbb{N}$, $i$, $j$ denote natural numbers, $a$, $b$, $c$ denote objects, $X$, $Y$, $Z$ denote sets, $D$, $D_1$, $D_2$ denote non empty sets, and $p$, $q$, $r$, $s$ denote finite sequences.

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The functor VAR yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all $\langle 0, k\rangle$ where $k$ is an element of $\mathbb{N}$.

Note that VAR is non empty and antichain-like.

A variable is an element of VAR. The functors: $'\mathrm{not}'$, &, and $'='$ yielding finite sequences are defined by terms

(Def. 2) $\langle 1\rangle$,

(Def. 3) $\langle 2\rangle$,

(Def. 4) $\langle 3\rangle$,

respectively. The functor GRZ-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) $\{'\mathrm{not}', \&, '='\}$.

Let us note that the functor GRZ-ops yields a Polish language. The functor GRZ-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 6) $\mathrm{VAR} \cup \mathrm{GRZ\text{-}ops}$.

The functors: $'\mathrm{not}'$, &, and $'='$ yield elements of GRZ-symbols. Now we state the proposition:

(1) (i) $'\mathrm{not}' \neq \&$, and

(ii) $'\mathrm{not}' \neq '='$, and

(iii) $\& \neq '='$.

Observe that GRZ-symbols is non trivial and antichain-like.

The functor GRZ-op-arity yielding a function from GRZ-ops into $\mathbb{N}$ is defined by

(Def. 7) $it('\mathrm{not}') = 1$ and $it(\&) = 2$ and $it('=') = 2$.

The functor GRZ-arity yielding a Polish arity-function of GRZ-symbols is defined by

(Def. 8) for every $a$ such that $a \in$ GRZ-symbols holds if $a \in$ GRZ-ops, then $it(a) =$ GRZ-op-arity$(a)$ and if $a \notin$ GRZ-ops, then $it(a) = 0$.

Now we state the propositions:

(2) (i) GRZ-arity$('\mathrm{not}') = 1$, and

(ii) GRZ-arity$(\&) = 2$, and

(iii) GRZ-arity$('=') = 2$.

(3) The Polish atoms( GRZ-symbols , GRZ-arity ) = VAR. The theorem is a consequence of (2).

The functor GRZ-formula-set yielding a Polish language of GRZ-symbols is defined by the term

(Def. 9) $\text{Polish-WFF-set}(\text{GRZ-symbols}, \text{GRZ-arity})$.

A GRZ-formula is a Polish WFF of GRZ-symbols and GRZ-arity. One can verify that there exists a subset of GRZ-formula-set which is non empty.

Let us consider $n$. The functor $\mathrm{x}_n$ yielding a GRZ-formula is defined by the term

(Def. 10) $\langle 0, n \rangle$.

From now on $\varphi$, $\psi$, $\vartheta$, $\eta$ denote GRZ-formulas.

Let us consider $\varphi$. The functor $\neg\varphi$ yielding a GRZ-formula is defined by the term

(Def. 11) $(\text{Polish-unOp}(\text{GRZ-symbols}, \text{GRZ-arity}, '\text{not}'))(\varphi)$.

Let us consider $\psi$. The functors: $\varphi \wedge \psi$ and $\varphi{=}\psi$ yielding GRZ-formulas are defined by terms

(Def. 12) $(\text{Polish-binOp}(\text{GRZ-symbols}, \text{GRZ-arity}, \&))(\varphi, \psi)$,

(Def. 13) $(\text{Polish-binOp}(\text{GRZ-symbols}, \text{GRZ-arity}, '{=}'))(\varphi, \psi)$,

respectively. The functors: $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ yielding GRZ-formulas are defined by terms

(Def. 14) $\neg(\neg\varphi \wedge \neg\psi)$,

(Def. 15) $\varphi{=}(\varphi \wedge \psi)$,

respectively. The functor $\varphi \Leftrightarrow \psi$ yielding a GRZ-formula is defined by the term

(Def. 16) $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

We say that $\varphi$ is atomic if and only if

(Def. 17) $\varphi \in$ the Polish atoms( GRZ-symbols , GRZ-arity ).

We say that $\varphi$ is negative if and only if

(Def. 18) $\text{Polish-WFF-head}\,\varphi = '\text{not}'$.

We say that $\varphi$ is conjunctive if and only if

(Def. 19) $\text{Polish-WFF-head}\,\varphi = \&$.

We say that $\varphi$ is an equality if and only if

(Def. 20) $\text{Polish-WFF-head}\,\varphi = '{=}'$.

Let us consider $\varphi$. Now we state the propositions:

(4) $\varphi$ is atomic if and only if $\varphi \in \text{VAR}$.

(5) $\varphi$ is negative if and only if there exists $\psi$ such that $\varphi = \neg\psi$.

PROOF: If $\varphi$ is negative, then there exists $\psi$ such that $\varphi = \neg\psi$ by (2), [**?** , (80)]. $\square$

(6) $\varphi$ is conjunctive if and only if there exists $\psi$ and there exists $\vartheta$ such that $\varphi = \psi \wedge \vartheta$.
PROOF: If $\varphi$ is conjunctive, then there exists $\psi$ and there exists $\vartheta$ such that $\varphi = \psi \wedge \vartheta$ by (2), [**?** , (82)]. □

(7) $\varphi$ is an equality if and only if there exists $\psi$ and there exists $\vartheta$ such that $\varphi = \psi{=}\vartheta$.
PROOF: If $\varphi$ is an equality, then there exists $\psi$ and there exists $\vartheta$ such that $\varphi = \psi{=}\vartheta$ by (2), [**?** , (82)]. □

(8) $\varphi$ is atomic or negative or conjunctive or an equality. The theorem is a consequence of (3).

Let us observe that every GRZ-formula which is atomic is also non negative and every GRZ-formula which is atomic is also non conjunctive and every GRZ-formula which is atomic is also non equality and every GRZ-formula which is negative is also non conjunctive and every GRZ-formula which is negative is also non equality and every GRZ-formula which is conjunctive is also non equality.

## 2. Axioms and Rules

The functors: GRZ-axioms and LD-specific axioms yielding non empty subsets of GRZ-formula-set are defined by conditions

(Def. 21) for every $a$, $a \in$ GRZ-axioms iff there exists $\varphi$ and there exists $\psi$ and there exists $\vartheta$ such that $a = \neg(\varphi \wedge \neg\varphi)$ or $a = (\neg\neg\varphi){=}\varphi$ or $a = \varphi{=}(\varphi \wedge \varphi)$ or $a = (\varphi \wedge \psi){=}(\psi \wedge \varphi)$ or $a = (\varphi \wedge (\psi \wedge \vartheta)){=}((\varphi \wedge \psi) \wedge \vartheta)$ or $a = (\varphi \wedge (\psi \vee \vartheta)){=}(\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ or $a = (\neg(\varphi \wedge \psi)){=}(\neg\varphi \vee \neg\psi)$ or $a = (\varphi{=}\psi){=}(\psi{=}\varphi)$ or $a = (\varphi{=}\psi){=}((\neg\varphi){=}(\neg\psi))$,

(Def. 22) for every $a$, $a \in$ LD-specific axioms iff there exists $\varphi$ and there exists $\psi$ and there exists $\vartheta$ such that $a = \varphi{=}\psi \Rightarrow (\varphi \wedge \vartheta){=}(\psi \wedge \vartheta)$ or $a = \varphi{=}\psi \Rightarrow (\varphi \vee \vartheta){=}(\psi \vee \vartheta)$ or $a = \varphi{=}\psi \Rightarrow (\varphi{=}\vartheta){=}(\psi{=}\vartheta)$,

respectively. The functor LD-axioms yielding a non empty subset of GRZ-formula-set is defined by the term

(Def. 23) GRZ-axioms $\cup$ LD-specific axioms.

A GRZ-rule is a relation between $2^{\text{GRZ-formula-set}}$ and GRZ-formula-set. In the sequel $R$, $R_1$, $R_2$ denote GRZ-rules.

Let us consider $R_1$ and $R_2$. Note that the functor $R_1 \cup R_2$ yields a GRZ-rule. The functors: GRZ-MP, GRZ-ConjIntro, GRZ-ConjElimL, and GRZ-ConjElimR yielding GRZ-rules are defined by terms

(Def. 24) the set of all $\langle\{\varphi, \varphi{=}\psi\}, \psi\rangle$ where $\varphi$ is a GRZ-formula, $\psi$ is a GRZ-formula, (Def. 25) the set of all $\langle\{\varphi, \psi\}, \varphi \wedge \psi\rangle$ where $\varphi$ is a GRZ-formula, $\psi$ is a GRZ-formula,

(Def. 26) the set of all $\langle\{\varphi \wedge \psi\}, \varphi\rangle$ where $\varphi$ is a GRZ-formula, $\psi$ is a GRZ-formula,

(Def. 27) the set of all $\langle\{\varphi \wedge \psi\}, \psi\rangle$ where $\varphi$ is a GRZ-formula, $\psi$ is a GRZ-formula,

respectively. The functor GRZ-rules yielding a GRZ-rule is defined by

(Def. 28) for every $a$, $a \in it$ iff $a \in$ GRZ-MP or $a \in$ GRZ-ConjIntro or $a \in$ GRZ-ConjElimL or $a \in$ GRZ-ConjElimR.

A GRZ-formula sequence is a finite sequence of elements of GRZ-formula-set.

A finite GRZ-formula set is a finite subset of GRZ-formula-set. From now on $\Gamma$, $\Gamma_1$, $\Gamma_2$ denote non empty subsets of GRZ-formula-set, $\Delta$, $\Delta_1$, $\Delta_2$ denote subsets of GRZ-formula-set, $P$, $P_1$, $P_2$ denote GRZ-formula sequences, and $\Sigma$, $\Sigma_1$, $\Sigma_2$ denote finite GRZ-formula sets.

Let us consider $\Sigma_1$ and $\Sigma_2$. Observe that the functor $\Sigma_1 \cup \Sigma_2$ yields a finite GRZ-formula set. Let us consider $\Gamma$, $R$, $P$, and $n$. We say that $(P, n)$ is a correct step w.r.t. $\Gamma$, $R$ if and only if

(Def. 29) $P(n) \in \Gamma$ or there exists a finite GRZ-formula set $Q$ such that $\langle Q, P(n)\rangle \in R$ and for every $q$ such that $q \in Q$ there exists $k$ such that $k \in \operatorname{dom} P$ and $k < n$ and $P(k) = q$.

We say that $P$ is $(\Gamma, R)$-correct if and only if

(Def. 30) for every $k$ such that $k \in \operatorname{dom} P$ holds $(P, k)$ is a correct step w.r.t. $\Gamma$, $R$.

Let $a$ be an element of $\Gamma$. One can verify that the functor $\langle a\rangle$ yields a GRZ-formula sequence. Now we state the proposition:

(9) Let us consider an element $a$ of $\Gamma$. Then $\langle a\rangle$ is $(\Gamma, R)$-correct.

Let us consider $\Gamma$ and $R$. Note that there exists a GRZ-formula sequence which is non empty and $(\Gamma, R)$-correct.

Let us consider $\Sigma$. We say that $\Sigma$ is $(\Gamma, R)$-correct if and only if

(Def. 31) there exists $P$ such that $\Sigma = \operatorname{rng} P$ and $P$ is $(\Gamma, R)$-correct.

Now we state the propositions:

(10) If $P$ is $(\Gamma, R)$-correct and $P = P_1 \frown P_2$, then $P_1$ is $(\Gamma, R)$-correct.

(11) If $P_1$ is $(\Gamma, R)$-correct and $P_2$ is $(\Gamma, R)$-correct, then $P_1 \frown P_2$ is $(\Gamma, R)$-correct.

(12) If $\Sigma_1$ is $(\Gamma, R)$-correct and $\Sigma_2$ is $(\Gamma, R)$-correct, then $\Sigma_1 \cup \Sigma_2$ is $(\Gamma, R)$-correct. The theorem is a consequence of (11).

(13) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and $P$ is $(\Gamma, R)$-correct, then $P$ is $(\Gamma_1, R_1)$-correct.

Let us consider $\Gamma$, $R$, and $\varphi$. We say that $\Gamma, R \vdash \varphi$ if and only if

(Def. 32) there exists $P$ such that $\varphi \in \operatorname{rng} P$ and $P$ is $(\Gamma, R)$-correct.

Let us consider $\Delta$. We say that $\Gamma, R \vdash \Delta$ if and only if

(Def. 33) for every $\varphi$ such that $\varphi \in \Delta$ holds $\Gamma, R \vdash \varphi$.

Let us consider $\Gamma$, $R$, and $\varphi$. Now we state the propositions:

(14) $\Gamma, R \vdash \varphi$ if and only if there exists $\Sigma$ such that $\varphi \in \Sigma$ and $\Sigma$ is ($\Gamma$, $R$)-correct.

(15) If $\varphi \in \Gamma$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (9).

Now we state the propositions:

(16) If $\Gamma, R \vdash \Sigma$, then there exists $\Sigma_1$ such that $\Sigma \subseteq \Sigma_1$ and $\Sigma_1$ is ($\Gamma$, $R$)-correct.

PROOF: Define $\mathcal{X}[\text{set}] \equiv$ there exists $\Sigma_1$ such that $\$_1 \subseteq \Sigma_1$ and $\Sigma_1$ is ($\Gamma$, $R$)-correct. $\mathcal{X}[\emptyset]$. For every sets $x$, $\Delta$ such that $x \in \Sigma$ and $\Delta \subseteq \Sigma$ and $\mathcal{X}[\Delta]$ holds $\mathcal{X}[\Delta \cup \{x\}]$. $\mathcal{X}[\Sigma]$ from [8, Sch. 2]. $\square$

(17) If $\Gamma, R \vdash \Sigma$ and $\langle \Sigma, \varphi \rangle \in R$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (16).

(18) If $\Gamma, R \vdash \varphi$, then $\varphi \in \Gamma$ or there exists $\Sigma$ such that $\langle \Sigma, \varphi \rangle \in R$ and $\Gamma, R \vdash \Sigma$.

(19) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and $\Gamma, R \vdash \varphi$, then $\Gamma_1, R_1 \vdash \varphi$.

Let us consider $\Gamma$, $\varphi$, and $\psi$. Now we state the propositions:

(20) $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$ if and only if $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(21) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi = \psi$. Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(22) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi \Rightarrow \psi$. Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (21) and (20).

(23) If $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$, then $\Gamma, \text{GRZ-rules} \vdash \psi \wedge \varphi$. The theorem is a consequence of (20).

Let us consider $\varphi$. We say that $\varphi$ is GRZ-axiomatic if and only if

(Def. 34) $\varphi \in$ GRZ-axioms.

We say that $\varphi$ is GRZ-provable if and only if

(Def. 35) $\text{GRZ-axioms}, \text{GRZ-rules} \vdash \varphi$.

We say that $\varphi$ is LD-axiomatic if and only if

(Def. 36) $\varphi \in$ LD-axioms.

We say that $\varphi$ is LD-provable if and only if

(Def. 37) $\text{LD-axioms}, \text{GRZ-rules} \vdash \varphi$.

Observe that $\neg(\varphi \wedge \neg\varphi)$ is GRZ-axiomatic and $(\neg\neg\varphi) = \varphi$ is GRZ-axiomatic and $\varphi = (\varphi \wedge \varphi)$ is GRZ-axiomatic.

Let us consider $\psi$. Observe that $(\varphi \wedge \psi)=(\psi \wedge \varphi)$ is GRZ-axiomatic and $(\neg(\varphi \wedge \psi))=(\neg\varphi \vee \neg\psi)$ is GRZ-axiomatic and $(\varphi=\psi)=(\psi=\varphi)$ is GRZ-axiomatic and $(\varphi=\psi)=((\neg\varphi)=(\neg\psi))$ is GRZ-axiomatic.

Let us consider $\vartheta$. Observe that $(\varphi \wedge (\psi \wedge \vartheta))=((\varphi \wedge \psi) \wedge \vartheta)$ is GRZ-axiomatic and $(\varphi \wedge (\psi \vee \vartheta))=(\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ is GRZ-axiomatic and $\varphi=\psi \Rightarrow (\varphi \wedge \vartheta)=(\psi \wedge \vartheta)$ is LD-axiomatic and $\varphi=\psi \Rightarrow (\varphi \vee \vartheta)=(\psi \vee \vartheta)$ is LD-axiomatic and $\varphi=\psi \Rightarrow (\varphi=\vartheta)=(\psi=\vartheta)$ is LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also GRZ-provable and every GRZ-formula which is LD-axiomatic is also LD-provable and every GRZ-formula which is GRZ-provable is also LD-provable and there exists a GRZ-formula which is GRZ-axiomatic, GRZ-provable, LD-axiomatic, and LD-provable.

Now we state the proposition:

(24) Suppose GRZ-axioms $\subseteq \Gamma$ and $\Gamma$, GRZ-rules $\vdash \varphi=\psi$. Then $\Gamma$, GRZ-rules $\vdash \psi=\varphi$. The theorem is a consequence of (15) and (21).

## 3. PROVABILITY

Let us consider $\varphi$ and $\psi$. Now we state the propositions:

(25) If $\varphi=\psi$ is GRZ-provable, then $\psi=\varphi$ is GRZ-provable.

(26) If $\varphi=\psi$ is LD-provable, then $\psi=\varphi$ is LD-provable.

Now we state the propositions:

(27) If $\varphi=\psi$ is LD-provable and $\psi=\vartheta$ is LD-provable, then $\varphi=\vartheta$ is LD-provable. The theorem is a consequence of (24), (22), and (21).

(28) $\varphi=\varphi$ is LD-provable. The theorem is a consequence of (24) and (27).

Let us consider $\varphi$ and $\psi$. We say that $\varphi =_{\rm LD} \psi$ if and only if

(Def. 38) $\varphi=\psi$ is LD-provable.

One can check that the predicate is reflexive and symmetric.

Now we state the proposition:

(29) If $\varphi =_{\rm LD} \psi$, then $\neg\varphi =_{\rm LD} \neg\psi$. The theorem is a consequence of (21).

The scheme *BinReplace* deals with a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 1) For every elements $a$, $b$, $c$, $d$ of $\mathcal{X}$ such that $\mathcal{R}[a, b]$ and $\mathcal{R}[c, d]$ holds $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, d)]$

provided

- for every elements $a$, $b$, $c$ of $\mathcal{X}$ such that $\mathcal{R}[a, b]$ and $\mathcal{R}[b, c]$ holds $\mathcal{R}[a, c]$ and

- for every elements $a$, $b$ of $\mathcal{X}$, $\mathcal{R}[\mathcal{F}(a,b),\mathcal{F}(b,a)]$ and
- for every elements $a$, $b$, $c$ of $\mathcal{X}$ such that $\mathcal{R}[a,b]$ holds $\mathcal{R}[\mathcal{F}(a,c),\mathcal{F}(b,c)]$.

Let us consider $\varphi$, $\psi$, $\vartheta$, and $\eta$.

Let us assume that $\varphi =_{\rm LD} \psi$ and $\vartheta =_{\rm LD} \eta$. Now we state the propositions:

(30) $\varphi \wedge \vartheta =_{\rm LD} \psi \wedge \eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 \wedge \$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1{=}\$_2$ is LD-provable. For every $\varphi$, $\psi$, and $\vartheta$ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\psi,\vartheta]$ holds $\mathcal{P}[\varphi,\vartheta]$. For every $\varphi$, $\psi$, $\vartheta$, and $\eta$ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\vartheta,\eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi,\vartheta),\mathcal{F}(\psi,\eta)]$ from *BinReplace*. □

(31) $\varphi{=}\vartheta =_{\rm LD} \psi{=}\eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1{=}\$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1{=}\$_2$ is LD-provable. For every $\varphi$, $\psi$, and $\vartheta$ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\psi,\vartheta]$ holds $\mathcal{P}[\varphi,\vartheta]$. For every $\varphi$, $\psi$, $\vartheta$, and $\eta$ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\vartheta,\eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi,\vartheta),\mathcal{F}(\psi,\eta)]$ from *BinReplace*. □

The functor LD-IdR yielding an equivalence relation of GRZ-formula-set is defined by

(Def. 39) for every $\varphi$ and $\psi$, $\langle\varphi, \psi\rangle \in it$ iff $\varphi =_{\rm LD} \psi$.

Note that there exists a family of subsets of GRZ-formula-set which is non empty.

The functor LD-IdClasses yielding a non empty family of subsets of GRZ-formula-set is defined by the term

(Def. 40) Classes LD-IdR.

An LD-identity class is an element of LD-IdClasses. Let us consider $\varphi$. The functor LD-IdClassOf $\varphi$ yielding an LD-identity class is defined by the term

(Def. 41) $[\varphi]_{\text{LD-IdR}}$.

Now we state the proposition:

(32) $\varphi =_{\rm LD} \psi$ if and only if LD-IdClassOf $\varphi$ = LD-IdClassOf $\psi$.

PROOF: If $\varphi =_{\rm LD} \psi$, then LD-IdClassOf $\varphi$ = LD-IdClassOf $\psi$ by [13, (18), (23)]. □

The scheme *UnOpCongr* deals with a non empty set $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 2) There exists a unary operation $f$ on Classes $\mathcal{E}$ such that for every element $x$ of $\mathcal{X}$, $f([x]_{\mathcal{E}}) = [\mathcal{F}(x)]_{\mathcal{E}}$

provided

- for every elements $x$, $y$ of $\mathcal{X}$ such that $\langle x, y\rangle \in \mathcal{E}$ holds $\langle\mathcal{F}(x), \mathcal{F}(y)\rangle \in \mathcal{E}$.

The scheme *BinOpCongr* deals with a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 3) There exists a binary operation $f$ on Classes $\mathcal{E}$ such that for every elements $x$, $y$ of $\mathcal{X}$, $f([x]_{\mathcal{E}}, [y]_{\mathcal{E}}) = [\mathcal{F}(x, y)]_{\mathcal{E}}$

provided

- for every elements $x_1$, $x_2$, $y_1$, $y_2$ of $\mathcal{X}$ such that $\langle x_1, x_2\rangle$, $\langle y_1, y_2\rangle \in \mathcal{E}$ holds $\langle\mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2)\rangle \in \mathcal{E}$.

From now on $x$, $y$, $z$ denote LD-identity classes.

Now we state the proposition:

(33) There exists $\varphi$ such that $x =$ LD-IdClassOf $\varphi$.

Let us consider $x$. We say that $x$ is LD-provable if and only if

(Def. 42) there exists $\varphi$ such that $x =$ LD-IdClassOf $\varphi$ and $\varphi$ is LD-provable.

The functor $\neg x$ yielding an LD-identity class is defined by

(Def. 43) there exists $\varphi$ such that $x =$ LD-IdClassOf $\varphi$ and $it =$ LD-IdClassOf $\neg\varphi$.

One can verify that the functor is involutive. Let us consider $y$. The functor $x \wedge y$ yielding an LD-identity class is defined by

(Def. 44) there exists $\varphi$ and there exists $\psi$ such that $x =$ LD-IdClassOf $\varphi$ and $y =$ LD-IdClassOf $\psi$ and $it =$ LD-IdClassOf$(\varphi \wedge \psi)$.

Note that the functor is commutative and idempotent. The functor $x{=}y$ yielding an LD-identity class is defined by

(Def. 45) there exists $\varphi$ and there exists $\psi$ such that $x =$ LD-IdClassOf $\varphi$ and $y =$ LD-IdClassOf $\psi$ and $it =$ LD-IdClassOf $\varphi{=}\psi$.

One can check that the functor is commutative.

The functor $x \vee y$ yielding an LD-identity class is defined by the term

(Def. 46) $\neg(\neg x \wedge \neg y)$.

Let us observe that the functor is commutative and idempotent. The functor $x \Rightarrow y$ yielding an LD-identity class is defined by the term

(Def. 47) $x{=}(x \wedge y)$.

Let $\varphi$ be an LD-provable GRZ-formula. Let us observe that LD-IdClassOf $\varphi$ is LD-provable.

Now we state the proposition:

(34) If LD-IdClassOf $\varphi$ is LD-provable, then $\varphi$ is LD-provable. The theorem is a consequence of (32) and (21).

Let us consider $x$ and $y$. Now we state the propositions:

(35) $x \wedge y$ is LD-provable if and only if $x$ is LD-provable and $y$ is LD-provable. The theorem is a consequence of (34) and (20).

(36) $x{=}y$ is LD-provable if and only if $x = y$. The theorem is a consequence of (34) and (32).

Now we state the proposition:

(37) LD-IdClassOf $\neg\varphi = \neg$ LD-IdClassOf $\varphi$.

Let us consider $\varphi$ and $\psi$. Now we state the propositions:

(38) LD-IdClassOf$(\varphi \wedge \psi) =$ LD-IdClassOf $\varphi \wedge$ LD-IdClassOf $\psi$.

(39) LD-IdClassOf $\varphi{=}\psi =$ (LD-IdClassOf $\varphi$)=(LD-IdClassOf $\psi$).

(40) LD-IdClassOf$(\varphi \vee \psi) =$ LD-IdClassOf $\varphi \vee$ LD-IdClassOf $\psi$.

(41) LD-IdClassOf$(\varphi \Rightarrow \psi) =$ LD-IdClassOf $\varphi \Rightarrow$ LD-IdClassOf $\psi$.

Now we state the propositions:

(42) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. The theorem is a consequence of (33) and (32).

(43) $x \Rightarrow y$ is LD-provable if and only if $x = x \wedge y$.

(44) If $x \Rightarrow y$ is LD-provable and $y \Rightarrow z$ is LD-provable, then $x \Rightarrow z$ is LD-provable. The theorem is a consequence of (36) and (42).

(45) If $\varphi \Rightarrow \psi$ is LD-provable and $\psi \Rightarrow \vartheta$ is LD-provable, then $\varphi \Rightarrow \vartheta$ is LD-provable. The theorem is a consequence of (41), (34), and (44).

Let us consider $x$, $y$, and $z$. Now we state the propositions:

(46) $x \vee (y \vee z) = (x \vee y) \vee z$.

(47) $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$. The theorem is a consequence of (33), (32), and (40).

(48) $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$. The theorem is a consequence of (47).

Let us consider $x$ and $y$. Now we state the propositions:

(49) $x \Rightarrow y$ is LD-provable and $y \Rightarrow x$ is LD-provable if and only if $x = y$. The theorem is a consequence of (36).

(50) If $x$ is LD-provable, then $x \vee y$ is LD-provable. The theorem is a consequence of (33), (35), (47), and (48).

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# Convergent Filter Bases

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**Summary.** We are inspired by the work of Henri Cartan [16], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: *filtre des sections*).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [9], [19], [27], and [15].

## 1. Filters – Set-Theoretical Approach

From now on $X$ denotes a non empty set, $\mathcal{F}$ denotes a filter of $X$, and $S$ denotes a family of subsets of $X$.

Let $X$ be a set and $S$ be a family of subsets of $X$. We say that $S$ is upper if and only if

(Def. 1) for every subsets $Y_1$, $Y_2$ of $X$ such that $Y_1 \in S$ and $Y_1 \subseteq Y_2$ holds $Y_2 \in S$.

Let us note that there exists a $\cap$-closed family of subsets of $X$ which is non empty and there exists a non empty, $\cap$-closed family of subsets of $X$ which is upper.

Let $X$ be a non empty set. Let us note that there exists a non empty, upper, $\cap$-closed family of subsets of $X$ which has non empty elements.

Now we state the propositions:




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(1) $S$ is a non empty, upper, $\cap$-closed family of subsets of $X$ with non empty elements if and only if $S$ is a filter of $X$.

(2) Let us consider non empty sets $X_1$, $X_2$, a filter $\mathcal{F}_1$ of $X_1$, and a filter $\mathcal{F}_2$ of $X_2$. Then the set of all $f_1 \times f_2$ where $f_1$ is an element of $\mathcal{F}_1$, $f_2$ is an element of $\mathcal{F}_2$ is a non empty family of subsets of $X_1 \times X_2$.

Let $X$ be a non empty set. We say that $X$ is $\cap$-finite closed if and only if

(Def. 2) for every finite, non empty subset $S_1$ of $X$, $\bigcap S_1 \in X$.

One can check that there exists a non empty set which is $\cap$-finite closed.

Now we state the proposition:

(3) Let us consider a non empty set $X$. If $X$ is $\cap$-finite closed, then $X$ is $\cap$-closed.

Note that every non empty set which is $\cap$-finite closed is also $\cap$-closed.

(4) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $S$ is $\cap$-closed and $X \in S$ if and only if $\text{FinMeetCl}(S) \subseteq S$.

(5) Let us consider a non empty set $X$, and a non empty subset $A$ of $X$. Then $\{B$, where $B$ is a subset of $X : A \subseteq B\}$ is a filter of $X$.

Let $X$ be a non empty set. Note that every filter of $X$ is $\cap$-closed.

(6) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $B = \{X\}$, then $B$ is upper.

(7) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\mathcal{F}' \neq 2^X$.

Let $X$ be a non empty set. The functor $\text{Filt}(X)$ yielding a non empty set is defined by the term

(Def. 3) the set of all $\mathcal{F}'$ where $\mathcal{F}'$ is a filter of $X$.

Let $I$ be a non empty set and $M$ be a $(\text{Filt}(X))$-valued many sorted set indexed by $I$. The intersection of the family of filters $M$ yielding a filter of $X$ is defined by the term

(Def. 4) $\bigcap \text{rng}\, M$.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters of $X$. We say that $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if

(Def. 5) $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

One can verify that the predicate is reflexive. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 6) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Observe that the predicate is reflexive.

Now we state the propositions:

(8) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a filter $\mathcal{F}$ of $X$. Suppose $\mathcal{F} = \{X\}$. Then $\mathcal{F}$ is coarser than $\mathcal{F}'$.

(9) Let us consider a non empty set $X$, a non empty set $I$, a (Filt($X$))-valued many sorted set $M$ indexed by $I$, an element $i$ of $I$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}' = M(i)$. Then the intersection of the family of filters $M$ is coarser than $\mathcal{F}'$.

(10) Let us consider a set $X$, and a family $S$ of subsets of $X$. Suppose FinMeetCl($S$) has non empty elements. Then $S$ has non empty elements.

(11) Let us consider a non empty set $X$, a family $G$ of subsets of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $G \subseteq \mathcal{F}'$. Then

(i) FinMeetCl($G$) $\subseteq \mathcal{F}'$, and

(ii) FinMeetCl($G$) has non empty elements.

The theorem is a consequence of (4).

Let $X$ be a non empty set, $\mathcal{F}'$ be a filter of $X$, and $B$ be a non empty subset of $\mathcal{F}'$. We say that $B$ is filter basis if and only if

(Def. 7) for every element $f$ of $\mathcal{F}'$, there exists an element $b$ of $B$ such that $b \subseteq f$.

Now we state the proposition:

(12) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a non empty subset $B$ of $\mathcal{F}'$. Then $\mathcal{F}'$ is coarser than $B$ if and only if $B$ is filter basis.

Let $X$ be a non empty set and $\mathcal{F}'$ be a filter of $X$. Observe that there exists a non empty subset of $\mathcal{F}'$ which is filter basis.

A generalized basis of $\mathcal{F}'$ is a filter basis, non empty subset of $\mathcal{F}'$. Now we state the proposition:

(13) Let us consider a non empty set $X$. Then every filter of $X$ is a generalized basis of $\mathcal{F}'$.

Let $X$ be a set and $B$ be a family of subsets of $X$. The functor $[B]$ yielding a family of subsets of $X$ is defined by

(Def. 8) for every subset $x$ of $X$, $x \in it$ iff there exists an element $b$ of $B$ such that $b \subseteq x$.

Now we state the propositions:

(14) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $[S] = \{x$, where $x$ is a subset of $X$ : there exists an element $b$ of $S$ such that $b \subseteq x\}$.

(15) Let us consider a set $X$, and an empty family $B$ of subsets of $X$. Then $[B] = 2^X$.

(16) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $\emptyset \in B$, then $[B] = 2^X$.

## 2. Filters – Lattice-Theoretical Approach

Now we state the propositions:

(17) Let us consider a set $X$, a non empty family $B$ of subsets of $X$, and a subset $L$ of $2^X_\subseteq$. If $B = L$, then $[B] = \uparrow L$.

(18) Let us consider a set $X$, and a family $B$ of subsets of $X$. Then $B \subseteq [B]$.

Let $X$ be a set and $B_1$, $B_2$ be families of subsets of $X$. We say that $B_1$ and $B_2$ are equivalent generators if and only if

(Def. 9) for every element $b_1$ of $B_1$, there exists an element $b_2$ of $B_2$ such that $b_2 \subseteq b_1$ and for every element $b_2$ of $B_2$, there exists an element $b_1$ of $B_1$ such that $b_1 \subseteq b_2$.

Let us note that the predicate is reflexive and symmetric.

Let us consider a set $X$ and families $B_1$, $B_2$ of subsets of $X$.

Let us assume that $B_1$ and $B_2$ are equivalent generators. Now we state the propositions:

(19) $[B_1] \subseteq [B_2]$.

(20) $[B_1] = [B_2]$.

Let $X$ be a non empty set, $\mathcal{F}'$ be a filter of $X$, and $B$ be a non empty subset of $\mathcal{F}'$. The functor $\# B$ yielding a non empty family of subsets of $X$ is defined by the term

(Def. 10) $B$.

Now we state the propositions:

(21) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then $\mathcal{F}' = [\# B]$.

(22) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a family $B$ of subsets of $X$. If $\mathcal{F}' = [B]$, then $B$ is a generalized basis of $\mathcal{F}'$.

(23) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, a generalized basis $B$ of $\mathcal{F}'$, a family $S$ of subsets of $X$, and a subset $S_1$ of $\mathcal{F}'$. Suppose $S = S_1$ and $\# B$ and $S$ are equivalent generators. Then $S_1$ is a generalized basis of $\mathcal{F}'$. The theorem is a consequence of (19), (21), and (22).

(24) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and generalized bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathcal{F}'$. Then $\# \mathcal{B}_1$ and $\# \mathcal{B}_2$ are equivalent generators. The theorem is a consequence of (21).

Let $X$ be a set and $B$ be a family of subsets of $X$. We say that $B$ is quasi basis if and only if

(Def. 11) for every elements $b_1$, $b_2$ of $B$, there exists an element $b$ of $B$ such that $b \subseteq b_1 \cap b_2$.

Let $X$ be a non empty set. Let us note that there exists a non empty family of subsets of $X$ which is quasi basis and there exists a non empty, quasi basis family of subsets of $X$ which has non empty elements.

A filter base of $X$ is a non empty, quasi basis family of subsets of $X$ with non empty elements. Now we state the proposition:

(25) Let us consider a non empty set $X$, and a filter base $B$ of $X$. Then $[B]$ is a filter of $X$.

Let $X$ be a non empty set and $B$ be a filter base of $X$. The functor $[B)$ yielding a filter of $X$ is defined by the term

(Def. 12) $[B]$.

Now we state the propositions:

(26) Let us consider a non empty set $X$, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $X$. Suppose $[\mathcal{B}_1) = [\mathcal{B}_2)$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent generators.

(27) Let us consider a non empty set $X$, a filter base $\mathcal{F}$ of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F} \subseteq \mathcal{F}'$. Then $[\mathcal{F})$ is coarser than $\mathcal{F}'$.

(28) Let us consider a non empty set $X$, and a family $G$ of subsets of $X$. Suppose FinMeetCl($G$) has non empty elements. Then

(i) FinMeetCl($G$) is a filter base of $X$, and

(ii) there exists a filter $\mathcal{F}'$ of $X$ such that FinMeetCl($G$) $\subseteq \mathcal{F}'$.

The theorem is a consequence of (4).

(29) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then every generalized basis of $\mathcal{F}'$ is a filter base of $X$.

(30) Let us consider a non empty set $X$. Then every filter base of $X$ is a generalized basis of $[B)$.

(31) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, a generalized basis $B$ of $\mathcal{F}'$, and a subset $L$ of $2^X_\subseteq$. If $L = \# B$, then $\mathcal{F}' = \uparrow L$. The theorem is a consequence of (21) and (17).

(32) Let us consider a non empty set $X$, a filter base $B$ of $X$, and a subset $L$ of $2^X_\subseteq$. If $L = B$, then $[B) = \uparrow L$.

(33) Let us consider a non empty set $X$, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$, a generalized basis $\mathcal{B}_1$ of $\mathcal{F}_1$, and a generalized basis $\mathcal{B}_2$ of $\mathcal{F}_2$. Then $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. The theorem is a consequence of (21).

(34) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a filter base of $Y$, and

(ii) $[f^\circ(\# B)]$ is a filter of $Y$, and

(iii) $[f^\circ(\# B)] = \{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

PROOF: Set $\mathcal{F} = f^\circ(\# B)$. $\mathcal{F}$ is a quasi basis, non empty family of subsets of $Y$ by (29), [35, (123), (121)]. $\mathcal{F}$ has non empty elements by [35, (118)]. $[\mathcal{F}] = \{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$ by [35, (143)], [12, (42)], (21), [35, (123)]. □

Let $X$, $Y$ be non empty sets, $f$ be a function from $X$ into $Y$, and $\mathcal{F}'$ be a filter of $X$. The image of filter $\mathcal{F}'$ under $f$ yielding a filter of $Y$ is defined by the term

(Def. 13) $\{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

Now we state the propositions:

(35) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and a filter $\mathcal{F}'$ of $X$. Then

(i) $f^\circ\mathcal{F}'$ is a filter base of $Y$, and

(ii) $[f^\circ\mathcal{F}'] =$ the image of filter $\mathcal{F}'$ under $f$.

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set $X$, and a filter base $B$ of $X$. If $B = [B)$, then $B$ is a filter of $X$.

(37) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a generalized basis of the image of filter $\mathcal{F}'$ under $f$, and

(ii) $[f^\circ(\# B)] =$ the image of filter $\mathcal{F}'$ under $f$.

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $X$. Suppose $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. Then $[\mathcal{B}_1)$ is coarser than $[\mathcal{B}_2)$. The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and a filter $\mathcal{F}'$ of $X$. Then $f^\circ\mathcal{F}'$ is a filter of $Y$ if and only if $Y = \operatorname{rng} f$.
PROOF: Reconsider $f_3 = f^\circ\mathcal{F}'$ as a filter base of $Y$. $[f_3) \subseteq f_3$ by [35, (143)], [11, (76), (77)]. □

(40) Let us consider a non empty set $X$, a non empty subset $A$ of $X$, a filter $\mathcal{F}'$ of $A$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $(\overset{A}{\hookrightarrow})^\circ(\# B)$ is a filter base of $X$, and

(ii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)]$ is a filter of $X$, and

(iii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)] = \{M$, where $M$ is a subset of $X : (\overset{A}{\hookrightarrow})^{-1}(M) \in \mathcal{F}'\}$.

Let $L$ be a non empty relational structure. The functor $\operatorname{Tails}(L)$ yielding a non empty family of subsets of $L$ is defined by the term

(Def. 14) the set of all $\uparrow i$ where $i$ is an element of $L$.

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then $[\mathrm{Tails}(L)]$ is a filter of $\Omega_L$.
PROOF: Tails($L$) is non empty family of subsets of $L$ and quasi basis and has non empty elements by [6, (22)]. □

Let $L$ be a non empty, transitive, reflexive relational structure. Assume $\Omega_L$ is directed. The functor TailsFilter $L$ yielding a filter of $\Omega_L$ is defined by the term

(Def. 15) $[\mathrm{Tails}(L)]$.

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then Tails($L$) is a generalized basis of TailsFilter $L$. The theorem is a consequence of (22).

Let $L$ be a relational structure and $x$ be a family of subsets of $L$. The functor $\# x$ yielding a family of subsets of $\Omega_L$ is defined by the term

(Def. 16) $x$.

Now we state the proposition:

(43) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, and a function $f$ from $\Omega_L$ into $X$. Suppose $\Omega_L$ is directed. Then $f^\circ(\#\,\mathrm{Tails}(L))$ is a generalized basis of the image of filter TailsFilter $L$ under $f$. The theorem is a consequence of (42) and (37).

Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a subset $x$ of $X$. Now we state the propositions:

(44) Suppose $\Omega_L$ is directed and $x \in f^\circ(\#\,\mathrm{Tails}(L))$. Then there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geqslant j$ holds $f(i) \in x$.

(45) Suppose $\Omega_L$ is directed and there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geqslant j$ holds $f(i) \in x$. Then there exists an element $b$ of Tails($L$) such that $f^\circ b \subseteq x$.

(46) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Suppose $\Omega_L$ is directed. Then $\mathcal{F}'$ is coarser than the image of filter TailsFilter $L$ under $f$ if and only if $B$ is coarser than $f^\circ(\#\,\mathrm{Tails}(L))$. The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a filter base $B$ of

$X$. Suppose $\Omega_L$ is directed. Then $B$ is coarser than $f^\circ(\#\,\mathrm{Tails}(L))$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (44) and (45).

Let $X$ be a non empty set and $s$ be a sequence of $X$. The elementary filter of $s$ yielding a filter of $X$ is defined by the term

(Def. 17) the image of filter FrechetFilter($\mathbb{N}$) under $s$.

Now we state the propositions:

(48) There exists a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$ such that for every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{y$, where $y$ is an element of $\mathbb{N}$ : there exists an element $x_0$ of $\mathbb{N}$ such that $x_0 = \$_1$ and $x_0 \leqslant y\}$. There exists a function $f$ from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object $x$ such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [12, Sch. 2]. Consider $\mathcal{F}'$ being a function from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object $x$ such that $x \in \mathbb{N}$ holds $\mathcal{F}'(x) = \mathcal{F}(x)$. For every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$. □

(49) Let us consider a natural number $n$. Then $\mathbb{N} \setminus \{t$, where $t$ is an element of $\mathbb{N} : n \leqslant t\}$ is finite.

PROOF: $\mathbb{N} \setminus \{t$, where $t$ is an element of $\mathbb{N} : n \leqslant t\} \subseteq n+1$ by [8, (3), (5)], [32, (4)]. □

(50) Let us consider an element $p$ of the ordered $\mathbb{N}$. Then $\{x$, where $x$ is an element of $\mathbb{N}$ : there exists an element $p_0$ of $\mathbb{N}$ such that $p = p_0$ and $p_0 \leqslant x\} = \uparrow p$.

PROOF: For every element $p$ of the carrier of the ordered $\mathbb{N}$, $\{x$, where $x$ is an element of the carrier of the ordered $\mathbb{N} : p \leqslant x\} = \uparrow p$ by [6, (18)]. □

Observe that $\Omega_{\text{the ordered } \mathbb{N}}$ is directed and the ordered $\mathbb{N}$ is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set $X$. Then FrechetFilter($X$) = the set of all $X \setminus A$ where $A$ is a finite subset of $X$.

Let us consider a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$.

Let us assume that for every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$. Now we state the propositions:

(52) $\mathrm{rng}\,\mathcal{F}'$ is a generalized basis of FrechetFilter($\mathbb{N}$).

PROOF: FrechetFilter($\mathbb{N}$) = the set of all $\mathbb{N} \setminus A$ where $A$ is a finite subset of $\mathbb{N}$. For every object $t$ such that $t \in \mathrm{rng}\,\mathcal{F}'$ holds $t \in$ FrechetFilter($\mathbb{N}$). Reconsider $\mathcal{F}_1 = \mathrm{rng}\,\mathcal{F}'$ as a non empty subset of FrechetFilter($\mathbb{N}$). $\mathcal{F}_1$ is filter basis by [21, (2)], [4, (44)], [11, (3)]. □

(53) $\#$ Tails(the ordered $\mathbb{N}$) = rng $\mathcal{F}'$. The theorem is a consequence of (50).

Now we state the proposition:

(54) (i) $\#$ Tails(the ordered $\mathbb{N}$) is a generalized basis of FrechetFilter($\mathbb{N}$), and

(ii) TailsFilter the ordered $\mathbb{N}$ = FrechetFilter($\mathbb{N}$).

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of $\mathbb{N}$ is defined by the term

(Def. 18) $\#$ Tails(the ordered $\mathbb{N}$).

Now we state the propositions:

(55) $\mathbb{N} \in$ the base of Frechet filter.

(56) The base of Frechet filter is a generalized basis of FrechetFilter($\mathbb{N}$).

(57) Let us consider a non empty set $X$, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}'$ is finer than $\mathcal{F}_1$ and $\mathcal{F}'$ is finer than $\mathcal{F}_2$. Let us consider an element $M_1$ of $\mathcal{F}_1$, and an element $M_2$ of $\mathcal{F}_2$. Then $M_1 \cap M_2$ is not empty.

(58) Let us consider a non empty set $X$, and filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$. Suppose for every element $M_1$ of $\mathcal{F}_1$ for every element $M_2$ of $\mathcal{F}_2$, $M_1 \cap M_2$ is not empty. Then there exists a filter $\mathcal{F}'$ of $X$ such that

(i) $\mathcal{F}'$ is finer than $\mathcal{F}_1$, and

(ii) $\mathcal{F}'$ is finer than $\mathcal{F}_2$.

Let $X$ be a set and $x$ be a subset of $X$. The functor SubsetToBooleSubset $x$ yielding an element of $2^X_\subseteq$ is defined by the term

(Def. 19) $x$.

Now we state the propositions:

(59) Let us consider an infinite set $X$. Then $X \in$ the set of all $X \setminus A$ where $A$ is a finite subset of $X$.

(60) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\} = \{B$, where $B$ is a subset of $X : A \subseteq B\}$.

(61) Let us consider a set $X$, and an element $a$ of $2^X_\subseteq$. Then $\uparrow a = \{Y$, where $Y$ is a subset of $X : a \subseteq Y\}$.

(62) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\} = \uparrow$SubsetToBooleSubset $A$. The theorem is a consequence of (60).

(63) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\bigcup \mathcal{F}' = X$.

(64) Let us consider an infinite set $X$. Then the set of all $X \setminus A$ where $A$ is a finite subset of $X$ is a filter of $X$. The theorem is a consequence of (59).

Let us consider a set $X$. Now we state the propositions:

(65) $2^X$ is a filter of $2^X_\subseteq$.

(66) $\{X\}$ is a filter of $2^X_\subseteq$.

(67) Let us consider a non empty set $X$. Then $\{X\}$ is a filter of $X$.

Let us consider an element $A$ of $2^X_\subseteq$. Now we state the propositions:

(68) $\{Y$, where $Y$ is a subset of $X : A \subseteq Y\}$ is a filter of $2^X_\subseteq$.

(69) $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (60) and (68).

Now we state the proposition:

(70) Let us consider a non empty set $X$, and a non empty subset $B$ of $2^X_\subseteq$. Then for every elements $x$, $y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ if and only if $B$ is filtered.

PROOF: For every elements $x$, $y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ by [19, (2)]. □

Let us consider a non empty set $X$ and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Now we state the propositions:

(71) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every elements $p$, $q$ of $\mathcal{F}'$, $p \cap q \in \mathcal{F}'$ and for every element $p$ of $\mathcal{F}'$ and for every element $q$ of the lattice of subsets of $X$ such that $p \subseteq q$ holds $q \in \mathcal{F}'$.

(72) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every subsets $Y_1$, $Y_2$ of $X$, if $Y_1, Y_2 \in \mathcal{F}'$, then $Y_1 \cap Y_2 \in \mathcal{F}'$ and if $Y_1 \in \mathcal{F}'$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \mathcal{F}'$. The theorem is a consequence of (71).

Now we state the propositions:

(73) Let us consider a non empty set $X$, and a non empty family $\mathcal{F}$ of subsets of $X$. Suppose $\mathcal{F}$ is a filter of the lattice of subsets of $X$. Then $\mathcal{F}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (71).

(74) Let us consider a non empty set $X$. Then every filter of $2^X_\subseteq$ is a filter of the lattice of subsets of $X$. The theorem is a consequence of (72).

(75) Let us consider a non empty set $X$, and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Then $\mathcal{F}'$ is filter of the lattice of subsets of $X$ and has non empty elements if and only if $\mathcal{F}'$ is a filter of $X$. The theorem is a consequence of (72).

(76) Let us consider a non empty set $X$. Then every proper filter of $2^X_\subseteq$ is a filter of $X$.

PROOF: $\mathcal{F}'$ has non empty elements by [19, (18)], [7, (4)]. □

(77) Let us consider a non empty topological space $T$, and a point $x$ of $T$. Then the neighborhood system of $x$ is a filter of the carrier of $T$.

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a proper filter of $2^{\Omega_T}_{\subseteq}$. The functor BooleanFilterToFilter($\mathcal{F}'$) yielding a filter of the carrier of $T$ is defined by the term

(Def. 20) $\mathcal{F}'$.

Let $\mathcal{F}_1$ be a filter of the carrier of $T$ and $\mathcal{F}_2$ be a proper filter of $2^{\Omega_T}_{\subseteq}$. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 21) BooleanFilterToFilter($\mathcal{F}_2$) $\subseteq \mathcal{F}_1$.

## 3. Limit of a Filter

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor LimFilter($\mathcal{F}'$) yielding a subset of $T$ is defined by the term

(Def. 22) $\{x$, where $x$ is a point of $T$ : $\mathcal{F}'$ is finer than the neighborhood system of $x\}$.

Let $B$ be a filter base of the carrier of $T$. The functor Lim $B$ yielding a subset of $T$ is defined by the term

(Def. 23) LimFilter($[B)$).

Now we state the proposition:

(78) Let us consider a non empty topological space $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Then there exists a proper filter $\mathcal{F}_1$ of $2^{\alpha}_{\subseteq}$ such that $\mathcal{F}' = \mathcal{F}_1$, where $\alpha$ is the carrier of $T$. The theorem is a consequence of (73) and (75).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor FilterToBooleanFilter($\mathcal{F}', T$) yielding a proper filter of $2^{\Omega_T}_{\subseteq}$ is defined by the term

(Def. 24) $\mathcal{F}'$.

Let us consider a non empty topological space $T$, a point $x$ of $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:

(79) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x$ is a convergence point of FilterToBooleanFilter($\mathcal{F}', T$) and $T$.

(80) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x \in$ LimFilter($\mathcal{F}'$). The theorem is a consequence of (78).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of $2^{\Omega_T}_{\subseteq}$. The functor LimFilterB($\mathcal{F}'$) yielding a subset of $T$ is defined by the term

(Def. 25) $\{x$, where $x$ is a point of $T$ : the neighborhood system of $x \subseteq \mathcal{F}'\}$.

Let us consider a non empty topological space $T$ and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:

(81) LimFilter($\mathcal{F}'$) = LimFilterB(FilterToBooleanFilter($\mathcal{F}', T$)).

(82) Lim(the net of FilterToBooleanFilter($\mathcal{F}', T$)) = LimFilter($\mathcal{F}'$).

(83) Let us consider a Hausdorff, non empty topological space $T$, a filter $\mathcal{F}'$ of the carrier of $T$, and points $p$, $q$ of $T$. If $p$, $q \in$ LimFilter($\mathcal{F}'$), then $p = q$.

Let $T$ be a Hausdorff, non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. Note that LimFilter($\mathcal{F}'$) is trivial.

Let $X$ be a non empty set, $T$ be a non empty topological space, $f$ be a function from $X$ into the carrier of $T$, and $\mathcal{F}'$ be a filter of $X$. The functor $\lim_{\mathcal{F}'} f$ yielding a subset of $\Omega_T$ is defined by the term

(Def. 26) LimFilter(the image of filter $\mathcal{F}'$ under $f$).

Let $L$ be a non empty, transitive, reflexive relational structure and $f$ be a function from $\Omega_L$ into the carrier of $T$. The functor LimF($f$) yielding a subset of $\Omega_T$ is defined by the term

(Def. 27) LimFilter(the image of filter TailsFilter $L$ under $f$).

Now we state the proposition:

(84) Let us consider a non empty topological space $T$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Suppose $\Omega_L$ is directed. Then $x \in$ LimF($f$) if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (46), (29), and (47).

Let $T$ be a non empty topological space and $s$ be a sequence of $T$. The functor LimF($s$) yielding a subset of $T$ is defined by the term

(Def. 28) LimFilter(the elementary filter of $s$).

Now we state the proposition:

(85) Let us consider a non empty topological space $T$, and a sequence $s$ of $T$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} s$ = LimF($s$).

Let us consider a non empty topological space $T$ and a point $x$ of $T$.

(86) The neighborhood system of $x$ is a filter base of $\Omega_T$. The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of BooleanFilterToFilter(the neighborhood system of $x$) is a filter base of $\Omega_T$.

(88) Let us consider a non empty set $X$, a sequence $s$ of $X$, and a filter base $B$ of $X$. Then $B$ is coarser than $s^\circ$(the base of Frechet filter) if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$.

(89) Let us consider a non empty topological space $T$, a sequence $s$ of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N})} s$ if and only if $B$ is coarser than $s^{\circ}$(the base of Frechet filter). The theorem is a consequence of (46) and (54).

(90) Let us consider a non empty topological space $T$, a sequence $s$ of $\Omega_T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $B$ is coarser than $s^{\circ}$(the base of Frechet filter) if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$).

(91) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N})} s$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (89) and (90).

(92) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

## 4. NETS

Let $L$ be a 1-sorted structure and $s$ be a sequence of the carrier of $L$. The net of $s$ yielding a non empty, strict net structure over $L$ is defined by the term

(Def. 29) $\langle \mathbb{N}, \leqslant_{\mathbb{N}}, s \rangle$.

Let $L$ be a non empty 1-sorted structure. Let us note that the net of $s$ is non empty.

Now we state the proposition:

(93) Let us consider a non empty 1-sorted structure $L$, a set $B$, and a sequence $s$ of the carrier of $L$. Then the net of $s$ is eventually in $B$ if and only if there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leqslant j$ holds (the net of $s$)$(j) \in B$.

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Now we state the propositions:

(94) for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$ if

and only if for every element $b$ of $B$, there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leqslant j$ holds (the net of $s$)$(j) \in b$.

(95) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, the net of $s$ is eventually in $b$. The theorem is a consequence of (92), (94), and (93).

(96) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $\mathbb{N}$ such that for every element $j$ of $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

(97) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (96).

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# Polynomially Bounded Sequences and Polynomial Sequences

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**Summary.** In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

## 1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers $m$, $k$. If $1 \leqslant m$, then $1 \leqslant m^k$.

Let us consider natural numbers $m$, $n$. Now we state the propositions:

(2) $m \leqslant m^{n+1}$.

(3) If $2 \leqslant m$, then $n+1 \leqslant m^n$.




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(4) Let us consider a natural number $k$. Then $2 \cdot k \leqslant 2^k$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv 2 \cdot \$_1 \leqslant 2^{\$_1}$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (25)], [24, (5)], [1, (14)], (2). For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(5) Let us consider natural numbers $k$, $n$. If $k \leqslant n$, then $n + k \leqslant 2^n$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \$_1 + k + k \leqslant 2^{\$_1 + k}$. $2 \cdot k \leqslant 2^k$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (27), (25), (24)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(6) Let us consider natural numbers $k$, $m$. If $2 \cdot k + 1 \leqslant m$, then $2^k \leqslant 2^m/_m$. The theorem is a consequence of (5).

(7) Let us consider real numbers $a$, $b$, $c$. If $1 < a$ and $0 < b \leqslant c$, then $\log_a b \leqslant \log_a c$.

Let us consider a natural number $n$ and a real number $a$. Now we state the propositions:

(8) If $1 < a$, then $a^n < a^{n+1}$.

(9) If $1 \leqslant a$, then $a^n \leqslant a^{n+1}$.

(10) There exists a partial function $g$ from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $\operatorname{dom} g = ]0, +\infty[$, and

(ii) for every real number $x$ such that $x \in ]0, +\infty[$ holds $g(x) = \log_2 x$, and

(iii) $g$ is differentiable on $]0, +\infty[$, and

(iv) for every real number $x$ such that $x \in ]0, +\infty[$ holds $g$ is differentiable in $x$ and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$.

PROOF: Set $g = \log_2 e \cdot$ (the function ln). For every real number $d$ such that $d \in ]0, +\infty[$ holds $g(d) = \log_2 d$ by [20, (56)]. For every real number $x$ such that $x \in ]0, +\infty[$ holds $g$ is differentiable in $x$ and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$ by [23, (18)], [22, (15)], [20, (57)], [23, (11)]. □

(11) There exists a partial function $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $]e, +\infty[ = \operatorname{dom} f$, and

(ii) for every real number $x$ such that $x \in \operatorname{dom} f$ holds $f(x) = x/_{\log_2 x}$, and

(iii) $f$ is differentiable on $]e, +\infty[$, and

(iv) for every real number $x_0$ such that $x_0 \in ]e, +\infty[$ holds $0 \leqslant f'(x_0)$, and

(v) $f$ is non-decreasing.

PROOF: Consider $g$ being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} g = ]0, +\infty[$ and for every real number $x$ such that $x \in ]0, +\infty[$ holds $g(x) = \log_2 x$ and $g$ is differentiable on $]0, +\infty[$ and for every real number $x$ such that $x \in ]0, +\infty[$ holds $g$ is differentiable in $x$ and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$. Set $g_0 = g\restriction]e, +\infty[$. For every object $x$ such that $x \in ]e, +\infty[$ holds $x \in ]0, +\infty[$ by [23, (11)]. Set $f = \mathrm{id}_{\Omega_{\mathbb{R}}}/_{g_0}$. ${g_0}^{-1}(\{0\}) = \emptyset$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number $x$ such that $x \in \operatorname{dom} f$ holds $f(x) = x/_{\log_2 x}$ by [7, (49)]. For every real number $x$ such that $x \in ]e, +\infty[$ holds $f$ is differentiable in $x$ and $f'(x) = \log_2 x - \log_2 e/_{(\log_2 x)^2}$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number $x$ such that $x \in ]e, +\infty[$ holds $0 \leqslant f'(x)$ by [20, (57)], [23, (11)]. □

(12) Let us consider real numbers $x$, $y$. If $e < x \leqslant y$, then $x/_{\log_2 x} \leqslant y/_{\log_2 y}$. The theorem is a consequence of (11).

(13) Let us consider a natural number $k$. Suppose $e < k$. Then there exists a natural number $N$ such that for every natural number $n$ such that $N \leqslant n$ holds $2^k \leqslant n/_{\log_2 n}$. The theorem is a consequence of (12) and (6).

Let us consider a natural number $x$. Let us assume that $1 < x$.

(14) There exists a natural number $N$ such that for every natural number $n$ such that $N \leqslant n$ holds $4 < n/_{\log_x n}$.

(15) There exist natural numbers $N$, $c$ such that for every natural number $n$ such that $N \leqslant n$ holds $n^x \leqslant c \cdot x^n$.

(16) Let us consider a natural number $x$. Suppose $1 < x$. Then there exist no natural numbers $N$, $c$ such that for every natural number $n$ such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$.

PROOF: Consider $N$ being a natural number such that there exists a natural number $c$ such that for every natural number $n$ such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$. $N \neq 0$ by [20, (42), (24)]. Consider $c$ being a natural number such that for every natural number $n$ such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$. There exists an element $n$ of $\mathbb{N}$ such that $N \leqslant n$ and $0 < n - (x/_4)$ by [24, (6), (3)]. Consider $n$ being an element of $\mathbb{N}$ such that $N \leqslant n$ and $0 < n - (x/_4)$. $0 < c$ by [20, (34)]. For every natural number $k$ such that $1 \leqslant k$ holds $2^{k \cdot n} \leqslant c \cdot (k \cdot n)^x$. For every natural number $k$ such that $1 \leqslant k$ holds $k \cdot n \leqslant \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n$ by [20, (34)], (7), [20, (55), (52), (53)]. Consider $Z$ being an element of $\mathbb{N}$ such that for every natural number $k$ such that $Z \leqslant k$ holds $4 < k/_{\log_2 k}$. There exists a natural number $k$ such that $Z \leqslant k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$ by [24, (6), (3)]. There exists a natural number $k$ such that $Z \leqslant k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$ and $1 < k$ by [1, (11)]. Consider $k$ being a natural number such that $Z \leqslant k$ and $1 < k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$. □

(17) Let us consider natural numbers $a$, $b$. If $a \leqslant b$, then $\{n^a\}_{n\in\mathbb{N}} \in O(\{n^b\}_{n\in\mathbb{N}})$.

(18) Let us consider a natural number $x$. Suppose $1 < x$. Then there exist no natural numbers $N$, $c$ such that for every natural number $n$ such that $N \leqslant n$ holds $x^n \leqslant c \cdot n^x$.

PROOF: There exist natural numbers $N$, $c$ such that for every natural number $n$ such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$ by [24, (7)]. □

(19) Let us consider a non negative real number $a$, and a natural number $n$. If $1 \leqslant n$, then $0 < \{n^a\}_{n\in\mathbb{N}}(n)$.

## 2. Polynomially Bounded Sequences

Let $p$ be a sequence of real numbers. We say that $p$ is polynomially bounded if and only if

(Def. 1) there exists a natural number $k$ such that $p \in O(\{n^k\}_{n\in\mathbb{N}})$.

Now we state the propositions:

(20) Let us consider a sequence $f$ of real numbers. Suppose $f$ is not polynomially bounded. Let us consider a natural number $k$. Then $f \notin O(\{n^k\}_{n\in\mathbb{N}})$.

(21) Let us consider a sequence $f$ of real numbers. Suppose for every natural number $k$, $f \notin O(\{n^k\}_{n\in\mathbb{N}})$. Then $f$ is not polynomially bounded.

(22) Let us consider a positive real number $a$. Then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is positive.

Let us consider a real number $a$. Now we state the propositions:

(23) If $1 \leqslant a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).

(24) If $1 < a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is increasing. The theorem is a consequence of (8).

(25) Let us consider a natural number $a$. If $1 < a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is not polynomially bounded.

PROOF: Consider $k$ being a natural number such that $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}} \in O(\{n^k\}_{n\in\mathbb{N}})$. Reconsider $f = \{n^k\}_{n\in\mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ as an eventually non-negative sequence of real numbers. $t \in O(f)$ and for every element $n$ of $\mathbb{N}$ such that $1 \leqslant n$ holds $0 < f(n)$. Consider $c$ being a real number such that $c > 0$ and for every element $n$ of $\mathbb{N}$ such that $n \geqslant 1$ holds $(\{a^{1\cdot n+0}\}_{n\in\mathbb{N}})(n) \leqslant c \cdot \{n^k\}_{n\in\mathbb{N}}(n)$. For every natural number $n$ such that $n \geqslant 1$ holds $2^n \leqslant c \cdot n^k$ by [24, (7)]. There exist natural numbers $N$, $b$ such that for every natural number $n$ such that $N \leqslant n$ holds $2^n \leqslant b \cdot n^k$ by [24, (3)]. □

## 3. Polynomial Sequences

Now we state the proposition:

(26) Let us consider a finite 0-sequence $x$ of $\mathbb{R}$, and a sequence $y$ of real numbers. Then

(i) $x \cdot y$ is a finite transfinite sequence of elements of $\mathbb{R}$, and

(ii) $\operatorname{dom}(x \cdot y) = \operatorname{dom} x$, and

(iii) for every object $i$ such that $i \in \operatorname{dom} x$ holds $(x \cdot y)(i) = x(i) \cdot y(i)$.

Let $x$ be a finite 0-sequence of $\mathbb{R}$ and $y$ be a sequence of real numbers. Observe that the functor $x \cdot y$ yields a finite 0-sequence of $\mathbb{R}$. Now we state the proposition:

(27) Let us consider a finite 0-sequence $d$ of $\mathbb{R}$, and natural numbers $x$, $i$. Suppose $i \in \operatorname{dom} d$. Then $(d \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i$. The theorem is a consequence of (26).

Let $c$ be a finite 0-sequence of $\mathbb{R}$. The functor $\mathrm{Seq}_{\mathrm{poly}}(c)$ yielding a sequence of real numbers is defined by

(Def. 2) for every natural number $x$, $it(x) = \sum(c \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})$.

Let us consider a finite 0-sequence $d$ of $\mathbb{R}$ and a natural number $k$. Now we state the propositions:

(28) Suppose $\operatorname{len} d = k+1$. Then there exists a real number $a$ and there exists a finite 0-sequence $d_1$ of $\mathbb{R}$ and there exists a sequence $y$ of real numbers such that $\operatorname{len} d_1 = k$ and $d_1 = d \restriction k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$ and $\mathrm{Seq}_{\mathrm{poly}}(d) = \mathrm{Seq}_{\mathrm{poly}}(d_1) + y$ and for every natural number $i$, $y(i) = a \cdot i^k$.
Proof: Consider $a$ being a real number, $d_1$ being a finite 0-sequence of $\mathbb{R}$ such that $\operatorname{len} d_1 = k$ and $d_1 = d \restriction k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$. Define $\mathcal{F}(\text{natural number}) = a \cdot \$_1^k$. Consider $y$ being a sequence of real numbers such that for every natural number $x$, $y(x) = \mathcal{F}(x)$ from [15, Sch. 1]. For every element $x$ of $\mathbb{N}$, $(\mathrm{Seq}_{\mathrm{poly}}(d))(x) = (\mathrm{Seq}_{\mathrm{poly}}(d_1) + y)(x)$ by (26), [1, (13), (44)], (27). $\square$

(29) If $\operatorname{len} d = 1$, then there exists a real number $a$ such that $a = d(0)$ and for every natural number $x$, $(\mathrm{Seq}_{\mathrm{poly}}(d))(x) = a$. The theorem is a consequence of (26).

(30) If $\operatorname{len} d = 1$ and $d$ is non-negative yielding, then $\mathrm{Seq}_{\mathrm{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. The theorem is a consequence of (29).

(31) Let us consider a natural number $k$, a real number $a$, and a sequence $y$ of real numbers. Suppose $0 \leqslant a$ and for every natural number $i$, $y(i) = a \cdot i^k$. Then $y \in O(\{n^k\}_{n \in \mathbb{N}})$.

(32) Let us consider natural numbers $k$, $n$. If $k \leqslant n$, then $O(\{n^k\}_{n\in\mathbb{N}}) \subseteq O(\{n^n\}_{n\in\mathbb{N}})$.
PROOF: Consider $i$ being a natural number such that $n = k + i$. Define $\mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n\in\mathbb{N}}) \subseteq O(\{n^{(k+\$_1)}\}_{n\in\mathbb{N}})$. For every natural number $x$ such that $\mathcal{P}[x]$ holds $\mathcal{P}[x+1]$. For every natural number $x$, $\mathcal{P}[x]$ from [1, Sch. 2]. □

(33) Let us consider a natural number $k$, and a non-negative yielding finite 0-sequence $c$ of $\mathbb{R}$. Suppose $\operatorname{len} c = k + 1$. Then $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^k\}_{n\in\mathbb{N}})$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non-negative yielding finite 0-sequence $c$ of $\mathbb{R}$ such that $\operatorname{len} c = \$_1 + 1$ holds $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^{\$_1}\}_{n\in\mathbb{N}})$. $\mathcal{P}[0]$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [1, (13), (39)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(34) Let us consider a natural number $k$, and a finite 0-sequence $c$ of $\mathbb{R}$. Then there exists a finite 0-sequence $d$ of $\mathbb{R}$ such that

(i) $\operatorname{len} d = \operatorname{len} c$, and

(ii) for every natural number $i$ such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$.

PROOF: Define $\mathcal{F}(\text{natural number}) = |c(\$_1)|(\in \mathbb{R})$. Consider $d$ being a finite 0-sequence of $\mathbb{R}$ such that $\operatorname{len} d = \operatorname{len} c$ and for every natural number $j$ such that $j \in \operatorname{len} c$ holds $d(j) = \mathcal{F}(j)$ from [18, Sch. 1]. □

(35) Let us consider a finite 0-sequence $c$ of $\mathbb{R}$, and a finite 0-sequence $d$ of $\mathbb{R}$. Suppose $\operatorname{len} d = \operatorname{len} c$ and for every natural number $i$ such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$. Let us consider a natural number $n$. Then $(\mathrm{Seq}_{\mathrm{poly}}(c))(n) \leqslant (\mathrm{Seq}_{\mathrm{poly}}(d))(n)$.
PROOF: $\operatorname{dom}(d \cdot \{x^{1\cdot n+0}\}_{n\in\mathbb{N}}) = \operatorname{dom} d$. For every natural number $i$ such that $i \in \operatorname{dom}(c\cdot\{x^{1\cdot n+0}\}_{n\in\mathbb{N}})$ holds $(c\cdot\{x^{1\cdot n+0}\}_{n\in\mathbb{N}})(i) \leqslant (d\cdot\{x^{1\cdot n+0}\}_{n\in\mathbb{N}})(i)$ by (26), (27), [19, (4)]. □

(36) Let us consider a natural number $k$, and a finite 0-sequence $c$ of $\mathbb{R}$. Suppose $\operatorname{len} c = k + 1$ and $\mathrm{Seq}_{\mathrm{poly}}(c)$ is eventually nonnegative. Then $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^k\}_{n\in\mathbb{N}})$.
PROOF: Consider $d$ being a finite 0-sequence of $\mathbb{R}$ such that $\operatorname{len} d = \operatorname{len} c$ and for every natural number $i$ such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$. For every natural number $i$ such that $i \in \operatorname{dom} d$ holds $0 \leqslant d(i)$ by [6, (46)]. For every real number $r$ such that $r \in \operatorname{rng} d$ holds $0 \leqslant r$. $\mathrm{Seq}_{\mathrm{poly}}(d) \in O(\{n^k\}_{n\in\mathbb{N}})$. Consider $t$ being an element of $\mathbb{R}^{\mathbb{N}}$ such that $\mathrm{Seq}_{\mathrm{poly}}(d) = t$ and there exists a real number $c$ and there exists an element $N$ of $\mathbb{N}$ such that $c > 0$ and for every element $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds $t(n) \leqslant c\cdot\{n^k\}_{n\in\mathbb{N}}(n)$ and $t(n) \geqslant 0$. Consider $N_1$ being a natural number such that for every natural number $n$ such that $N_1 \leqslant n$ holds $0 \leqslant (\mathrm{Seq}_{\mathrm{poly}}(c))(n)$.

Consider $a$ being a real number, $N_2$ being an element of $\mathbb{N}$ such that $a > 0$ and for every element $n$ of $\mathbb{N}$ such that $n \geqslant N_2$ holds $t(n) \leqslant a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geqslant 0$. Set $N = N_1 + N_2$. For every element $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds $(\mathrm{Seq}_{\mathrm{poly}}(c))(n) \leqslant a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\mathrm{Seq}_{\mathrm{poly}}(c))(n) \geqslant 0$ by [1, (11)], (35). □

(37) Let us consider natural numbers $k$, $n$. If $0 < n$, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.

(38) Let us consider a finite 0-sequence $c$ of $\mathbb{R}$. Suppose $\mathrm{len}\, c = 0$. Let us consider a natural number $x$. Then $(\mathrm{Seq}_{\mathrm{poly}}(c))(x) = 0$.

(39) Let us consider an eventually nonnegative sequence $f$ of real numbers, and a natural number $k$. Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number $N$ such that for every natural number $n$ such that $N \leqslant n$ holds $f(n) \leqslant \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).

(40) Let us consider a finite 0-sequence $c$ of $\mathbb{R}$. Then there exists a finite 0-sequence $a_1$ of $\mathbb{R}$ such that

(i) $a_1 = |c|$, and

(ii) for every natural number $n$, $(\mathrm{Seq}_{\mathrm{poly}}(c))(n) \leqslant (\mathrm{Seq}_{\mathrm{poly}}(a_1))(n)$.

PROOF: Reconsider $a_1 = |c|$ as a finite 0-sequence of $\mathbb{R}$. Set $m_1 = c \cdot \{n^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$. For every natural number $x$ such that $x \in \mathrm{dom}\, m_1$ holds $m_1(x) \leqslant m_2(x)$ by [19, (4)]. □

(41) Let us consider finite 0-sequences $c$, $a_1$ of $\mathbb{R}$. Suppose $a_1 = |c|$. Let us consider a natural number $n$. Then $|(\mathrm{Seq}_{\mathrm{poly}}(c))(n)| \leqslant (\mathrm{Seq}_{\mathrm{poly}}(a_1))(n)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite 0-sequences $c$, $a_1$ of $\mathbb{R}$ such that $\mathrm{len}\, c = \$_1$ and $a_1 = |c|$ for every natural number $x$, $|(\mathrm{Seq}_{\mathrm{poly}}(c))(x)| \leqslant (\mathrm{Seq}_{\mathrm{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), [6, (44)]. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [15, (7)], [6, (56), (65)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(42) Let us consider a real number $a$. Suppose $0 < a$. Let us consider a natural number $k$, and a non-negative yielding finite 0-sequence $d$ of $\mathbb{R}$. Suppose $\mathrm{len}\, d = k$. Then there exists a natural number $N$ such that for every natural number $x$ such that $N \leqslant x$ for every natural number $i$ such that $i \in \mathrm{dom}\, d$ holds $d(i) \cdot x^i \cdot k \leqslant a \cdot x^k$.
PROOF: For every natural number $i$ such that $i \in \mathrm{dom}\, d$ holds $0 \leqslant d(i)$ by [7, (3)]. □

(43) Let us consider a natural number $k$, a finite 0-sequence $d$ of $\mathbb{R}$, a real number $a$, and a sequence $y$ of real numbers. Suppose $0 < a$ and $\mathrm{len}\, d = k$ and for every natural number $x$, $y(x) = a \cdot x^k$. Then there exists a natural number $N$ such that for every natural number $x$ such that $N \leqslant x$ holds

$|(\mathrm{Seq_{poly}}(d))(x)| \leqslant y(x)$. The theorem is a consequence of (38), (42), (26), (27), and (41).

(44) Let us consider a natural number $k$, and a finite 0-sequence $d$ of $\mathbb{R}$. Suppose $\mathrm{len}\, d = k+1$ and $0 < d(k)$. Then $\mathrm{Seq_{poly}}(d)$ is eventually nonnegative.

PROOF: Consider $a$ being a real number, $d_1$ being a finite 0-sequence of $\mathbb{R}$, $y$ being a sequence of real numbers such that $\mathrm{len}\, d_1 = k$ and $d_1 = d{\restriction}k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$ and $\mathrm{Seq_{poly}}(d) = \mathrm{Seq_{poly}}(d_1) + y$ and for every natural number $i$, $y(i) = a \cdot i^k$. Consider $N$ being a natural number such that for every natural number $i$ such that $N \leqslant i$ holds $|(\mathrm{Seq_{poly}}(d_1))(i)| \leqslant y(i)$. For every natural number $i$ such that $N \leqslant i$ holds $0 \leqslant (\mathrm{Seq_{poly}}(d))(i)$ by [19, (4)], [15, (7)]. $\square$

Let us consider a natural number $k$ and a finite 0-sequence $c$ of $\mathbb{R}$.

Let us assume that $\mathrm{len}\, c = k+1$ and $0 < c(k)$. Now we state the propositions:

(45) $\mathrm{Seq_{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

(46) $\mathrm{Seq_{poly}}(c)$ is polynomially bounded. The theorem is a consequence of (36) and (44).

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# Fermat's Little Theorem via Divisibility of Newton's Binomial

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**Summary.** Solving equations in integers is an important part of the number theory [29]. In many cases it can be conducted by the factorization of equation's elements, such as the Newton's binomial. The article introduces several simple formulas, which may facilitate this process. Some of them are taken from relevant books [28], [14].

In the second section of the article, Fermat's Little Theorem is proved in a classical way, on the basis of divisibility of Newton's binomial. Although slightly redundant in its content (another proof of the theorem has earlier been included in [12]), the article provides a good example, how the application of registrations could shorten the length of Mizar proofs [9], [17].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [1], [5], [4], [10], [6], [30], [22], [21], [3], [15], [32], [19], [7], [23], and [8].

## 1. Divisibility of Newton's Binomial

From now on $a$, $b$, $c$, $d$, $m$, $x$, $n$, $j$, $k$, $l$ denote natural numbers, $t$, $u$, $v$, $z$ denote integers, $f$, $F$ denote finite sequences of elements of $\mathbb{N}$, $p$, $q$, $r$, $s$ denote real numbers.

Let $a$ be a complex. Note that $1 \cdot a^0$ reduces to 1.

Let $n$ be a non zero natural number. One can check that $0^n$ reduces to 0.

Let $a$ be a natural number. Let us observe that $|a|$ reduces to $a$.

Let us note that $\gcd(a, 0)$ reduces to $a$.

Let us consider $t$ and $z$. Let us note that $(t \bmod z) \bmod z$ reduces to $t \bmod z$.

Observe that $0 \bmod t$ reduces to 0.

Let us consider $u$ and $z$. One can check that $0 + u \cdot z \bmod z$ reduces to 0.

Let $r$ be a non zero real number and $n$ be an even, natural number. One can verify that $r^n$ is positive.

Now we state the propositions:

(1) $\gcd(t, z) = \gcd(-t, z)$.

(2) If $t \mid z$ and $u \mid v$, then $t \cdot u \mid z \cdot v$.

(3) $t \mid z$ if and only if $\gcd(t, z) = |t|$.

(4) $t \cdot u \mid z \cdot u$ if and only if $|u| \cdot (\gcd(t, z)) = |u| \cdot |t|$. The theorem is a consequence of (3).

(5) (i) $\gcd(t + u \cdot z, z) = \gcd(t, z)$, and

(ii) $\gcd(t - u \cdot z, z) = \gcd(t, z)$.

(6) If $n > 0$, then $t \mid t^n$.

(7) $\gcd(a^n, b^n) = (\gcd(a, b))^n$.

Proof: If $\gcd(a, b) = k$, then $\gcd(a^n, b^n) = k^n$ by [22, (21)], [16, (12)], [11, (15)], [21, (7), (11), (4)]. □

(8) If $a > b$ and $a$ and $b$ are relatively prime, then $\gcd(a + b, a - b) \leqslant 2$. The theorem is a consequence of (5).

(9) $\gcd(t, z)$ is even if and only if $t$ is even and $z$ is even.

Proof: If $\gcd(t, z)$ is even, then $t$ is even and $z$ is even by [22, (21)]. □

(10) (i) $t \mid (t + z)^n - z^n$, and

(ii) $z \mid (t + z)^n - t^n$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv t \mid (t + z)^{\$_1} - z^{\$_1}$ and $z \mid (t + z)^{\$_1} - t^{\$_1}$. $\mathcal{P}[0]$ by [21, (4)], [30, (11)]. If $\mathcal{P}[x]$, then $\mathcal{P}[x + 1]$ by [16, (4)], [21, (8)]. For every $m$, $\mathcal{P}[m]$ from [2, Sch. 2]. □

(11) $u \mid (u + z)^n$ if and only if $u \mid z^n$. The theorem is a consequence of (10).

(12) If $t \mid (t + z)^n$, then $t \mid (t + z)^n + z^n$. The theorem is a consequence of (11).

(13) $t + u \mid (t + 2 \cdot u)^n - u^n$. The theorem is a consequence of (10).

(14) If $l > 0$ and $t \mid z$, then $t \mid z^l$.

(15) If $t \mid z$, then $t^n \mid z^n$. The theorem is a consequence of (7) and (3).

(16) If $n > 0$ and $t \nmid (t + z)^n$, then $t \nmid z$. The theorem is a consequence of (14).

(17) If $m > 0$, then $t \cdot z \mid (t+z)^m - (t^m + z^m)$.
PROOF: Consider $n$ such that $m = 1 + n$. $t \cdot z \mid (t+z)^{n+1} - (t^{n+1} + z^{n+1})$ by [22, (12), (2)], [21, (6)], [22, (1)]. □

(18) $t - z \mid t^m - z^m$. The theorem is a consequence of (11) and (10).

(19) If $n > 0$, then $t \cdot z \mid (t-z)^n - (t^n + (-z)^n)$. The theorem is a consequence of (17).

(20) $t \cdot z \mid (t+z)^n - (t-z)^n + ((-z)^n - z^n)$. The theorem is a consequence of (17) and (19).

(21) If $n > 0$, then $t \mid (t+z)^n + (t^n - z^n)$. The theorem is a consequence of (6) and (10).

(22) If $u \mid t + z$ and $u \mid t - z$, then $u \mid 2 \cdot t$ and $u \mid 2 \cdot z$.

(23) $t \cdot z \mid (t+z)^{2 \cdot n} - (t-z)^{2 \cdot n}$. The theorem is a consequence of (20).

(24) If $n > 0$, then $t \cdot z \mid (t-z)^{2 \cdot n} - (t^{2 \cdot n} + z^{2 \cdot n})$. The theorem is a consequence of (19).

(25) $t \cdot z \mid (t-z)^{2 \cdot n+1} - (t^{2 \cdot n+1} - z^{2 \cdot n+1})$. The theorem is a consequence of (19).

(26) If $k > 0$ and $x \mid a + k$ and $x \mid a - k$, then $x \leqslant 2 \cdot k$. The theorem is a consequence of (22).

(27) If $k > 0$, then $\gcd(a, b) \leqslant \gcd(a, b \cdot k)$.

(28) If $n > 0$, then $\gcd(\gcd(a, b), b^n) = \gcd(a, b)$.

(29) $t + z$ and $t$ are relatively prime if and only if $t + z$ and $z$ are relatively prime.

(30) If $a$ and $b$ are relatively prime and $a \cdot b = c^n$, then there exists $k$ such that $k^n = a$. The theorem is a consequence of (7).

(31) If $a$ and $b$ are relatively prime and $a + b > 2$, then $a + b \mid a^n + b^n$ iff $a + b \nmid a^n - b^n$.
PROOF: $b > 0$. If $a + b \mid a^n - b^n$, then $a + b \nmid a^n + b^n$ by [16, (4)]. □

(32) If $a$ and $b$ are relatively prime and $a + b > 2$ and $n$ is odd, then $a + b \nmid a^n - b^n$. The theorem is a consequence of (31).

(33) If $a$ and $b$ are relatively prime and $a + b > 2$ and $n$ is even, then $a + b \nmid a^n + b^n$. The theorem is a consequence of (31).

Let us assume that $a$ and $b$ are relatively prime. Now we state the propositions:

(34) $a \cdot b$ and $a^{n+1} + b^{n+1}$ are relatively prime. The theorem is a consequence of (5).

(35) $a \cdot b$ and $a^{n+1} - b^{n+1}$ are relatively prime. The theorem is a consequence of (5).

(36) If $q > 0$ and $n > 0$, then there exists $r$ such that $q = r^n$.

(37) If $k > 0$ and $a + b > k$ and $a + b \mid k \cdot a$, then $a$ and $b$ are not relatively prime.

(38) If $k > 1$, then $k \nmid (k + 1)^n$. The theorem is a consequence of (11).

(39) If $a > 1$ and $b > 0$ and $\gcd(a, b) = 1$, then $a \nmid (a + b)^n$. The theorem is a consequence of (11).

(40) If $c > 0$, then for every non negative real numbers $r$, $s$, $r < s$ iff $r^c < s^c$.
PROOF: if $r < s$, then $r^c < s^c$ and if $r^c < s^c$, then $r < s$ by [24, (6)], [2, (14)], [21, (11)], [25, (37)]. □

(41) Let us consider non negative real numbers $r$, $s$. If $r \geqslant s$, then $r^n \geqslant s^n$. The theorem is a consequence of (40).

(42) If $a > 0$ and $n > 0$, then there exists $r$ such that $a^n + b^n = r^n$.

(43) There exists $b$ such that $b^{n+1} \leqslant a < (b + 1)^{n+1}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ there exists $b$ such that $b^{n+1} \leqslant \$_1 < (b + 1)^{n+1}$. $\mathcal{P}[0]$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$ by [2, (13)], (40). For every $x$, $\mathcal{P}[x]$ from [2, Sch. 2]. □

(44) If $n > 0$ and $a > b$ and $a$ and $b$ are relatively prime, then $\gcd(a^n + b^n, a^n - b^n) \leqslant 2$. The theorem is a consequence of (40) and (8).

(45) If $a + b \mid c$ and $a$ and $c$ are relatively prime, then $a$ and $b$ are relatively prime. The theorem is a consequence of (5).

(46) If $t$ and $z$ are relatively prime and $t$ and $u$ are relatively prime and $t$ is even, then $u + z$ is even and $u - z$ is even and $u \cdot z$ is odd.

(47) If $a$ and $b$ are relatively prime and $c$ is even and $a^n + b^n = c^n$, then $a + b$ is even and $a - b$ is even.

(48) If $a$ is even and $a$ and $b$ are relatively prime, then $a - b$ and $a + b$ are relatively prime. The theorem is a consequence of (9), (8), and (1).

(49) If $a$ and $b$ are relatively prime, then $a + b$ and $a \cdot b$ are relatively prime. The theorem is a consequence of (5).

(50) If $3 \nmid a \cdot b$, then $3 \mid (a + b) \cdot (a - b)$. The theorem is a consequence of (3).

(51) $3 \mid (a + b) \cdot (a - b) + a \cdot b$ if and only if $3 \mid a$ and $3 \mid b$.

(52) If $b^2 = a \cdot (a - b)$, then $3 \mid a$ and $3 \mid b$. The theorem is a consequence of (51).

(53) If $a$ and $b$ are relatively prime, then $3 \nmid (a+b) \cdot (a-b) + a \cdot b$. The theorem is a consequence of (51).

(54) If $a > b$ and $a + b \geqslant 2^{n+1}$, then $a > 2^n$.

(55) If $a \neq b$, then $2 \cdot a \cdot b < a^2 + b^2$.

(56) If $n > 0$ and $a \neq b$, then $2 \cdot a^n \cdot b^n < a^{2 \cdot n} + b^{2 \cdot n}$. The theorem is a consequence of (55).

(57) If $b > 0$, then there exists $n$ such that $b \geqslant 2^n$ and $b < 2^{n+1}$.
Proof: Consider $a$ such that $b = 1+a$. There exists $n$ such that $a+1 \geqslant 2^n$ and $a+1 < 2^{n+1}$ by [21, (6)]. □

(58) Let us consider odd natural numbers $a$, $b$. Then $4 \mid a+b$ if and only if $4 \nmid a-b$.
Proof: Consider $t$, $z$ such that $a+b = 2 \cdot t$ and $a-b = 2 \cdot z$. $t$ is odd iff $z$ is even. If $2 \cdot 2 \mid a+b$, then $2 \cdot 2 \nmid a-b$ by (3), [27, (16)]. If $2 \cdot 2 \nmid a+b$, then $2 \cdot 2 \mid a-b$ by (3), [27, (16)]. □

(59) If $\gcd(b+c, b) = 1$ and $c$ is odd, then $\gcd(2 \cdot b + c, c) = 1$.

(60) If $a+b = k \cdot a + k \cdot b$ and $a \cdot b > 0$, then $k = 1$.

(61) If $t \cdot z = t + z$, then $t = z$.

(62) $(2 \cdot n + 1)^2 = 4 \cdot n \cdot (n+1) + 1$.

(63) If $a$ is odd and $b$ is odd, then $8 \mid a^2 - b^2$. The theorem is a consequence of (62).

(64) Let us consider odd natural numbers $a$, $b$. If $4 \mid a-b$, then $4 \mid a^n - b^n$.

(65) Let us consider odd natural numbers $a$, $b$, and an even natural number $m$. Then $4 \mid a^m - b^m$.
Proof: Consider $n$ such that $m = 2 \cdot n$. If $4 \mid a+b$, then $4 \mid a^m - b^m$ by [34, (36)], [22, (9)]. If $4 \mid a-b$, then $4 \mid a^m - b^m$. □

(66) If $t$ is even and $4 \nmid t$, then there exists $u$ such that $u = t/2$ and $u$ is odd.

(67) If $a$ is odd and $2^n \mid a \cdot b$, then $2^n \mid b$.

Let us consider odd natural numbers $a$, $b$, $m$. Now we state the propositions:

(68) $4 \mid a^m + b^m$ if and only if $4 \mid a+b$.
Proof: Consider $n$ such that $m = 2 \cdot n + 1$. If $4 \mid a^{2 \cdot n+1} + b^{2 \cdot n+1}$, then $4 \mid a+b$ by [21, (81)], (65), [22, (2)], (58). □

(69) $4 \mid a-b$ if and only if $4 \nmid a^m + b^m$. The theorem is a consequence of (58) and (68).

Now we state the propositions:

(70) If $a^2 + b^2 = c^2$, then there exists $t$ such that $b^2 = (2 \cdot a + t) \cdot t$.

(71) If $b^2 = (2 \cdot a + t) \cdot t$, then there exists $c$ such that $a^2 + b^2 = c^2$.

(72) If $a$ is odd and $b$ is odd and $m$ is even, then $a^m + b^m \neq c^m$.
Proof: If $a$ is odd and $b$ is odd, then $a^{2 \cdot n} + b^{2 \cdot n} \neq c^{2 \cdot n}$ by [21, (9)]. □

(73) If $t$ and $z^n$ are relatively prime and $n > 0$, then $t$ and $z$ are relatively prime. The theorem is a consequence of (6).

Let us assume that $a$ and $b$ are relatively prime. Now we state the propositions:

(74) $\gcd((a+b)^2, a^2+b^2-(n-2)\cdot a\cdot b) = \gcd(a^2+b^2-(n-2)\cdot a\cdot b, n)$. The theorem is a consequence of (34) and (5).

(75) $a+b$ and $a^2+b^2+a\cdot b$ are relatively prime. The theorem is a consequence of (74) and (73).

(76) $\gcd((a-b)^2, a^2+b^2+(n-2)\cdot a\cdot b) = \gcd(a^2+b^2+(n-2)\cdot a\cdot b, n)$. The theorem is a consequence of (35), (5), and (1).

Now we state the propositions:

(77) $a \mid k\cdot(a\cdot n+1)$ if and only if $a \mid k$.
PROOF: If $a \mid k\cdot(a\cdot n+1)$, then $a \mid k$ by [22, (1)]. □

(78) Let us consider a positive natural number $n$. Then $a \mid k\cdot(a^n+1)$ if and only if $a \mid k$. The theorem is a consequence of (77).

(79) Let us consider positive natural numbers $a$, $b$. If $a \bmod b = b \bmod a$, then $a = b$.

(80) $k\cdot(a\cdot n+1) \bmod a = k \bmod a$.

Let us consider a positive natural number $n$. Now we state the propositions:

(81) $k\cdot(a^n+1) \bmod a = k \bmod a$. The theorem is a consequence of (80).

(82) $k\cdot(a^n+1)^m \bmod a = k \bmod a$. The theorem is a consequence of (81).

(83) $b\cdot(a^n+1)^m + c\cdot(a^n+1)^l \bmod a = b+c \bmod a$. The theorem is a consequence of (82).

Now we state the propositions:

(84) Let us consider positive natural numbers $a$, $n$. Then $a \mid b\cdot(a^n+1)^m + c\cdot(a^n+1)^l$ if and only if $a \mid b+c$. The theorem is a consequence of (83).

(85) If $|t| < a$, then $t \bmod a = |t|$ or $t \bmod a = a-|t|$.

(86) $-t \bmod a = u\cdot a - (t \bmod a) \bmod a$.

(87) Let us consider an odd natural number $n$. Then $t^n \bmod 3 = t \bmod 3$.

(88) $t + (u \bmod z) \bmod z = t+u \bmod z$.

(89) Let us consider an odd natural number $n$. Then $a+b-c \bmod 3 = a^n+b^n-c^n \bmod 3$. The theorem is a consequence of (87).

(90) Let us consider a positive natural number $k$. Then $t \bmod k = k-1$ if and only if $t+1 \bmod k = 0$. The theorem is a consequence of (88).

(91) If $a^2+b^2=c^2$, then $3 \mid a\cdot b\cdot c$. The theorem is a consequence of (14) and (50).

(92) Let us consider non zero natural numbers $a$, $n$. Suppose $t \bmod a = z \bmod a$. Then $t^n \bmod a = z^n \bmod a$.

(93) If $3 \mid t-z$, then $3 \mid t^n - z^n$. The theorem is a consequence of (18).

(94) Let us consider an odd natural number $n$. Then $3 \mid a+b-c$ if and only if $3 \mid a^n+b^n-c^n$.
PROOF: If $3 \mid a+b-c$, then $3 \mid a^n+b^n-c^n$ by [30, (62)], (89). $a+b-c \bmod 3 = 0$. $\square$

(95) $(t+u-z)^2 \equiv t^2+u^2+z^2 \pmod 2$.

(96) $(t+u-z)^3 \equiv t^3+u^3-z^3 \pmod 3$.

(97) $6 \mid a^3-a$. The theorem is a consequence of (50).

(98) Let us consider odd natural numbers $a$, $b$, $c$. Then $3 \mid t^a+t^b+t^c$. The theorem is a consequence of (87) and (88).

(99) (i) $2^m-1 \mid 2^{2\cdot m+1}-2$, and
(ii) $2^m+1 \mid 2^{2\cdot m+1}-2$.

(100) If $u+t+z$ is even, then $u\cdot t\cdot z$ is even.

(101) If $t^n+u^n=z^n$, then $2^n \mid (t\cdot u\cdot z)^n$. The theorem is a consequence of (100) and (15).

(102) $t^n \equiv t^m \pmod{t-1}$. The theorem is a consequence of (18).

## 2. FERMAT'S LITTLE THEOREM REVISITED

In the sequel $a$, $b$, $c$, $d$, $m$, $x$, $n$, $k$, $l$ denote natural numbers, $t$, $z$ denote integers, $f$, $F$, $G$ denote finite sequences of elements of $\mathbb{R}$, $q$, $r$, $s$ denote real numbers, and $D$ denotes a set.

Now we state the propositions:

(103) Let us consider a finite sequence $f$. Then $f$ is $D$-valued if and only if $f$ is a finite sequence of elements of $D$.

(104) $k+1 \in \operatorname{Seg} n$ if and only if $k<n$.
PROOF: If $k+1 \in \operatorname{Seg} n$, then $k<n$ by [4, (1)], [2, (13)]. $\square$

(105) $n+1 \leqslant \operatorname{len} f$ if and only if $n+1 \in \operatorname{dom} f$.
PROOF: If $n+1 \leqslant \operatorname{len} f$, then $n+1 \in \operatorname{dom} f$ by [2, (13)], (104). $n < \operatorname{len} f$. $\square$

(106) $k \in \mathbb{Z}_n$ if and only if $k+1 \in \operatorname{Seg} n$.

(107) If $n \in \operatorname{dom} f$ and $1 \leqslant m \leqslant n$, then $f(m) = (f{\restriction}n)(m)$.

(108) Suppose $f$ is a finite sequence of elements of $D$. Then
(i) $f{\restriction}n$ is a finite sequence of elements of $D$, and
(ii) $f_{\lfloor n}$ is a finite sequence of elements of $D$.

(109) If $n \in \operatorname{dom} f$, then $(f{\restriction}n)(1) = f(1)$. The theorem is a consequence of (107).

(110) Let us consider a finite sequence $f$ of elements of $\mathbb{R}$. If $n \in \operatorname{dom} f$, then $\operatorname{len}(f \restriction n) = n$.

Let us consider $s$. Observe that $\langle s \rangle$ is $\mathbb{R}$-valued.

Let us consider $D$. Let $f$ be a $D$-valued finite sequence. Let us consider $n$. Let us note that $f \restriction n$ is $D$-valued.

Let $f$ be a finite sequence of elements of $D$. Observe that $f_{\downharpoonleft n}$ is $D$-valued.

Now we state the proposition:

(111) Let us consider a finite sequence $f$ of elements of $\mathbb{C}$. If $k \in \operatorname{dom}(f \restriction n)$, then $k \in \operatorname{dom} f$.

Let us consider $n$. Note that $\emptyset_{\downharpoonleft n}$ is empty.

Let us consider $f$. One can check that $(f \restriction n)_{\downharpoonleft n}$ is empty.

Let us consider $D$. Let $f$ be a $D$-valued finite sequence. One can verify that $f_{\downharpoonleft n}$ is $D$-valued.

Let $f$ be a finite sequence of elements of $\mathbb{N}$. Observe that $f(n)$ is natural.

Let us consider $k$. One can verify that $(f \restriction n)(k)$ is natural and $(f \restriction n)_{\downharpoonleft 1}(k)$ is natural.

Now we state the propositions:

(112) $\sum(f \frown F) = \sum f + \sum F$.

(113) Let us consider a finite sequence $f$ of elements of $\mathbb{R}$. Suppose $k \in \operatorname{dom} f_{\downharpoonleft n}$ and $n \in \operatorname{dom} f$. Then $n + k \in \operatorname{dom} f$. The theorem is a consequence of (110).

(114) Let us consider a positive natural number $k$. If $n + k \in \operatorname{dom} f$, then $k \in \operatorname{dom} f_{\downharpoonleft n}$.

(115) Let us consider a positive natural number $n$. Suppose $n + 1 = \operatorname{len} f$. Then $\sum f = \sum(f \restriction n)_{\downharpoonleft 1} + f(1) + f(n+1)$. The theorem is a consequence of (112) and (109).

(116) If $n + 1 = \operatorname{len} f$, then $f_{\downharpoonleft n} = \langle f(n+1) \rangle$.

Let us assume that $(f \restriction n)_{\downharpoonleft 1}$ is not empty. Now we state the propositions:

(117) $\operatorname{len}(f \restriction n)_{\downharpoonleft 1} \leqslant \operatorname{len} f - 1$. The theorem is a consequence of (110).

(118) $\operatorname{len}(f \restriction n)_{\downharpoonleft 1} < n$. The theorem is a consequence of (110).

Now we state the propositions:

(119) If $n$ is prime and $k \neq 0$ and $k \neq n$, then $n \mid \binom{n}{k}$.

(120) If $b \geqslant 2$, then $(b+1)! > 2^b$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\$_1 + 1)! > 2^{\$_1}$. $\mathcal{P}[2]$ by [21, (14), (15), (81)]. For every natural number $k$ such that $k \geqslant 2$ and $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [21, (6), (15)]. For every natural number $x$ such that $x \geqslant 2$ holds $\mathcal{P}[x]$ from [2, Sch. 8]. □

(121) $b > 1$ if and only if $b! > 1$.

PROOF: If $b > 1$, then $b! > 1$ by [2, (13)], [20, (55)]. $\square$

(122) If $b \geqslant 2$, then $b! < b^b$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \$_1! < {\$_1}^{\$_1}$. $\mathcal{P}[2]$ by [21, (81), (14)]. For every natural number $k$ such that $k \geqslant 2$ and $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [24, (10)], [21, (15), (6)]. For every natural number $x$ such that $x \geqslant 2$ holds $\mathcal{P}[x]$ from [2, Sch. 8]. $\square$

(123) $(b+1)! \geqslant 2^b$. The theorem is a consequence of (120).

(124) $b! \leqslant b^b$. The theorem is a consequence of (122).

(125) If $b > 0$ and $a$ and $b!$ are relatively prime, then $a$ and $b$ are relatively prime. The theorem is a consequence of (121).

(126) If $a$ and $(a+b)!$ are relatively prime, then $a = 1$ or $a = 0$ and ($b = 0$ or $b = 1$). The theorem is a consequence of (121).

(127) If $n \in \operatorname{dom} f$ and $m \in \operatorname{dom}(f{\restriction}n)_{\downharpoonright 1}$, then $(f{\restriction}n)_{\downharpoonright 1}(m) = f(m+1)$. The theorem is a consequence of (113), (105), (110), and (107).

Let us consider $n$. One can verify that $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ is non empty.

Let us consider $m$. One can check that $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(m)$ is natural and $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ is $\mathbb{N}$-valued.

Let $h$ be a finite sequence of elements of $\mathbb{N}$. One can verify that $\sum h$ is natural.

Now we state the propositions:

(128) If $n > 0$, then $n \in \operatorname{dom}\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$.

(129) $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ is a finite sequence of elements of $\mathbb{N}$.

(130) $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(n+1) = 1$. The theorem is a consequence of (105).

(131) $\langle \binom{k}{0}, \ldots, \binom{k}{k} \rangle(1) = 1$.

PROOF: $\langle \binom{k}{0}1^0 1^k, \ldots, \binom{k}{k}1^k 1^0 \rangle(1) = 1$ by [21, (28)]. $\square$

Let us consider a positive natural number $n$. Now we state the propositions:

(132) $\sum\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = \sum(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n)_{\downharpoonright 1} + \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(1) + \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(n+1)$. The theorem is a consequence of (128), (112), and (109).

(133) $\sum\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = \sum(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n)_{\downharpoonright 1} + 2$. The theorem is a consequence of (132), (130), and (131).

Now we state the propositions:

(134) $\sum\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = \sum(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n) + 1$. The theorem is a consequence of (103).

(135) $\operatorname{len}(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n) = n$.

(136) Suppose $m \in \operatorname{dom}(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n)_{\downharpoonright 1}$. Then $(\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle{\restriction}n)_{\downharpoonright 1}(m) = \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(m+1)$. The theorem is a consequence of (128) and (127).

(137) If $n$ is prime, then $n \mid (\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle \restriction n)_{\downharpoonleft 1}(k)$.

PROOF: If $n$ is prime and $k \geqslant n$, then $n \mid (\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle \restriction n)_{\downharpoonleft 1}(k)$ by (128), (110), [2, (13)], [31, (25)]. If $n$ is prime and $k < n$, then $n \mid (\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle \restriction n)_{\downharpoonleft 1}(k)$ by [31, (25)], (128), (110), [2, (13)]. □

(138) Let us consider a prime natural number $n$. Then $n \mid 2^n - 2$. The theorem is a consequence of (103), (137), and (133).

Let $k$ be a positive natural number. Let us consider $n$. Let us note that $n^k - n$ is natural.

Now we state the propositions:

(139) Let us consider prime natural numbers $k$, $n$. Then $n \cdot k \mid (2^n - 2) \cdot (2^k - 2)$. The theorem is a consequence of (138).

(140) Let us consider an odd prime number $n$. If $n = 2 \cdot k + 1$, then $n \mid 2^k - 1$ iff $n \nmid 2^k + 1$.

PROOF: $n \mid 2^k - 1$ or $n \mid 2^k + 1$ by (138), [21, (6)], [33, (7)], [21, (9)]. □

Let $n$ be a natural number. The functor $n \setminus$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the term

(Def. 1) $\langle \binom{n}{0} 1^0 1^n, \ldots, \binom{n}{n} 1^n 1^0 \rangle$.

Let us consider $n$. We identify $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ with $n \setminus$. We identify $n \setminus$ with $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$. Now we state the proposition:

(141) If $n > 0$, then $n \in \operatorname{dom} \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$.

Let us consider $a$, $b$, $n$, and $m$. Let us observe that $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(m)$ is natural and $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ is $\mathbb{N}$-valued.

Now we state the propositions:

(142) If $k + l$ is prime and $k > 0$ and $l > 0$, then $k + l \mid \langle \binom{k+l}{0} a^0 b^{k+l}, \ldots, \binom{k+l}{k+l} a^{k+l} b^0 \rangle(k + 1)$. The theorem is a consequence of (119).

(143) If $a \neq 0$, then $\langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle(1) \neq 0$.

(144) Let us consider a non zero natural number $m$. Then $a = 0$ if and only if $\langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle(1) = 0$.

PROOF: For every non zero natural number $m$ such that $a = 0$ holds $\langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle(1) = 0$ by [21, (28)]. □

(145) If $\langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle(1) = 0$, then $m \neq 0$.

(146) Let us consider a positive natural number $m$. Then $\sum \langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle = a^m + b^m + \sum (\langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle \restriction m)_{\downharpoonleft 1}$. The theorem is a consequence of (115).

(147) $\sum \langle \binom{m+n}{0} a^0 b^{m+n}, \ldots, \binom{m+n}{m+n} a^{m+n} b^0 \rangle = \sum \langle \binom{m}{0} a^0 b^m, \ldots, \binom{m}{m} a^m b^0 \rangle \cdot \sum \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$.

(148) If $l > 0$, then there exists $x$ such that $\langle\binom{k+l}{0}a^0b^{k+l}, \ldots, \binom{k+l}{k+l}a^{k+l}b^0\rangle(k+1) = a \cdot x$.

(149) If $m > 0$, then there exists $k$ such that $\langle\binom{m}{0}a^0b^m, \ldots, \binom{m}{m}a^mb^0\rangle(1) = a \cdot k$. The theorem is a consequence of (148).

(150) If $l > 0$, then there exists $x$ such that $\langle\binom{k+l}{0}a^0b^{k+l}, \ldots, \binom{k+l}{k+l}a^{k+l}b^0\rangle(l) = a \cdot x$.

(151) If $n = \langle\binom{k+l}{0}a^0b^{k+l}, \ldots, \binom{k+l}{k+l}a^{k+l}b^0\rangle(k+1)$ and $l > 0$, then $a \mid n$. The theorem is a consequence of (148).

Let us consider a prime natural number $n$ and positive natural numbers $a$, $b$. Now we state the propositions:

(152) $n \cdot a \cdot b \mid (\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle\restriction n)_{\downharpoonleft 1}(k)$.
PROOF: If $k \notin \mathrm{dom}(\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle\restriction n)_{\downharpoonleft 1}$, then $n \cdot a \cdot b \mid (\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle\restriction n)_{\downharpoonleft 1}(k)$. If $n$ is prime and $k \in \mathrm{dom}(\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle\restriction n)_{\downharpoonleft 1}$, then $n \cdot a \cdot b \mid (\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle\restriction n)_{\downharpoonleft 1}(k)$ by [31, (25)], (118), [2, (13), (10)]. □

(153) $n \cdot a \cdot b \mid (a+b)^n - (a^n + b^n)$. The theorem is a consequence of (103), (152), and (146).

Now we state the propositions:

(154) Let us consider a prime natural number $n$. Then $n \cdot a \mid (a+1)^n - (a^n + 1)$. The theorem is a consequence of (153).

(155) Let us consider positive natural numbers $a$, $b$. Then $2 \cdot a \cdot b \mid (a+b)^2 - (a^2 + b^2)$.

(156) Let us consider a prime natural number $n$. Then $n \mid a^n - a$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv n \mid \$_1{}^n - \$_1$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$ by (154), [22, (2)], [16, (4)]. For every natural number $x$, $\mathcal{P}[x]$ from [2, Sch. 2]. □

(157) Let us consider a natural number $k$. If $k+1$ is prime and $k+1 \nmid a$, then $k+1 \mid a^k - 1$. The theorem is a consequence of (156).

(158) Let us consider a prime natural number $n$. Then $n \mid a+b$ if and only if $n \mid a^n + b^n$. The theorem is a consequence of (156).

(159) $163 \mid a+b$ if and only if $163 \mid a^{163} + b^{163}$.

Let us consider a prime natural number $n$. Now we state the propositions:

(160) $n \mid a$ if and only if $n \mid a^n$.

(161) $n \mid a^n + 1$ if and only if $n \mid a+1$. The theorem is a consequence of (158).

(162) $n \mid a^n + b^n$ if and only if $n \mid (a+b)^n$.

Now we state the propositions:

(163) $7 \mid a^7 + 1$ if and only if $7 \mid a+1$.

(164) If $7 \nmid a$, then $7 \mid a^6 - 1$. The theorem is a consequence of (156).

Let us consider a prime natural number $n$ and positive natural numbers $a$, $b$. Now we state the propositions:

(165) $n \cdot a \cdot b \mid (a+b)^{k \cdot n} - (a^n + b^n)^k$. The theorem is a consequence of (153).

(166) If $n \cdot a \cdot b \mid (a+t)^n - (a^n + b^n)$, then $n \cdot a \cdot b \mid (a+b)^n - (a+t)^n$. The theorem is a consequence of (153).

Now we state the proposition:

(167) Let us consider a prime natural number $n$, and positive natural numbers $a$, $b$, $c$. If $n \cdot a \cdot b \mid c - b$, then $n \cdot a \cdot b \mid a^n + b^n - (a+c)^n$. The theorem is a consequence of (153).

Let us consider a prime natural number $p$. Now we state the propositions:

(168) If $p = 2 \cdot n + 1$, then $p \mid a$ or $p \mid a^n - 1$ or $p \mid a^n + 1$. The theorem is a consequence of (156).

(169) If $p \nmid a$, then there exists $n$ such that $p \mid a^n - 1$ and $0 < n < p$. The theorem is a consequence of (157).

Now we state the propositions:

(170) $5 \mid a^3 - 1$ if and only if $5 \mid a - 1$.
PROOF: If $5 \mid a^3 - 1$, then $5 \mid a - 1$ by [13, (59)], (156), [22, (13)], [18, (3)]. □

(171) If $k+1$ is prime, then $k+1 \mid a^{n \cdot k + 1} - a$. The theorem is a consequence of (157).

(172) $2 \mid a^{n+1} - a$. The theorem is a consequence of (171).

(173) $3 \mid a^{2 \cdot n + 1} - a$. The theorem is a consequence of (171).

(174) $5 \mid a^{4 \cdot n + 1} - a$. The theorem is a consequence of (171).

(175) $7 \mid a^{6 \cdot n + 1} - a$. The theorem is a consequence of (171).

(176) If $k \neq l$ and $k+1$ is odd and prime and $l+1$ is odd and prime, then $2 \cdot (k+1) \cdot (l+1) \mid a^{k \cdot l + 1} - a$. The theorem is a consequence of (171) and (172).

(177) $154 \mid a^{61} - a$. The theorem is a consequence of (176).

(178) $6 \mid a^{2 \cdot n + 1} - a$. The theorem is a consequence of (172) and (173).

(179) $30 \mid a^{4 \cdot n + 1} - a$. The theorem is a consequence of (172), (173), and (174).

(180) $42 \mid a^{6 \cdot n + 1} - a$. The theorem is a consequence of (172), (173), and (175).

(181) Let us consider a prime natural number $n$. Then $n \mid a^{n+k} - a^{k+1}$. The theorem is a consequence of (156).

(182) If $2 \cdot n + 1$ is prime, then for every $k$ such that $2 \cdot n > k > 1$ holds $2 \cdot n + 1 \nmid a^n - k$ and $2 \cdot n + 1 \nmid a^n + k$. The theorem is a consequence of (168).

(183) (i) $5 \nmid a^2 - 2$, and

(ii) $5 \nmid a^2 + 2$, and

(iii) $5 \nmid a^2 - 3$, and

(iv) $5 \nmid a^2 + 3$.

The theorem is a consequence of (182).

(184) If $a^2 + b^2 = c^2$, then $5 \mid a$ or $5 \mid b$ or $5 \mid c$. The theorem is a consequence of (168) and (183).

(185) (i) $7 \nmid a^3 - 2$, and

(ii) $7 \nmid a^3 + 2$, and

(iii) $7 \nmid a^3 - 3$, and

(iv) $7 \nmid a^3 + 3$, and

(v) $7 \nmid a^3 - 4$, and

(vi) $7 \nmid a^3 + 4$, and

(vii) $7 \nmid a^3 - 5$, and

(viii) $7 \nmid a^3 + 5$.

The theorem is a consequence of (182).

(186) $2 \mid 2^n - 1$ if and only if $n = 0$.

PROOF: If $2 \mid 2^n - 1$, then $n = 0$ by [18, (3)], [22, (13)]. □

(187) If $2^{k+l} \mid 2^{n+k} - 2^k$, then $l = 0$ or $n = 0$. The theorem is a consequence of (186).

(188) (i) $3 \mid b$, or

(ii) $3 \mid b - 1$, or

(iii) $3 \mid b + 1$.

The theorem is a consequence of (168).

(189) If $3 \nmid b$, then $3 \nmid b^2 + c^2$.

PROOF: If $3 \nmid b$ and $3 \nmid c$, then $3 \nmid b^2 + c^2$ by (157), [13, (41)], [16, (4)], [22, (1), (27)]. If $3 \nmid b$ and $3 \mid c$, then $3 \nmid b^2 + c^2$ by [18, (3)], [22, (9)], [18, (5)], [13, (41)]. □

(190) (i) $3 \nmid b^2 + 1$, and

(ii) $3 \nmid b^2 - 2$.

The theorem is a consequence of (189).

(191) $3 \nmid b^3 + b^2 - b + 1$. The theorem is a consequence of (190) and (156).

(192) Let us consider a positive natural number $a$. If $b$ and $c$ are relatively prime and $a + 1 \mid b$, then $a + 1 \nmid c$.

(193) If $b$ and $c$ are relatively prime, then $3 \nmid b^2 + c^2$. The theorem is a consequence of (192) and (189).

(194) Let us consider a prime natural number $p$. If $p \mid a$, then $p \mid a^{n+1}$.

(195) If $b$ and $c$ are relatively prime and $b^2 + c^2 = a^2$, then $3 \nmid a$. The theorem is a consequence of (193) and (194).

Let us consider a prime natural number $p$. Now we state the propositions:

(196) If $p \mid a + b$, then $p \mid a^{2 \cdot n+1} + b^{2 \cdot n+1}$.

(197) If $p \nmid a^{2 \cdot n+1} + b^{2 \cdot n+1}$ and $p \mid a^2 - b^2$, then $p \mid a - b$. The theorem is a consequence of (196).

Now we state the propositions:

(198) (i) $3 \mid a \cdot b$, or

(ii) $3 \mid a + b$, or

(iii) $3 \mid a - b$.

The theorem is a consequence of (188).

(199) If $3 \nmid a$ and $3 \nmid b$, then $3 \mid a^{2 \cdot n+1} + b^{2 \cdot n+1}$ or $3 \mid a^{2 \cdot n+1} - b^{2 \cdot n+1}$. The theorem is a consequence of (188).

(200) If $a^3 + b^3 = c^3$, then $7 \mid a$ or $7 \mid b$ or $7 \mid c$. The theorem is a consequence of (168) and (185).

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# Weak Convergence and Weak* Convergence[1]

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**Summary.** In this article, we deal with weak convergence on sequences in real normed spaces, and weak* convergence on sequences in dual spaces of real normed spaces. In the first section, we proved some topological properties of dual spaces of real normed spaces. We used these theorems for proofs of Section 3. In Section 2, we defined weak convergence and weak* convergence, and proved some properties. By `RNS_Real` Mizar functor, real normed spaces as real number spaces already defined in the article [18], we regarded sequences of real numbers as sequences of `RNS_Real`. So we proved the last theorem in this section using the theorem (8) from [25]. In Section 3, we defined weak sequential compactness of real normed spaces. We showed some lemmas for the proof and proved the theorem of weak sequential compactness of reflexive real Banach spaces. We referred to [36], [23], [24] and [3] in the formalization.

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The notation and terminology used in this paper have been introduced in the following articles: [4], [19], [17], [18], [28], [5], [6], [21], [30], [26], [25], [29], [1], [22], [16], [2], [7], [34], [35], [37], [31], [27], [14], [33], and [8].

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## 1. Some Properties about Dual Spaces of Real Normed Spaces

Let $X$ be a non empty set, $F$ be a sequence of $X^{\mathbb{N}}$, and $k$ be a natural number. One can check that the functor $F(k)$ yields a sequence of $X$. Now we state the propositions:

(1) Let us consider a strict real normed space $X$, and a non empty subset $A$ of $X$. Suppose for every point $f$ of DualSp $X$ such that for every point $x$ of $X$ such that $x \in A$ holds $(\text{Bound2Lipschitz}(f, X))(x) = 0$ holds $\text{Bound2Lipschitz}(f, X) = 0_{\text{DualSp}\,X}$. Then $\text{ClNLin}(A) = X$.
PROOF: Set $M = \text{ClNLin}(A)$. Consider $Z$ being a subset of $X$ such that $Z =$ the carrier of $\text{Lin}(A)$ and $M = \langle \overline{Z}, \text{Zero}(\overline{Z}, X), \text{Add}(\overline{Z}, X), \text{Mult}(\overline{Z}, X),$ the norm of $\overline{Z}$ induced by $X\rangle$. Reconsider $Y =$ the carrier of $M$ as a non empty subset of $X$. $Y =$ the carrier of $X$ by [18, (2)], [32, (15)], [16, (4)], [17, (25)]. □

(2) Let us consider a strict real normed space $X$. If DualSp $X$ is separable, then $X$ is separable.
PROOF: Set $Y = \text{DualSp}\,X$. Consider $Y_1$ being a sequence of $Y$ such that $\text{rng}\,Y_1$ is dense. Define $\mathcal{P}[\text{natural number}, \text{point of } X] \equiv \|Y_1(\$_1)\|/_2 \leqslant |Y_1(\$_1)(\$_2)|$ and $\|\$_2\| \leqslant 1$. For every element $n$ of $\mathbb{N}$, there exists a point $x$ of $X$ such that $\mathcal{P}[n, x]$ by [4, (46)], [15, (45)], [17, (24)]. Consider $X_2$ being a function from $\mathbb{N}$ into the carrier of $X$ such that for every element $n$ of $\mathbb{N}$, $\mathcal{P}[n, X_2(n)]$ from [6, Sch. 3]. For every natural number $n$, $\|Y_1(n)\|/_2 \leqslant |Y_1(n)(X_2(n))|$ and $\|X_2(n)\| \leqslant 1$. Consider $X_2$ being a sequence of $X$ such that for every natural number $n$, $\|Y_1(n)\|/_2 \leqslant |Y_1(n)(X_2(n))|$ and $\|X_2(n)\| \leqslant 1$. Set $X_1 = \text{rng}\,X_2$. For every point $f$ of $Y$ such that for every point $x$ of $X$ such that $x \in X_1$ holds $(\text{Bound2Lipschitz}(f, X))(x) = 0$ holds $\text{Bound2Lipschitz}(f, X) = 0_Y$ by [17, (23)], [16, (14)], [22, (24)], [26, (20)]. $M = X$. □

(3) Let us consider a real number $x$, and a point $x_1$ of the real normed space of $\mathbb{R}$. If $x = x_1$, then $-x = -x_1$.

(4) Let us consider real numbers $x$, $y$, and points $x_1$, $y_1$ of the real normed space of $\mathbb{R}$. If $x = x_1$ and $y = y_1$, then $x - y = x_1 - y_1$. The theorem is a consequence of (3).

Let us consider a sequence $s_2$ of real numbers and a sequence $s_3$ of the real normed space of $\mathbb{R}$. Now we state the propositions:

(5) If $s_2 = s_3$, then $s_2$ is convergent iff $s_3$ is convergent. The theorem is a consequence of (4).

(6) If $s_2 = s_3$ and $s_2$ is convergent, then $\lim s_2 = \lim s_3$. The theorem is a consequence of (5) and (4).

(7) Let us consider a sequence $s_3$ of the real normed space of $\mathbb{R}$. If $s_3$ is Cauchy sequence by norm, then $s_3$ is convergent.
PROOF: Reconsider $s_2 = s_3$ as a sequence of real numbers. For every real number $s$ such that $0 < s$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|s_2(m) - s_2(n)| < s$ by [27, (8)], (4). □

Let us note that the real normed space of $\mathbb{R}$ is complete.

Let $X$ be a real normed space, $g$ be a sequence of DualSp $X$, and $x$ be a point of $X$. The functor $g\#x$ yielding a sequence of real numbers is defined by

(Def. 1) for every natural number $i$, $it(i) = g(i)(x)$.

## 2. WEAK CONVERGENCE AND WEAK* CONVERGENCE

Let $X$ be a real normed space and $x$ be a sequence of $X$. We say that $x$ is weakly convergent if and only if

(Def. 2) there exists a point $x_0$ of $X$ such that for every Lipschitzian linear functional $f$ in $X$, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$.

Now we state the proposition:

(8) Let us consider a real normed space $X$, and a sequence $x$ of $X$. If $\operatorname{rng} x \subseteq \{0_X\}$, then $x$ is weakly convergent.
PROOF: Reconsider $x_0 = 0_X$ as a point of $X$. For every Lipschitzian linear functional $f$ in $X$, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$ by [6, (4), (15)], [4, (44)]. □

Let $X$ be a real normed space and $x$ be a sequence of $X$. Assume $x$ is weakly convergent. The functor w-lim$(x)$ yielding a point of $X$ is defined by

(Def. 3) for every Lipschitzian linear functional $f$ in $X$, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(it)$.

Let us consider a real normed space $X$ and a sequence $x$ of $X$. Now we state the propositions:

(9) If $x$ is convergent, then $x$ is weakly convergent and w-lim$(x) = \lim x$.
PROOF: Reconsider $x_0 = \lim x$ as a point of $X$. For every Lipschitzian linear functional $f$ in $X$, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$ by [21, (19)], [20, (46)]. □

(10) Suppose $X$ is not trivial and $x$ is weakly convergent. Then

(i) $\|x\|$ is bounded, and

(ii) $\|\text{w-lim}(x)\| \leqslant \liminf \|x\|$, and

(iii) w-lim$(x) \in$ ClNLin$(\operatorname{rng} x)$.

PROOF: Reconsider $x_0 = \text{w-lim}(x)$ as a point of $X$. For every point $f$ of DualSp $X$, there exists a real number $K_1$ such that $0 \leqslant K_1$ and for every point $y$ of $X$ such that $y \in \text{rng}\, x$ holds $|f(y)| \leqslant K_1$ by [14, (3)], [20, (6)], [6, (15)]. Consider $K$ being a real number such that $0 \leqslant K$ and for every point $y$ of $X$ such that $y \in \text{rng}\, x$ holds $\|y\| \leqslant K$. For every natural number $n$, $\|x\|(n) \leqslant K$ by [6, (4)]. For every natural number $n$, $|\|x\|(n)| < K+1$. For every point $f$ of DualSp $X$, $|f(x_0)| \leqslant \liminf\|x\| \cdot \|f\|$ by [17, (26)], [6, (15)], [13, (12), (9)]. Consider $Y$ being a non empty subset of $\mathbb{R}$ such that $Y = \{|(\text{Bound2Lipschitz}(F, X))(x_0)|$, where $F$ is a point of DualSp $X : \|F\| \leqslant 1\}$ and $\|x_0\| = \sup Y$. $x_0 \in \text{ClNLin}(\text{rng}\, x)$ by [16, (29)], [18, (2)], [17, (23)], [32, (15)]. □

Let $X$ be a real normed space and $g$ be a sequence of DualSp $X$. We say that $g$ is weakly* convergent if and only if

(Def. 4) there exists a point $g_0$ of DualSp $X$ such that for every point $x$ of $X$, $g \# x$ is convergent and $\lim(g \# x) = g_0(x)$.

Assume $g$ is weakly* convergent. The functor w*-lim($g$) yielding a point of DualSp $X$ is defined by

(Def. 5) for every point $x$ of $X$, $g \# x$ is convergent and $\lim(g \# x) = it(x)$.

Now we state the proposition:

(11) Let us consider a real normed space $X$, and a sequence $g$ of DualSp $X$. Suppose $g$ is convergent. Then

(i) $g$ is weakly* convergent, and

(ii) w*-lim($g$) $= \lim g$.

PROOF: Reconsider $g_0 = \lim g$ as a point of DualSp $X$. For every point $x$ of $X$, $g \# x$ is convergent and $\lim(g \# x) = g_0(x)$ by [17, (33), (26)]. □

Let us consider a real normed space $X$ and a sequence $f$ of DualSp $X$. Now we state the propositions:

(12) If $f$ is weakly convergent, then $f$ is weakly* convergent.
PROOF: Reconsider $f_0 = \text{w-lim}(f)$ as a point of DualSp $X$. For every point $x$ of $X$, $f \# x$ is convergent and $\lim(f \# x) = f_0(x)$ by [6, (15)]. □

(13) If $X$ is reflexive, then $f$ is weakly convergent iff $f$ is weakly* convergent.
PROOF: If $f$ is weakly* convergent, then $f$ is weakly convergent by [18, (21)], [6, (15)]. □

(14) Let us consider a real Banach space $X$, and a subset $T$ of DualSp $X$. Suppose for every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every point $f$ of DualSp $X$ such that $f \in T$ holds $|f(x)| \leqslant K$. Then there exists a real number $L$ such that

(i) $0 \leqslant L$, and

(ii) for every point $f$ of DualSp $X$ such that $f \in T$ holds $\|f\| \leqslant L$.

PROOF: Reconsider $T_1 = T$ as a subset of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$. For every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every point $f$ of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$ such that $f \in T_1$ holds $\|f(x)\| \leqslant K$. Consider $L$ being a real number such that $0 \leqslant L$ and for every point $f$ of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$ such that $f \in T_1$ holds $\|f\| \leqslant L$. For every point $f$ of DualSp $X$ such that $f \in T$ holds $\|f\| \leqslant L$ by [18, (18)]. □

(15) Let us consider a real Banach space $X$, and a sequence $f$ of DualSp $X$. Suppose $f$ is weakly* convergent. Then

(i) $\|f\|$ is bounded, and

(ii) $\|\text{w*-lim}(f)\| \leqslant \liminf \|f\|$.

PROOF: Reconsider $f_0 = \text{w*-lim}(f)$ as a point of DualSp $X$. For every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every point $g$ of DualSp $X$ such that $g \in \operatorname{rng} f$ holds $|g(x)| \leqslant K$ by [6, (11)], [13, (12)], [4, (46)]. Consider $L$ being a real number such that $0 \leqslant L$ and for every point $g$ of DualSp $X$ such that $g \in \operatorname{rng} f$ holds $\|g\| \leqslant L$. For every natural number $n$, $|\|f\|(n)| < L + 1$ by [6, (4)]. For every point $x$ of $X$, $|f_0(x)| \leqslant \liminf \|f\| \cdot \|x\|$ by [13, (12), (9)], [17, (26)], [25, (1)]. □

(16) Let us consider a real normed space $X$, a point $x$ of $X$, a sequence $v$ of DualSp $X$, and a sequence $v_1$ of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$. If $v = v_1$, then $v \# x = v_1 \# x$.

(17) Let us consider a real Banach space $X$, a subset $X_1$ of $X$, and a sequence $v$ of DualSp $X$. Suppose $\|v\|$ is bounded and $X_1$ is dense and for every point $x$ of $X$ such that $x \in X_1$ holds $v \# x$ is convergent. Then $v$ is weakly* convergent.

PROOF: Reconsider $v_1 = v$ as a sequence of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$. Reconsider $X_2 = X_1$ as a subset of LinearTopSpaceNorm $X$. For every point $x$ of $X$ such that $x \in X_2$ holds $v_1 \# x$ is convergent. For every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every natural number $n$, $\|(v_1 \# x)(n)\| \leqslant K$ by [14, (3)], [17, (26)], (16). Consider $t$ being a point of the real norm space of bounded linear operators from $X$ into the real normed space of $\mathbb{R}$ such that for every point $x$ of $X$, $v_1 \# x$ is convergent and $t(x) = \lim(v_1 \# x)$ and $\|t(x)\| \leqslant \liminf \|v_1\| \cdot \|x\|$ and $\|t\| \leqslant \liminf \|v_1\|$.

Reconsider $g_0 = t$ as a point of DualSp $X$. For every point $x$ of $X$, $v\#x$ is convergent and $\lim(v\#x) = g_0(x)$. □

(18) Let us consider a real Banach space $X$, and a sequence $f$ of DualSp $X$. Then $f$ is weakly* convergent if and only if $\|f\|$ is bounded and there exists a subset $X_1$ of $X$ such that $X_1$ is dense and for every point $x$ of $X$ such that $x \in X_1$ holds $f\#x$ is convergent. The theorem is a consequence of (15) and (17).

## 3. Weak Sequential Compactness of Real Banach Spaces

Let $X$ be a real normed space and $X_1$ be a non empty subset of $X$. We say that $X_1$ is weakly sequentially compact if and only if

(Def. 6) for every sequence $s_2$ of $X_1$, there exists a sequence $s_3$ of $X$ such that $s_3$ is subsequence of $s_2$ and weakly convergent and w-lim$(s_3) \in X$.

Now we state the proposition:

(19) Let us consider a real normed space $X$, and a sequence $x$ of $X$. Suppose $X$ is reflexive. Then $x$ is weakly convergent if and only if BidualFunc $X \cdot x$ is weakly* convergent.

Proof: Set $f =$ BidualFunc $X \cdot x$. Consider $f_0$ being a point of DualSp DualSp $X$ such that for every point $h$ of DualSp $X$, $f\#h$ is convergent and $\lim(f\#h) = f_0(h)$. Consider $x_0$ being a point of $X$ such that for every point $g$ of DualSp $X$, $f_0(g) = g(x_0)$. For every Lipschitzian linear functional $g$ in $X$, $g \cdot x$ is convergent and $\lim(g \cdot x) = g(x_0)$ by [6, (15)]. □

Let us consider a real normed space $X$, a sequence $f$ of DualSp $X$, and a point $x$ of $X$.

Let us assume that $\|f\|$ is bounded. Now we state the propositions:

(20) There exists a sequence $f_0$ of DualSp $X$ such that

(i) $f_0$ is a subsequence of $f$, and

(ii) $\|f_0\|$ is bounded, and

(iii) $f_0\#x$ is convergent.

Proof: Consider $r_0$ being a real number such that $0 < r_0$ and for every natural number $m$, $|\|f\|(m)| < r_0$. Set $r = r_0 \cdot \|x\| + 1$. For every natural number $m$, $|(f\#x)(m)| < r$ by [17, (26)]. Reconsider $s_2 = f\#x$ as a sequence of real numbers. Consider $s_3$ being a sequence of real numbers such that $s_3$ is subsequence of $s_2$ and convergent. Consider $N$ being an increasing sequence of $\mathbb{N}$ such that $s_3 = s_2 \cdot N$. Set $f_0 = f \cdot N$. For every natural number $k$, $(f_0\#x)(k) = s_3(k)$ by [6, (15)]. For every natural number $n$, $|\|f_0\|(n)| < r_0$ by [6, (15)]. □

(21) There exists a sequence $f_0$ of DualSp $X$ such that

(i) $f_0$ is a subsequence of $f$, and

(ii) $\|f_0\|$ is bounded, and

(iii) $f_0\#x$ is convergent and subsequence of $f\#x$.

Proof: Consider $r_0$ being a real number such that $0 < r_0$ and for every natural number $m$, $|\|f\|(m)| < r_0$. Set $r = r_0 \cdot \|x\| + 1$. For every natural number $m$, $|(f\#x)(m)| < r$ by [17, (26)]. Reconsider $s_2 = f\#x$ as a sequence of real numbers. Consider $s_3$ being a sequence of real numbers such that $s_3$ is subsequence of $s_2$ and convergent. Consider $N$ being an increasing sequence of $\mathbb{N}$ such that $s_3 = s_2 \cdot N$. Reconsider $f_0 = f \cdot N$ as a sequence of DualSp $X$. For every natural number $n$, $|\|f_0\|(n)| < r_0$ by [6, (15)]. □

(22) There exists a sequence $f_0$ of DualSp $X$ and there exists an increasing sequence $N$ of $\mathbb{N}$ such that $f_0$ is a subsequence of $f$ and $\|f_0\|$ is bounded and $f_0\#x$ is convergent and subsequence of $f\#x$ and $f_0 = f \cdot N$. The theorem is a consequence of (21).

Let us consider a real normed space $X$, a sequence $f$ of DualSp $X$, and a sequence $x$ of $X$.

Let us assume that $\|f\|$ is bounded. Now we state the propositions:

(23) There exists a sequence $F$ of (the carrier of DualSp $X)^{\mathbb{N}}$ such that

(i) $F(0)$ is a subsequence of $f$, and

(ii) $F(0)\#x(0)$ is convergent, and

(iii) for every natural number $k$, $F(k+1)$ is a subsequence of $F(k)$, and

(iv) for every natural number $k$, $F(k+1)\#x(k+1)$ is convergent.

Proof: Set $D =$ (the carrier of DualSp $X)^{\mathbb{N}}$. Consider $f_0$ being a sequence of DualSp $X$ such that $f_0$ is a subsequence of $f$ and $\|f_0\|$ is bounded and $f_0\#x(0)$ is convergent. Reconsider $A = f_0$ as an element of $D$. Define $\mathcal{P}[\text{natural number}, \text{sequence of DualSp}\, X, \text{sequence of DualSp}\, X] \equiv$ if $\|\$_2\|$ is bounded, then $\$_3$ is a subsequence of $\$_2$ and $\|\$_3\|$ is bounded and $\$_3\#x(\$_1 + 1)$ is convergent. For every natural number $n$ and for every element $z$ of $D$, there exists an element $y$ of $D$ such that $\mathcal{P}[n, z, y]$ by (20), [6, (8)]. Consider $F$ being a sequence of $D$ such that $F(0) = A$ and for every natural number $n$, $\mathcal{P}[n, F(n), F(n+1)]$ from [10, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv F(\$_1+1)$ is a subsequence of $F(\$_1)$ and $\|F(\$_1+1)\|$ is bounded and $F(\$_1 + 1)\#x(\$_1 + 1)$ is convergent. For every natural number $n$, $\mathcal{Q}[n]$ from [1, Sch. 2]. □

(24) There exists a sequence $F$ of (the carrier of DualSp $X)^{\mathbb{N}}$ and there exists a sequence $N$ of $\mathbb{N}^{\mathbb{N}}$ such that $F(0)$ is a subsequence of $f$ and $F(0)\#x(0)$ is convergent and $N(0)$ is an increasing sequence of $\mathbb{N}$ and $F(0) = f \cdot N(0)$ and for every natural number $k$, $F(k+1)$ is a subsequence of $F(k)$ and for every natural number $k$, $F(k+1)\#x(k+1)$ is convergent and for every natural number $k$, $F(k+1)\#x(k+1)$ is a subsequence of $F(k)\#x(k+1)$ and for every natural number $k$, $N(k+1)$ is an increasing sequence of $\mathbb{N}$ and for every natural number $k$, $F(k+1) = F(k) \cdot N(k+1)$.

PROOF: Consider $f_0$ being a sequence of DualSp $X$ such that $f_0$ is a subsequence of $f$ and $\|f_0\|$ is bounded and $f_0\#x(0)$ is convergent and subsequence of $f\#x(0)$. Consider $N_0$ being an increasing sequence of $\mathbb{N}$ such that $f_0 = f \cdot N_0$. Set $D_1 =$ (the carrier of DualSp $X)^{\mathbb{N}}$. Set $D_2 = \mathbb{N}^{\mathbb{N}}$. Reconsider $A = f_0$ as an element of $D_1$. Reconsider $B = N_0$ as an element of $D_2$. Define $\mathcal{P}$[natural number, sequence of DualSp $X$, sequence of $\mathbb{N}$, sequence of DualSp $X$, sequence of $\mathbb{N}$] $\equiv$ if $\|\$_2\|$ is bounded, then $\$_4$ is a subsequence of $\$_2$ and $\|\$_4\|$ is bounded and $\$_4\#x(\$_1+1)$ is convergent and subsequence of $\$_2\#x(\$_1+1)$ and $\$_5$ is an increasing sequence of $\mathbb{N}$ and $\$_4 = \$_2 \cdot \$_5$. For every natural number $n$ and for every element $z$ of $D_1$ and for every element $y$ of $D_2$, there exists an element $z_1$ of $D_1$ and there exists an element $y_1$ of $D_2$ such that $\mathcal{P}[n, z, y, z_1, y_1]$ by (22), [6, (8)]. Consider $F$ being a sequence of $D_1$, $N$ being a sequence of $D_2$ such that $F(0) = A$ and $N(0) = B$ and for every natural number $n$, $\mathcal{P}[n, F(n), N(n), F(n+1), N(n+1)]$ from [11, Sch. 3]. Define $\mathcal{Q}$[natural number] $\equiv F(\$_1 + 1)$ is a subsequence of $F(\$_1)$ and $\|F(\$_1 + 1)\|$ is bounded and $F(\$_1 + 1)\#x(\$_1 + 1)$ is convergent and subsequence of $F(\$_1)\#x(\$_1 + 1)$ and $N(\$_1 + 1)$ is an increasing sequence of $\mathbb{N}$ and $F(\$_1 + 1) = F(\$_1) \cdot N(\$_1 + 1)$. For every natural number $n$, $\mathcal{Q}[n]$ from [1, Sch. 2]. □

(25) There exists a sequence $M$ of DualSp $X$ such that

(i) $M$ is a subsequence of $f$, and

(ii) for every natural number $k$, $M\#x(k)$ is convergent.

PROOF: Consider $F$ being a sequence of (the carrier of DualSp $X)^{\mathbb{N}}$, $N$ being a sequence of $\mathbb{N}^{\mathbb{N}}$ such that $F(0)$ is a subsequence of $f$ and $F(0)\#x(0)$ is convergent and $N(0)$ is an increasing sequence of $\mathbb{N}$ and $F(0) = f \cdot N(0)$ and for every natural number $k$, $F(k+1)$ is a subsequence of $F(k)$ and for every natural number $k$, $F(k+1)\#x(k+1)$ is convergent and for every natural number $k$, $F(k+1)\#x(k+1)$ is a subsequence of $F(k)\#x(k+1)$ and for every natural number $k$, $N(k+1)$ is an increasing sequence of $\mathbb{N}$ and for every natural number $k$, $F(k+1) = F(k) \cdot N(k+1)$. Define $\mathcal{F}$(element of $\mathbb{N}$) $= F(\$_1)(\$_1)$. Consider $M$ being a function from $\mathbb{N}$ into

DualSp $X$ such that for every element $k$ of $\mathbb{N}$, $M(k) = \mathcal{F}(k)$ from [6, Sch. 4]. For every natural number $k$, $M(k) = F(k)(k)$. Set $D = \mathbb{N}^{\mathbb{N}}$. Reconsider $A = N(0)$ as an element of $D$. Define $\mathcal{P}$[natural number, sequence of $\mathbb{N}$, sequence of $\mathbb{N}$] $\equiv \$_3 = \$_2 \cdot N(\$_1 + 1)$. For every natural number $n$ and for every element $x$ of $D$, there exists an element $y$ of $D$ such that $\mathcal{P}[n, x, y]$ by [6, (8)]. Consider $J$ being a sequence of $D$ such that $J(0) = A$ and for every natural number $n$, $\mathcal{P}[n, J(n), J(n+1)]$ from [10, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv J(\$_1)$ is an increasing sequence of $\mathbb{N}$. For every natural number $n$ such that $\mathcal{Q}[n]$ holds $\mathcal{Q}[n+1]$. For every natural number $n$, $\mathcal{Q}[n]$ from [1, Sch. 2]. Define $\mathcal{R}$[natural number] $\equiv F(\$_1) = f \cdot J(\$_1)$. For every natural number $n$ such that $\mathcal{R}[n]$ holds $\mathcal{R}[n+1]$ by [34, (36)]. For every natural number $n$, $\mathcal{R}[n]$ from [1, Sch. 2]. Define $\mathcal{H}$(element of $\mathbb{N}$) $= J(\$_1)(\$_1)$. Consider $L$ being a function from $\mathbb{N}$ into $\mathbb{N}$ such that for every element $k$ of $\mathbb{N}$, $L(k) = \mathcal{H}(k)$ from [6, Sch. 4]. For every natural number $k$, $L(k) = J(k)(k)$. Reconsider $L_0 = L$ as a sequence of real numbers. For every natural number $k$, $L_0(k) < L_0(k+1)$ by [6, (7), (15)], [12, (14), (1)]. For every natural number $k$, $M(k) = (f \cdot L)(k)$ by [6, (15)]. For every natural number $k$, $M\#x(k)$ is convergent by [1, (6), (11)], [12, (14)], [30, (3)]. □

Now we state the propositions:

(26) Let us consider a real Banach space $X$, and a sequence $f$ of DualSp $X$. Suppose $X$ is separable and $\|f\|$ is bounded. Then there exists a sequence $f_0$ of DualSp $X$ such that $f_0$ is subsequence of $f$ and weakly* convergent.
Proof: Consider $x_0$ being a sequence of $X$ such that rng $x_0$ is dense. Consider $f_0$ being a sequence of DualSp $X$ such that $f_0$ is a subsequence of $f$ and for every natural number $n$, $f_0\#x_0(n)$ is convergent. For every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every natural number $n$, $|(f\#x)(n)| \leqslant K$ by [14, (3)], [17, (26)]. Set $T = \operatorname{rng} f_0$. Consider $N$ being an increasing sequence of $\mathbb{N}$ such that $f_0 = f \cdot N$. For every point $x$ of $X$, there exists a real number $K$ such that $0 \leqslant K$ and for every point $g$ of DualSp $X$ such that $g \in T$ holds $|g(x)| \leqslant K$ by [6, (15), (11)]. Consider $L$ being a real number such that $0 \leqslant L$ and for every point $g$ of DualSp $X$ such that $g \in T$ holds $\|g\| \leqslant L$. Set $M = L + 1$. For every Lipschitzian linear functional $g$ in $X$ such that $g \in T$ for every points $x$, $y$ of $X$, $|g(x) - g(y)| \leqslant M \cdot \|x - y\|$ by [31, (16)], [17, (26)]. For every point $x$ of $X$, $f_0\#x$ is convergent by [9, (8), (16)], [22, (6)], [16, (17)]. Define $\mathcal{X}$[element of the carrier of $X$, object] $\equiv \$_2 = \lim(f_0\#\$_1)$. For every element $x$ of the carrier of $X$, there exists an element $y$ of $\mathbb{R}$ such that $\mathcal{X}[x, y]$. Consider $f_1$ being a function from the carrier of $X$ into $\mathbb{R}$ such that for every element $x$ of the carrier of $X$, $\mathcal{X}[x, f_1(x)]$ from [6,

Sch. 3]. $f_1$ is additive by [13, (7)], [14, (6)]. $f_1$ is homogeneous by [13, (9)], [14, (8)]. Consider $M$ being a real number such that $0 < M$ and for every natural number $n$, $|\|f\|(n)| < M$. □

(27) Let us consider a real Banach space $X$, and a sequence $x$ of $X$. Suppose $X$ is reflexive and $\|x\|$ is bounded. Then there exists a sequence $x_0$ of $X$ such that $x_0$ is subsequence of $x$ and weakly convergent.

Proof: Set $L = \mathrm{ClNLin}(\mathrm{rng}\, x)$. For every object $z$ such that $z \in \mathrm{rng}\, x$ holds $z \in$ the carrier of $L$ by [32, (15)], [16, (4)]. □

(28) Let us consider a real Banach space $X$, and a non empty subset $X_1$ of $X$. Suppose $X$ is non trivial and reflexive. Then $X_1$ is weakly sequentially compact if and only if there exists a non empty subset $S$ of $\mathbb{R}$ such that $S = \{\|x\|$, where $x$ is a point of $X : x \in X_1\}$ and $S$ is upper bounded.

Proof: For every sequence $s_2$ of $X_1$, there exists a sequence $s_3$ of $X$ such that $s_3$ is subsequence of $s_2$ and weakly convergent and w-lim$(s_3) \in X$ by [6, (7)], (27). □

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# The Orthogonal Projection and the Riesz Representation Theorem[1]

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**Summary.** In this article, the orthogonal projection and the Riesz representation theorem are mainly formalized. In the first section, we defined the norm of elements on real Hilbert spaces, and defined Mizar functor `RUSp2RNSp`, real normed spaces as real Hilbert spaces. By this definition, we regarded sequences of real Hilbert spaces as sequences of real normed spaces, and proved some properties of real Hilbert spaces. Furthermore, we defined the continuity and the Lipschitz the continuity of functionals on real Hilbert spaces.

Referring to the article [15], we also defined some definitions on real Hilbert spaces and proved some theorems for defining dual spaces of real Hilbert spaces. As to the properties of all definitions, we proved that they are equivalent properties of functionals on real normed spaces. In Sec. 2, by the definitions [11], we showed properties of the orthogonal complement. Then we proved theorems on the orthogonal decomposition of elements of real Hilbert spaces. They are the last two theorems of existence and uniqueness. In the third and final section, we defined the kernel of linear functionals on real Hilbert spaces. By the last three theorems, we showed the Riesz representation theorem, existence, uniqueness, and the property of the norm of bounded linear functionals on real Hilbert spaces. We referred to [36], [9], [24] and [3] in the formalization.

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The notation and terminology used in this paper have been introduced in the following articles: [20], [21], [22], [35], [4], [16], [15], [27], [5], [6], [18], [25], [28], [17], [23], [2], [7], [33], [34], [30], [31], [12], [26], [10], [11], [13], [14], [32], and [8].

## 1. Preliminaries

Let $X$ be a real unitary space. The norm of $X$ yielding a function from the carrier of $X$ into $\mathbb{R}$ is defined by

(Def. 1) for every point $x$ of $X$, $it(x) = \|x\|$.

The real normed space of $X$ yielding a real normed space is defined by the term

(Def. 2) $\langle$the carrier of $X$, the zero of $X$, the addition of $X$, the external multiplication of $X$, the norm of $X\rangle$.

Now we state the propositions:

(1) Let us consider a real unitary space $X$, a point $x$ of $X$, and a point $x_1$ of the real normed space of $X$. If $x = x_1$, then $-x = -x_1$.

(2) Let us consider a real unitary space $X$, points $x, y$ of $X$, and points $x_1, y_1$ of the real normed space of $X$. If $x = x_1$ and $y = y_1$, then $x - y = x_1 - y_1$.

(3) Let us consider a real unitary space $X$, a point $x$ of $X$, and a point $x_1$ of the real normed space of $X$. Suppose $x = x_1$. Then $\|x\| = \|x_1\|$.

Let us consider a real unitary space $X$, a sequence $s_1$ of $X$, and a sequence $s_2$ of the real normed space of $X$. Now we state the propositions:

(4) If $s_1 = s_2$, then $s_1$ is convergent iff $s_2$ is convergent. The theorem is a consequence of (1).

(5) If $s_1 = s_2$ and $s_1$ is convergent, then $\lim s_1 = \lim s_2$. The theorem is a consequence of (4) and (1).

(6) If $s_1 = s_2$, then $s_2$ is Cauchy sequence by norm iff $s_1$ is Cauchy.
Proof: For every real number $r$ such that $r > 0$ there exists a natural number $k$ such that for every natural numbers $n$, $m$ such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_2(n) - s_2(m)\| < r$ by [22, (2)], (1). □

(7) Let us consider a real unitary space $X$. Then $X$ is complete if and only if the real normed space of $X$ is complete. The theorem is a consequence of (6) and (4).

Let $X$ be a real Hilbert space. Note that the real normed space of $X$ is complete.

Let $X$ be a real unitary space and $Y$ be a subset of $X$. We say that $Y$ is open if and only if

(Def. 3) there exists a subset $Z$ of the real normed space of $X$ such that $Z = Y$ and $Z$ is open.

We say that $Y$ is closed if and only if

(Def. 4) there exists a subset $Z$ of the real normed space of $X$ such that $Z = Y$ and $Z$ is closed.

Let us consider a real unitary space $X$ and a subset $Y$ of $X$. Now we state the propositions:

(8) $Y$ is closed if and only if for every sequence $s$ of $X$ such that $\operatorname{rng} s \subseteq Y$ and $s$ is convergent holds $\lim s \in Y$. The theorem is a consequence of (4) and (5).

(9) $Y$ is open if and only if $Y^{\mathrm{c}}$ is closed.

Let $X$ be a real unitary space, $f$ be a partial function from the carrier of $X$ to $\mathbb{R}$, and $x_0$ be a point of $X$. We say that $f$ is continuous in $x_0$ if and only if

(Def. 5) $x_0 \in \operatorname{dom} f$ and for every sequence $s_1$ of $X$ such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $f_* s_1$ is convergent and $f_{x_0} = \lim(f_* s_1)$.

Let $Y$ be a set. We say that $f$ is continuous on $Y$ if and only if

(Def. 6) $Y \subseteq \operatorname{dom} f$ and for every point $x_0$ of $X$ such that $x_0 \in Y$ holds $f \restriction Y$ is continuous in $x_0$.

Now we state the propositions:

(10) Let us consider a real unitary space $X$, a function $f$ from $X$ into $\mathbb{R}$, a function $g$ from the real normed space of $X$ into $\mathbb{R}$, a point $x_0$ of $X$, and a point $y_0$ of the real normed space of $X$. Suppose $f = g$ and $x_0 = y_0$. Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. The theorem is a consequence of (4) and (5).

(11) Let us consider a real unitary space $X$, a function $f$ from $X$ into $\mathbb{R}$, and a function $g$ from the real normed space of $X$ into $\mathbb{R}$. Suppose $f = g$. Then $f$ is continuous on the carrier of $X$ if and only if $g$ is continuous on the carrier of the real normed space of $X$. The theorem is a consequence of (10).

(12) Let us consider a real unitary space $X$, a point $w$ of $X$, and a function $f$ from $X$ into $\mathbb{R}$. Suppose for every point $v$ of $X$, $f(v) = (w|v)$. Then $f$ is continuous on the carrier of $X$.

Proof: Set $Y =$ the real normed space of $X$. Reconsider $g = f$ as a function from $Y$ into $\mathbb{R}$. For every point $y_0$ of $Y$ such that $y_0 \in$ the carrier of $Y$ holds $g \restriction$(the carrier of $Y$) is continuous in $y_0$ by [20, (28)], (2), (3), [20, (12), (29)]. $\square$

Let $X$ be a real unitary space, $Y$ be a set, and $f$ be a partial function from the carrier of $X$ to $\mathbb{R}$. We say that $f$ is Lipschitzian on $Y$ if and only if

(Def. 7) $Y \subseteq \operatorname{dom} f$ and there exists a real number $r$ such that $0 < r$ and for every points $x_1$, $x_2$ of $X$ such that $x_1, x_2 \in Y$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Now we state the propositions:

(13) Let us consider a real unitary space $X$, a function $f$ from $X$ into $\mathbb{R}$, and a function $g$ from the real normed space of $X$ into $\mathbb{R}$. Suppose $f = g$. Then $f$ is Lipschitzian on the carrier of $X$ if and only if $g$ is Lipschitzian on the carrier of the real normed space of $X$. The theorem is a consequence of (2) and (3).

(14) Let us consider a real unitary space $X$, and a function $f$ from $X$ into $\mathbb{R}$. Suppose $f$ is Lipschitzian on the carrier of $X$. Then $f$ is continuous on the carrier of $X$. The theorem is a consequence of (13) and (11).

(15) Let us consider a real unitary space $X$, and a linear functional $F$ in $X$. Suppose $F =$ (the carrier of $X$) $\longmapsto 0$. Then $F$ is Lipschitzian.

Let $X$ be a real unitary space. Let us observe that there exists a linear functional in $X$ which is Lipschitzian.

The bounded linear functionals $X$ yielding a subset of $\overline{X}$ is defined by

(Def. 8) for every set $x$, $x \in it$ iff $x$ is a Lipschitzian linear functional in $X$.

One can check that the bounded linear functionals $X$ is non empty and linearly closed.

Let $f$ be an object. The functor Bound2Lipschitz$(f, X)$ yielding a Lipschitzian linear functional in $X$ is defined by the term

(Def. 9) $f(\in$ the bounded linear functionals $X)$.

Let $u$ be a linear functional in $X$. The functor PreNorms$(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 10) $\{|u(t)|$, where $t$ is a vector of $X : \|t\| \leqslant 1\}$.

Let $g$ be a Lipschitzian linear functional in $X$. Let us observe that PreNorms$(g)$ is upper bounded.

The bounded linear functionals norm $X$ yielding a function from the bounded linear functionals $X$ into $\mathbb{R}$ is defined by

(Def. 11) for every object $x$ such that $x \in$ the bounded linear functionals $X$ holds $it(x) = \sup \text{PreNorms}(\text{Bound2Lipschitz}(x, X))$.

Let $f$ be a Lipschitzian linear functional in $X$.

One can check that Bound2Lipschitz$(f, X)$ reduces to $f$.

Now we state the proposition:

(16) Let us consider a real unitary space $X$, and a Lipschitzian linear functional $f$ in $X$.

Then (the bounded linear functionals norm $X)(f) = \sup \mathrm{PreNorms}(f)$.

Let $X$ be a real unitary space. The functor DualSp $X$ yielding a non empty normed structure is defined by the term

(Def. 12) $\langle$the bounded linear functionals $X$, Zero(the bounded linear functionals $X, \overline{X}$), Add(the bounded linear functionals $X, \overline{X}$), Mult(the bounded linear functionals $X, \overline{X}$), the bounded linear functionals norm $X\rangle$.

Now we state the propositions:

(17) Let us consider a real unitary space $X$, a point $f$ of DualSp $X$, and a Lipschitzian linear functional $g$ in $X$. Suppose $g = f$. Let us consider a vector $t$ of $X$. Then $|g(t)| \leqslant \|f\| \cdot \|t\|$. The theorem is a consequence of (16).

(18) Let us consider a real unitary space $X$, and a point $f$ of DualSp $X$. Then $0 \leqslant \|f\|$. The theorem is a consequence of (16).

(19) Let us consider a real unitary space $X$, a function $f$ from $X$ into $\mathbb{R}$, and a function $g$ from the real normed space of $X$ into $\mathbb{R}$. Suppose $f = g$. Then $f$ is additive and homogeneous if and only if $g$ is additive and homogeneous.

(20) Let us consider a real unitary space $X$, a linear functional $f$ in $X$, and a linear functional $g$ in the real normed space of $X$. If $f = g$, then $f$ is Lipschitzian iff $g$ is Lipschitzian.

PROOF: Set $Y$ = the real normed space of $X$. Consider $K$ being a real number such that $0 \leqslant K$ and for every point $y$ of $Y$, $|g(y)| \leqslant K \cdot \|y\|$. For every point $x$ of $X$, $|f(x)| \leqslant (K+1) \cdot \|x\|$ by [20, (28)]. □

(21) Let us consider a real unitary space $X$. Then the bounded linear functionals $X$ = the bounded linear functionals the real normed space of $X$. The theorem is a consequence of (19) and (20).

(22) Let us consider a real unitary space $X$, a linear functional $u$ in $X$, and a linear functional $v$ in the real normed space of $X$. If $u = v$, then $\mathrm{PreNorms}(u) = \mathrm{PreNorms}(v)$.

Let us consider a real unitary space $X$. Now we state the propositions:

(23) The bounded linear functionals norm $X$ = the bounded linear functionals norm the real normed space of $X$. The theorem is a consequence of (21) and (22).

(24) The linear functionals of $X$ = the linear functionals of the real normed space of $X$. The theorem is a consequence of (19).

(25) $\overline{X} = \overline{\alpha}$, where $\alpha$ is the real normed space of $X$. The theorem is a consequence of (24).

(26) DualSp $X$ = DualSp(the real normed space of $X$). The theorem is a consequence of (25), (21), and (23).

## 2. The Orthogonal Projection

Now we state the propositions:

(27) Let us consider a real unitary space $X$, and subspaces $M$, $N$ of $X$. Suppose the carrier of $M \subseteq$ the carrier of $N$. Then the carrier of Ort_Comp $N \subseteq$ the carrier of Ort_Comp $M$.

(28) Let us consider a real unitary space $X$, and a subspace $M$ of $X$. Then the carrier of $M \subseteq$ the carrier of Ort_Comp Ort_Comp $M$.

(29) Let us consider a real unitary space $X$, a subspace $M$ of $X$, and a point $x$ of $X$. Suppose $x \in$ (the carrier of $M$) $\cap$ (the carrier of Ort_Comp $M$). Then $x = 0_X$.

(30) Let us consider a real unitary space $X$, a subspace $M$ of $X$, and a non empty subset $N$ of $X$. Suppose $N =$ the carrier of Ort_Comp $M$. Then $N$ is linearly closed and closed.

Proof: For every sequence $s$ of $X$ such that rng $s \subseteq N$ and $s$ is convergent holds $\lim s \in N$ by [6, (4)], [20, (28)], [21, (19)], [20, (12), (29)]. □

(31) Let us consider a real Hilbert space $X$, a subspace $M$ of $X$, a subset $N$ of $X$, a point $x$ of $X$, and a real number $d$. Suppose $N =$ the carrier of $M$ and $N$ is closed and there exists a non empty subset $Y$ of $\mathbb{R}$ such that $Y = \{\|x - y\|$, where $y$ is a point of $X : y \in M\}$ and $d = \inf Y \geqslant 0$. Then there exists a point $x_0$ of $X$ such that

(i) $d = \|x - x_0\|$, and

(ii) $x_0 \in M$.

Proof: Consider $Y$ being a non empty subset of $\mathbb{R}$ such that $Y = \{\|x - y\|$, where $y$ is a point of $X : y \in M\}$ and $d = \inf Y \geqslant 0$. Reconsider $r_0 = 0$ as a real number. For every extended real $r$ such that $r \in Y$ holds $r_0 \leqslant r$ by [20, (28)]. Define $\mathcal{P}$[natural number, real number] $\equiv \$_2 \in Y$ and $\$_2 < d + (1/_{\$_1+1})$. For every element $n$ of $\mathbb{N}$, there exists an element $r$ of $\mathbb{R}$ such that $\mathcal{P}[n, r]$. Consider $S$ being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element $n$ of $\mathbb{N}$, $\mathcal{P}[n, S(n)]$ from [6, Sch. 3]. For every natural number $n$, $|S(n) - d| \leqslant 1/_{n+1}$. For every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|S(m) - d| < p$ by [14, (3)], [1, (16)]. Define $\mathcal{Q}$[natural number, point of $X$] $\equiv \$_2 \in M$ and $S(\$_1) = \|x - \$_2\|$. For every element $n$ of $\mathbb{N}$, there exists a point $v$ of $X$ such that $\mathcal{Q}[n, v]$. Consider $z$ being a function from $\mathbb{N}$ into the carrier of $X$ such that for every element $n$ of $\mathbb{N}$, $\mathcal{Q}[n, z(n)]$ from [6, Sch. 3]. For every natural number $n$, $z(n) \in M$ and $S(n) = \|x - z(n)\|$. Consider $z$ being a sequence of $X$ such that for every natural number $n$, $z(n) \in M$ and $S(n) = \|x - z(n)\|$. Reconsider $S_1 = S \cdot S$, $S_2 = S \cdot S$ as a sequence of real numbers. Reconsider $S_3 = 2 \cdot S_1$, $S_4 = 2 \cdot S_2$ as a sequence of real numbers. For every real number $e$ such that $0 < e$ there exists a natural number $k$ such that for every natural numbers $n$, $m$ such that $n \geqslant k$ and $m \geqslant k$ holds $|S_3(m) + S_4(n) - 4 \cdot (d \cdot d)| < e$ by [4, (56)]. For every real number $p$ such that $p > 0$ there exists a natural number $k$ such that for every natural numbers $n$, $m$ such that $n \geqslant k$ and $m \geqslant k$ holds $\|z(n) - z(m)\| < p$ by [31, (31), (33), (5)]. Consider $x_0$ being a point of $X$ such that for every real number $r$ such that $r > 0$ there exists a natural number $m$ such that for every natural number $n$ such that $n \geqslant m$ holds $\|z(n) - x_0\| < r$. For every object $y$ such that $y \in \operatorname{rng} z$ holds $y \in N$ by [6, (11)]. $\lim z \in N$. There exists a natural number $k_0$ such that for every natural number $n$ such that $k_0 \leqslant n$ holds $S(n) = \|z - x\|(n)$ by [31, (33)], [20, (31), (56)]. □

(32) Let us consider a real Hilbert space $X$, a subspace $M$ of $X$, points $x$, $x_0$ of $X$, and a real number $d$. Suppose $x_0 \in M$ and there exists a non empty subset $Y$ of $\mathbb{R}$ such that $Y = \{\|x - y\|$, where $y$ is a point of $X : y \in M\}$ and $d = \inf Y \geqslant 0$. Then $d = \|x - x_0\|$ if and only if for every point $w$ of $X$ such that $w \in M$ holds $w$, $x - x_0$ are orthogonal.

PROOF: Consider $Y$ being a non empty subset of $\mathbb{R}$ such that $Y = \{\|x - y\|$, where $y$ is a point of $X : y \in M\}$ and $d = \inf Y \geqslant 0$. Reconsider $r_0 = 0$ as a real number. For every extended real $r$ such that $r \in Y$ holds $r_0 \leqslant r$ by [20, (28)]. For every point $y_0$ of $X$ such that $y_0 \in M$ holds $d \leqslant \|x - y_0\|$. For every point $y$ of $X$ such that $y \in M$ holds $\|x - x_0\| \leqslant \|x - y\|$ by [10, (17)], [31, (5)], [11, (30)], [29, (26)]. For every real number $s$ such that $s \in Y$ holds $\|x - x_0\| \leqslant s$. □

(33) Let us consider a real Hilbert space $X$, a subspace $M$ of $X$, a subset $N$ of $X$, and a point $x$ of $X$. Suppose $N =$ the carrier of $M$ and $N$ is closed. Then there exist points $y$, $z$ of $X$ such that

(i) $y \in M$, and

(ii) $z \in \text{Ort\_Comp}\, M$, and

(iii) $x = y + z$.

PROOF: Set $Y = \{\|x - y\|$, where $y$ is a point of $X : y \in M\}$. $Y \subseteq \mathbb{R}$. Set $d = \inf Y$. For every real number $r$ such that $r \in Y$ holds $0 \leqslant r$ by [20, (28)]. Consider $x_0$ being a point of $X$ such that $d = \|x - x_0\|$ and $x_0 \in M$.

For every point $w$ of $X$ such that $w \in M$ holds $w$, $x - x_0$ are orthogonal. □

(34) Let us consider a real unitary space $X$, a subspace $M$ of $X$, a point $x$ of $X$, and points $y_1$, $y_2$, $z_1$, $z_2$ of $X$. Suppose $y_1, y_2 \in M$ and $z_1, z_2 \in \mathrm{Ort\_Comp}\, M$ and $x = y_1 + z_1$ and $x = y_2 + z_2$. Then

(i) $y_1 = y_2$, and

(ii) $z_1 = z_2$.

The theorem is a consequence of (29).

## 3. Riesz Representation Theorem

Now we state the proposition:

(35) Let us consider a real unitary space $X$, a linear functional $f$ in $X$, and a point $y$ of $X$. If for every point $x$ of $X$, $f(x) = (x|y)$, then $f$ is Lipschitzian.

PROOF: Reconsider $K = \|y\| + 1$ as a real number. For every point $x$ of $X$, $|f(x)| \leqslant K \cdot \|x\|$ by [20, (29), (28)]. □

Let $X$ be a real unitary space and $f$ be a linear functional in $X$. One can check that $f^{-1}(\{0\})$ is non empty.

Now we state the proposition:

(36) Let us consider a real unitary space $X$, and a function $f$ from $X$ into $\mathbb{R}$. Suppose $f$ is additive and homogeneous. Then $f^{-1}(\{0\})$ is linearly closed.

PROOF: Set $X_1 = f^{-1}(\{0\})$. For every points $v$, $u$ of $X$ such that $v, u \in X_1$ holds $v + u \in X_1$ by [6, (38)]. For every real number $r$ and for every point $v$ of $X$ such that $v \in X_1$ holds $r \cdot v \in X_1$ by [6, (38)]. □

Let $X$ be a real unitary space and $f$ be a linear functional in $X$. The null space of $f$ yielding a strict subspace of $X$ is defined by

(Def. 13) the carrier of $it = f^{-1}(\{0\})$.

Now we state the propositions:

(37) Let us consider a real unitary space $X$, and a linear functional $f$ in $X$. If $f$ is Lipschitzian, then $f^{-1}(\{0\})$ is closed.

PROOF: Set $Y = f^{-1}(\{0\})$. For every sequence $s$ of $X$ such that $\mathrm{rng}\, s \subseteq Y$ and $s$ is convergent holds $\lim s \in Y$ by [18, (19)], (14), [6, (4), (38)]. □

(38) Let us consider a real unitary space $V$, a subspace $W$ of $V$, and a vector $v$ of $V$. If $v \neq 0_V$, then if $v \in \mathrm{Ort\_Comp}\, W$, then $v \notin W$.

(39) Let us consider a real Hilbert space $X$, and a linear functional $f$ in $X$. Suppose $f$ is Lipschitzian. Then there exists a point $y$ of $X$ such that for

every point $x$ of $X$, $f(x) = (x|y)$. The theorem is a consequence of (33), (37), and (38).

(40) Let us consider a real unitary space $X$, a linear functional $f$ in $X$, and points $y_1$, $y_2$ of $X$. If for every point $x$ of $X$, $f(x) = (x|y_1)$ and $f(x) = (x|y_2)$, then $y_1 = y_2$.

(41) Let us consider a real Hilbert space $X$, a point $f$ of $\mathrm{DualSp}\, X$, and a Lipschitzian linear functional $g$ in $X$. Suppose $g = f$. Then there exists a point $y$ of $X$ such that

(i) for every point $x$ of $X$, $g(x) = (x|y)$, and

(ii) $\|f\| = \|y\|$.

Proof: Consider $y$ being a point of $X$ such that for every point $x$ of $X$, $g(x) = (x|y)$. $\|f\| \leqslant \|y\|$. $\|y\| \leqslant \|f\|$ by (18), [19, (4)], (17), [20, (28)]. □

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# Extended Real-Valued Double Sequence and Its Convergence[1]

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**Summary.** In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou's lemma and the monotone convergence theorem for double sequences.

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Keywords: double sequence; Fatou's lemma for double sequence; monotone convergence theorem for double sequence

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The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

## 1. Preliminaries

Let $X$ be a non empty set. One can verify that there exists a function from $X$ into $\mathbb{R}$ which is non-negative and non-positive and there exists a function from $X$ into $\overline{\mathbb{R}}$ which is without $-\infty$, without $+\infty$, non-negative, and non-positive and every function from $X$ into $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every function from $X$ into $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$ and there exists a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ which is without $-\infty$.

Let $f$ be a function from $X$ into $\overline{\mathbb{R}}$. Let us observe that the functor $-f$ yields a function from $X$ into $\overline{\mathbb{R}}$. Let $f$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Note that $-f$ is without $+\infty$.

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Let $f$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let us observe that $-f$ is without $-\infty$.

Let $f$ be a non-negative function from $X$ into $\overline{\mathbb{R}}$. Note that $-f$ is non-positive.

Let $f$ be a non-positive function from $X$ into $\overline{\mathbb{R}}$. Let us observe that $-f$ is non-negative.

Let $A$, $B$ be non empty sets and $f$ be a without $-\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. Let us observe that $f^{\mathrm{T}}$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. One can verify that $f^{\mathrm{T}}$ is without $+\infty$.

Let $f$ be a non-negative function from $A \times B$ into $\overline{\mathbb{R}}$. One can check that $f^{\mathrm{T}}$ is non-negative.

Let $f$ be a non-positive function from $A \times B$ into $\overline{\mathbb{R}}$. Note that $f^{\mathrm{T}}$ is non-positive.

Now we state the propositions:

(1) Let us consider a sequence $s$ of extended reals. Then $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$.

Proof: Define $\mathcal{Q}[\text{natural number}] \equiv$ $(-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $n$, $\mathcal{Q}[n]$ from [1, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}}$ $(\$_1) = (-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(2) Let us consider a non empty set $X$, and a partial function $f$ from $X$ to $\overline{\mathbb{R}}$. Then $--f = f$.

(3) Let us consider non empty sets $X$, $Y$, and a function $f$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(-f)^{\mathrm{T}} = -f^{\mathrm{T}}$.

Let $s$ be a non-negative sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-negative.

Let $s$ be a non-positive sequence of extended reals. Let us observe that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-positive.

Now we state the propositions:

(4) Let us consider a non-negative sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv s(\$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

(5) Let us consider a non-positive sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \geqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$. The theorem is a consequence of (4), (1), and (2).

(6) Let us consider a non empty set $X$. Then every without $-\infty$, without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ is a function from $X$ into $\mathbb{R}$.

Let $X$ be a non empty set and $f_1$, $f_2$ be without $-\infty$ functions from $X$ into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$, $f_2$ be without $+\infty$ functions from $X$ into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$ and $f_2$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let us observe that the functor $f_1 - f_2$ yields a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ and $f_2$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(7) Let us consider a non empty set $X$, an element $x$ of $X$, and functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then
   (i) if $f_1$ is without $-\infty$ and $f_2$ is without $-\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
   (ii) if $f_1$ is without $+\infty$ and $f_2$ is without $+\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
   (iii) if $f_1$ is without $-\infty$ and $f_2$ is without $+\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$, and
   (iv) if $f_1$ is without $+\infty$ and $f_2$ is without $-\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

(8) Let us consider a non empty set $X$, and without $-\infty$ functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then
   (i) $f_1 + f_2 = f_1 - -f_2$, and
   (ii) $-(f_1 + f_2) = -f_1 - f_2$.

   The theorem is a consequence of (7).

(9) Let us consider a non empty set $X$, and without $+\infty$ functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then
   (i) $f_1 + f_2 = f_1 - -f_2$, and
   (ii) $-(f_1 + f_2) = -f_1 - f_2$.

   The theorem is a consequence of (7).

(10) Let us consider a non empty set $X$, a without $-\infty$ function $f_1$ from $X$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then
   (i) $f_1 - f_2 = f_1 + -f_2$, and
   (ii) $f_2 - f_1 = f_2 + -f_1$, and
   (iii) $-(f_1 - f_2) = -f_1 + f_2$, and

(iv) $-(f_2 - f_1) = -f_2 + f_1$.

The theorem is a consequence of (8), (2), and (9).

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and $n$, $m$ be natural numbers. One can check that the functor $f(n, m)$ yields an element of $\overline{\mathbb{R}}$. Now we state the propositions:

(11) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(12) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(13) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(f_1 - f_2)(n, m) = f_1(n, m) - f_2(n, m)$, and

(ii) $(f_2 - f_1)(n, m) = f_2(n, m) - f_1(n, m)$.

The theorem is a consequence of (7).

(14) Let us consider non empty sets $X$, $Y$, and without $-\infty$ functions $f_1$, $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(15) Let us consider non empty sets $X$, $Y$, and without $+\infty$ functions $f_1$, $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(16) Let us consider non empty sets $X$, $Y$, a without $-\infty$ function $f_1$ from $X \times Y$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then

(i) $(f_1 - f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} - {f_2}^{\mathrm{T}}$, and

(ii) $(f_2 - f_1)^{\mathrm{T}} = {f_2}^{\mathrm{T}} - {f_1}^{\mathrm{T}}$.

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to $+\infty$ is also convergent and every sequence of extended reals which is convergent to $-\infty$ is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without $-\infty$ sequence of extended reals which is convergent and there exists a without $+\infty$ sequence of extended reals which is convergent.

Now we state the proposition:

(17) Let us consider a convergent sequence $s$ of extended reals. Then

(i) $s$ is convergent to a finite limit iff $-s$ is convergent to a finite limit, and

(ii) $s$ is convergent to $+\infty$ iff $-s$ is convergent to $-\infty$, and

(iii) $s$ is convergent to $-\infty$ iff $-s$ is convergent to $+\infty$, and

(iv) $-s$ is convergent, and

(v) $\lim(-s) = -\lim s$.

The theorem is a consequence of (2).

Let us consider without $-\infty$ sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(18) Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $+\infty$. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

(19) Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Now we state the proposition:

(20) Let us consider without $+\infty$ sequences $s_1$, $s_2$ of extended reals. Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Let us consider without $-\infty$ sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(21) Suppose $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to $-\infty$. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(22) Suppose $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(23) Suppose $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to a finite limit and convergent, and

(ii) $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without $+\infty$ sequences $s_1$, $s_2$ of extended reals. Then

(i) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $+\infty$, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(ii) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(iii) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to $-\infty$, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(iv) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(v) if $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to a finite limit and convergent and $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without $-\infty$ sequence $s_1$ of extended reals, and a without $+\infty$ sequence $s_2$ of extended reals. Then

(i) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $-\infty$, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(ii) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(iii) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to $-\infty$ and convergent and $s_2 - s_1$ is convergent to $+\infty$ and convergent and $\lim(s_1 - s_2) = -\infty$ and $\lim(s_2 - s_1) = +\infty$, and

(iv) if $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to a finite limit and convergent and $s_2 - s_1$ is convergent to a finite limit and convergent and $\lim(s_1 - s_2) = \lim s_1 - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - \lim s_1$.

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

## 2. Subsequences of Convergent Extended Real-Valued Sequences

Let us consider sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(26) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to a finite limit. Then

(i) $s_2$ is convergent to a finite limit, and

(ii) $\lim s_1 = \lim s_2$.

PROOF: Consider $g$ being a real number such that $\lim s_1 = g$ and for every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|s_1(m) - \lim s_1| < p$ and $s_1$ is convergent to a finite limit. Reconsider $L = \lim s_1$ as an extended real number. There exists a real number $g$ such that for every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|(s_2(m) - g$ **qua** extended real$)| < p$ by [19, (14)], [7, (15)]. For every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|s_2(m) - L| < p$ by [19, (14)], [7, (15)]. □

(27) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to $+\infty$. Then

(i) $s_2$ is convergent to $+\infty$, and

(ii) $\lim s_2 = +\infty$.

(28) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to $-\infty$. Then

(i) $s_2$ is convergent to $-\infty$, and

(ii) $\lim s_2 = -\infty$.

## 3. Convergency for Extended Real-Valued Double Sequences

Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(29) Suppose the lim in the first coordinate of $R$ is convergent. Then the first coordinate major iterated lim of $R$ = lim(the lim in the first coordinate of $R$).

(30) Suppose the lim in the second coordinate of $R$ is convergent. Then the second coordinate major iterated lim of $R$ = lim(the lim in the second coordinate of $R$).

Let $E$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $E$ is P-convergent to a finite limit if and only if

(Def. 1) there exists a real number $p$ such that for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|E(n,m) - (p$ **qua** extended real$)| < e$.

We say that $E$ is P-convergent to $+\infty$ if and only if

(Def. 2) for every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $g \leqslant E(n,m)$.

We say that $E$ is P-convergent to $-\infty$ if and only if

(Def. 3) for every real number $g$ such that $g < 0$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $E(n,m) \leqslant g$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $f$ is convergent in the first coordinate to $+\infty$ if and only if

(Def. 4) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f, m)$ is convergent to $+\infty$.

We say that $f$ is convergent in the first coordinate to $-\infty$ if and only if

(Def. 5) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f, m)$ is convergent to $-\infty$.

We say that $f$ is convergent in the first coordinate to a finite limit if and only if

(Def. 6) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f, m)$ is convergent to a finite limit.

We say that $f$ is convergent in the first coordinate if and only if

(Def. 7) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f, m)$ is convergent.

We say that $f$ is convergent in the second coordinate to $+\infty$ if and only if

(Def. 8) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f, m)$ is convergent to $+\infty$.

We say that $f$ is convergent in the second coordinate to $-\infty$ if and only if

(Def. 9) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f, m)$ is convergent to $-\infty$.

We say that $f$ is convergent in the second coordinate to a finite limit if and only if

(Def. 10) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f, m)$ is convergent to a finite limit.

We say that $f$ is convergent in the second coordinate if and only if

(Def. 11) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f, m)$ is convergent.

Now we state the propositions:

(31) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) if $f$ is convergent in the first coordinate to $+\infty$ or convergent in the first coordinate to $-\infty$ or convergent in the first coordinate to a finite limit, then $f$ is convergent in the first coordinate, and

(ii) if $f$ is convergent in the second coordinate to $+\infty$ or convergent in the second coordinate to $-\infty$ or convergent in the second coordinate to a finite limit, then $f$ is convergent in the second coordinate.

(32) Let us consider non empty sets $X$, $Y$, $Z$, a function $F$ from $X \times Y$ into $Z$, and an element $x$ of $X$. Then $\mathrm{curry}(F, x) = \mathrm{curry}'(F^{\mathrm{T}}, x)$.

(33) Let us consider non empty sets $X$, $Y$, $Z$, a function $F$ from $X \times Y$ into $Z$, and an element $y$ of $Y$. Then $\mathrm{curry}'(F, y) = \mathrm{curry}(F^{\mathrm{T}}, y)$.

(34) Let us consider non empty sets $X$, $Y$, a function $F$ from $X \times Y$ into $\overline{\mathbb{R}}$, and an element $x$ of $X$. Then $\mathrm{curry}(-F, x) = -\mathrm{curry}(F, x)$.

(35) Let us consider non empty sets $X$, $Y$, a function $F$ from $X \times Y$ into $\overline{\mathbb{R}}$, and an element $y$ of $Y$. Then $\mathrm{curry}'(-F, y) = -\mathrm{curry}'(F, y)$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(36) (i) $f$ is convergent in the first coordinate to $+\infty$ iff $f^{\mathrm{T}}$ is convergent in the second coordinate to $+\infty$, and

(ii) $f$ is convergent in the second coordinate to $+\infty$ iff $f^{\mathrm{T}}$ is convergent in the first coordinate to $+\infty$, and

(iii) $f$ is convergent in the first coordinate to $-\infty$ iff $f^{\mathrm{T}}$ is convergent in the second coordinate to $-\infty$, and

(iv) $f$ is convergent in the second coordinate to $-\infty$ iff $f^{\mathrm{T}}$ is convergent in the first coordinate to $-\infty$, and

(v) $f$ is convergent in the first coordinate to a finite limit iff $f^{\mathrm{T}}$ is convergent in the second coordinate to a finite limit, and

(vi) $f$ is convergent in the second coordinate to a finite limit iff $f^{\mathrm{T}}$ is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i) $f$ is convergent in the first coordinate to $+\infty$ iff $-f$ is convergent in the first coordinate to $-\infty$, and

(ii) $f$ is convergent in the first coordinate to $-\infty$ iff $-f$ is convergent in the first coordinate to $+\infty$, and

(iii) $f$ is convergent in the first coordinate to a finite limit iff $-f$ is convergent in the first coordinate to a finite limit, and

(iv) $f$ is convergent in the second coordinate to $+\infty$ iff $-f$ is convergent in the second coordinate to $-\infty$, and

(v) $f$ is convergent in the second coordinate to $-\infty$ iff $-f$ is convergent in the second coordinate to $+\infty$, and

(vi) $f$ is convergent in the second coordinate to a finite limit iff $-f$ is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functors: the lim in the first coordinate of $f$ and the lim in the second coordinate of $f$ yielding sequences of extended reals are defined by conditions

(Def. 12) for every element $m$ of $\mathbb{N}$, the lim in the first coordinate of $f(m) = \lim \mathrm{curry}'(f, m)$,

(Def. 13) for every element $n$ of $\mathbb{N}$, the lim in the second coordinate of $f(n) = \lim \mathrm{curry}(f, n)$,

respectively. Now we state the proposition:

(38) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the lim in the first coordinate of $f =$ the lim in the second coordinate of $f^{\mathrm{T}}$, and

(ii) the lim in the second coordinate of $f =$ the lim in the first coordinate of $f^{\mathrm{T}}$.

The theorem is a consequence of (33) and (32).

Let $X$, $Y$ be non empty sets, $F$ be a without $+\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$, and $x$ be an element of $X$. Let us observe that $\mathrm{curry}(F, x)$ is without $+\infty$.

Let $y$ be an element of $Y$. One can verify that $\mathrm{curry}'(F, y)$ is without $+\infty$.

Let $F$ be a without $-\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$ and $x$ be an element of $X$. Let us note that $\mathrm{curry}(F, x)$ is without $-\infty$.

Let $y$ be an element of $Y$. Observe that $\mathrm{curry}'(F, y)$ is without $-\infty$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The partial sums in the second coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every natural numbers $n$, $m$, $it(n, 0) = f(n, 0)$ and $it(n, m+1) = it(n, m) + f(n, m+1)$.

The partial sums in the first coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 15) for every natural numbers $n$, $m$, $it(0,m) = f(0,m)$ and $it(n+1,m) = it(n,m) + f(n+1,m)$.

Let $f$ be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the second coordinate of $f$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the second coordinate of $f$ is without $+\infty$.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the second coordinate of $f$ is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the second coordinate of $f$ is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the first coordinate of $f$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the first coordinate of $f$ is without $+\infty$.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the first coordinate of $f$ is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the first coordinate of $f$ is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by the term

(Def. 16) the partial sums in the second coordinate of the partial sums in the first coordinate of $f$.

Now we state the propositions:

(39) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) (the partial sums in the first coordinate of $f)(n,m) =$ (the partial sums in the second coordinate of $f^{\mathrm{T}})(m,n)$, and

(ii) (the partial sums in the second coordinate of $f)(n,m) =$ (the partial sums in the first coordinate of $f^{\mathrm{T}})(m,n)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $f)(\$_1, m) =$ (the partial sums in the second coordinate of $f^{\mathrm{T}})(m, \$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(n, \$_1) =$ (the partial sums in the first coordinate of $f^{\mathrm{T}})(\$_1, n)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(40) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) (the partial sums in the first coordinate of $f)^{\mathrm{T}}$ = the partial sums in the second coordinate of $f^{\mathrm{T}}$, and

(ii) (the partial sums in the second coordinate of $f)^{\mathrm{T}}$ = the partial sums in the first coordinate of $f^{\mathrm{T}}$.

The theorem is a consequence of (39).

(41) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an extended real-valued function $g$, and a natural number $n$. Suppose for every natural number $k$, (the partial sums in the first coordinate of $f)(n, k) = g(k)$. Then

(i) for every natural number $k$, $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$, and

(ii) (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum g$.

(42) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $-f$ = $-$(the partial sums in the second coordinate of $f$), and

(ii) the partial sums in the first coordinate of $-f$ = $-$(the partial sums in the first coordinate of $f$).

Proof: For every element $z$ of $\mathbb{N} \times \mathbb{N}$, ($-$(the partial sums in the second coordinate of $f))(z)$ = (the partial sums in the second coordinate of $-f)(z)$ by [9, (87)]. For every element $z$ of $\mathbb{N} \times \mathbb{N}$, ($-$(the partial sums in the first coordinate of $f))(z)$ = (the partial sums in the first coordinate of $-f)(z)$ by [9, (87)]. □

(43) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and elements $m$, $n$ of $\mathbb{N}$. Then

(i) (the partial sums in the first coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and

(ii) (the partial sums in the second coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $f)(\$_1, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(m, \$_1) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(44) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of $f_1$)+(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of $f_1$) + (the partial sums in the first coordinate of $f_2$).

The theorem is a consequence of (11).

(45) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of $f_1$)+(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of $f_1$) + (the partial sums in the first coordinate of $f_2$).

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

(46) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 - f_2 =$ (the partial sums in the second coordinate of $f_1$)−(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 - f_2 =$ (the partial sums in the first coordinate of $f_1$) − (the partial sums in the first coordinate of $f_2$), and

(iii) the partial sums in the second coordinate of $f_2 - f_1 =$ (the partial sums in the second coordinate of $f_2$)−(the partial sums in the second coordinate of $f_1$), and

(iv) the partial sums in the first coordinate of $f_2 - f_1 =$ (the partial sums in the first coordinate of $f_2$) − (the partial sums in the first coordinate of $f_1$).

The theorem is a consequence of (10), (44), (42), and (45).

(47) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$ (the partial sums in the second coordinate of $f$)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m+1)$ = (the partial sums in the first coordinate of $f$)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m)$.

PROOF: Set $R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$. Set $C_1$ = the partial sums in the first coordinate of the partial sums in the second coordinate of $f$. Set $R_2$ = the partial sums in the first coordinate of $f$. Set $C_2$ = the partial sums in the second coordinate of $f$. Define $\mathcal{P}$[natural number] $\equiv R_1(n+1, \$_1) = C_2(n+1, \$_1) + R_1(n, \$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv C_1(\$_1, m+1) = R_2(\$_1, m+1) + C_1(\$_1, m)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(48) Let us consider a without $+\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n+1, m)$ = (the partial sums in the second coordinate of $f$)$(n+1, m)$ + $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m+1)$ = (the partial sums in the first coordinate of $f$)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m)$.

The theorem is a consequence of (2), (42), and (47).

(49) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $f$ is without $-\infty$ or without $+\infty$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$ = the partial sums in the first coordinate of the partial sums in the second coordinate of $f$.

(50) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (44).

(51) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (45).

(52) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) $(\sum_{\alpha=0}^{\kappa} (f_1 - f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) $(\sum_{\alpha=0}^{\kappa} (f_2 - f_1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$.

The theorem is a consequence of (46).

(53) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $k$ of $\mathbb{N}$. Then

(i) curry′(the partial sums in the first coordinate of $f, k$) = $(\sum_{\alpha=0}^{\kappa}(\text{curry}'(f,k))(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) curry(the partial sums in the second coordinate of $f, k$) = $(\sum_{\alpha=0}^{\kappa}(\text{curry}(f,k))(\alpha))_{\kappa\in\mathbb{N}}$.

The theorem is a consequence of (43).

(54) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $f$ is without $-\infty$ or without $+\infty$. Then

(i) curry$((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}, 0)$ = curry(the partial sums in the second coordinate of $f, 0$), and

(ii) curry′$((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}, 0)$ = curry′(the partial sums in the first coordinate of $f, 0$).

(55) Let us consider non empty sets $C$, $D$, without $-\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then curry$(F_1 + F_2, c)$ = curry$(F_1, c)$ + curry$(F_2, c)$. The theorem is a consequence of (7).

(56) Let us consider non empty sets $C$, $D$, without $-\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then curry′$(F_1 + F_2, d)$ = curry′$(F_1, d)$ + curry′$(F_2, d)$. The theorem is a consequence of (7).

(57) Let us consider non empty sets $C$, $D$, without $+\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then curry$(F_1 + F_2, c)$ = curry$(F_1, c)$ + curry$(F_2, c)$. The theorem is a consequence of (7).

(58) Let us consider non empty sets $C$, $D$, without $+\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then curry′$(F_1 + F_2, d)$ = curry′$(F_1, d)$ + curry′$(F_2, d)$. The theorem is a consequence of (7).

(59) Let us consider non empty sets $C$, $D$, a without $-\infty$ function $F_1$ from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then

(i) curry$(F_1 - F_2, c)$ = curry$(F_1, c)$ − curry$(F_2, c)$, and

(ii) curry$(F_2 - F_1, c)$ = curry$(F_2, c)$ − curry$(F_1, c)$.

The theorem is a consequence of (7).

(60) Let us consider non empty sets $C$, $D$, a without $-\infty$ function $F_1$ from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then

(i) curry′$(F_1 - F_2, d)$ = curry′$(F_1, d)$ − curry′$(F_2, d)$, and

(ii) curry′$(F_2 - F_1, d)$ = curry′$(F_2, d)$ − curry′$(F_1, d)$.

The theorem is a consequence of (7).

## 4. Non-Negative Extended Real-Valued Double Sequences

Now we state the propositions:

(61) Let us consider a non-negative sequence $s$ of extended reals. Suppose $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is not convergent to $+\infty$. Let us consider a natural number $n$. Then $s(n)$ is a real number.

(62) Let us consider a non-negative sequence $s$ of extended reals. Suppose $s$ is non-decreasing. Then $s$ is convergent to $+\infty$ or convergent to a finite limit.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and $n$ be an element of $\mathbb{N}$. Let us observe that curry$(f, n)$ is non-negative and curry$'(f, n)$ is non-negative.

Now we state the propositions:

(63) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $m$ of $\mathbb{N}$. Then curry(the partial sums in the second coordinate of $f, m$) is non-decreasing.

Proof: Set $P =$ curry(the partial sums in the second coordinate of $f, m$). For every natural numbers $n$, $j$ such that $j \leqslant n$ holds $P(j) \leqslant P(n)$ by [4, (51)], [1, (13), (20)]. □

(64) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $n$ of $\mathbb{N}$. Then curry$'$(the partial sums in the first coordinate of $f, n$) is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and $m$ be an element of $\mathbb{N}$. One can check that curry(the partial sums in the second coordinate of $f, m$) is non-decreasing and curry$'$(the partial sums in the first coordinate of $f, m$) is non-decreasing.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(65) (i) if $f$ is convergent in the first coordinate, then the lim in the first coordinate of $f$ is non-negative, and

(ii) if $f$ is convergent in the second coordinate, then the lim in the second coordinate of $f$ is non-negative.

(66) (i) the partial sums in the first coordinate of $f$ is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of $f$ is convergent in the second coordinate.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an element $m$ of $\mathbb{N}$, and a natural number $n$.

Let us assume that curry$'$(the partial sums in the first coordinate of $f, m$) is not convergent to $+\infty$. Now we state the propositions:

(67) $f(n, m)$ is a real number.

(68) $f(m, n)$ is a real number.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers $n$, $m$. Now we state the propositions:

(69) Suppose for every natural number $i$ such that $i \leqslant n$ holds $f(i, m)$ is a real number. Then (the partial sums in the first coordinate of $f)(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of $f)(\$_1, m) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(70) Suppose for every natural number $i$ such that $i \leqslant m$ holds $f(n, i)$ is a real number. Then (the partial sums in the second coordinate of $f)(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of $f)(n, \$_1) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Now we state the proposition:

(71) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to $-\infty$. Then there exists an element $m$ of $\mathbb{N}$ such that curry′(the partial sums in the first coordinate of $f, m)$ is convergent to $-\infty$. The theorem is a consequence of (54).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a natural number $m$. Now we state the propositions:

(72) for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds curry(the partial sums in the second coordinate of $f, k)$ is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry(the partial sums in the second coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(73) for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds curry′(the partial sums in the first coordinate of $f, k)$ is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry′(the partial sums in the first coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(74) $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate

of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (72), (65), and (4).

(75) $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry$'$(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

(76) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) (the partial sums in the first coordinate of $f)(n, m) \geqslant f(n, m)$, and

(ii) (the partial sums in the second coordinate of $f)(n, m) \geqslant f(n, m)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of $f)(\$_1, m) \geqslant f(\$_1, m)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of $f)(n, \$_1) \geqslant f(n, \$_1)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an element $m$ of $\mathbb{N}$. Now we state the propositions:

(77) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate of $f, k)$ is convergent to $+\infty$. Then

(i) curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m)$ is convergent to $+\infty$, and

(ii) lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m) = +\infty$.

Proof: For every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural number $n$ such that $N \leqslant n$ holds $g \leqslant$ (curry(the par- tial sums in the second coordinate of the partial sums in the first coordinate of $f, m))(n)$ by [26, (7)], (76). □

(78) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry$'$(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. Then

(i) curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m)$ is convergent to $+\infty$, and

(ii) lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m) = +\infty$.

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m$).

Proof: The partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. Define $\mathcal{P}$[natural number] $\equiv$ for every element $k$ of $\mathbb{N}$ such that $k \leqslant \$_1$ holds $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(k) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, k$). For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [1, (13)], [14, (7)], (47), [4, (51)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(81) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of $f$ is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m$). The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence $s$ of extended reals. Now we state the propositions:

(83) Suppose for every element $m$ of $\mathbb{N}$, $s(m) = \liminf \mathrm{curry}'(f, m)$. Then $\sum s \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of $f$).

Proof: For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$

such that $n \geqslant N$ holds (the inferior realsequence curry$'(f, m))(N) \leqslant f(n, m)$ by [26, (7), (8)]. Define $\mathcal{F}$(element of $\mathbb{N}$) = the inferior realsequence curry$'(f, \$_1)$. Define $\mathcal{G}$(element of $\mathbb{N}$, element of $\mathbb{N}$) = (the inferior realsequence curry$'(f, \$_2))(\$_1)$. Consider $g$ being a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$ and for every element $m$ of $\mathbb{N}$, $g(n, m) = \mathcal{G}(n, m)$ from [5, Sch. 4]. For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the partial sums in the second coordinate of $f)(n, m)$. For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the inferior realsequence the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37), (23)]. Define $\mathcal{Q}$[natural number] $\equiv$ for every element $m$ of $\mathbb{N}$ such that $m = \$_1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \text{curry}'$(the partial sums in the second coordinate of $g, m)$. For every element $m$ of $\mathbb{N}$, curry$'$(the partial sums in the second coordinate of $g, m)$ is convergent by [26, (7), (37)]. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [26, (37)], [4, (51), (52)], [14, (11)]. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. For every natural number $m$, $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of $f$) by [26, (37), (38)]. For every object $m$ such that $m \in \operatorname{dom} s$ holds $0 \leqslant s(m)$ by [4, (51), (52)], [26, (23)]. □

(84) Suppose for every element $m$ of $\mathbb{N}$, $s(m) = \liminf \text{curry}(f, m)$. Then $\sum s \leqslant \liminf$(the lim in the first coordinate of the partial sums in the first coordinate of $f$). The theorem is a consequence of (32), (83), (38), and (40).

Now we state the proposition:

(85) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a sequence $s$ of extended reals, and natural numbers $n$, $m$. Then

- (i) if for every natural numbers $i$, $j$, $f(i, j) \leqslant s(i)$, then (the partial sums in the first coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$, and
- (ii) if for every natural numbers $i$, $j$, $f(i, j) \leqslant s(j)$, then (the partial sums in the second coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(n, \$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Let us consider a sequence $s$ of extended reals and an extended real number $r$. Now we state the propositions:

(86) If for every natural number $n$, $s(n) \leqslant r$, then $\limsup s \leqslant r$.
PROOF: Define $\mathcal{F}$(element of $\mathbb{N}$) $= r$. Consider $f$ being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number $n$, $f(n) = r$. For every natural number $n$, $s(n) \leqslant f(n)$. □

(87) If for every natural number $n$, $r \leqslant s(n)$, then $r \leqslant \liminf s$.
PROOF: Define $\mathcal{F}$(element of $\mathbb{N}$) $= r$. Consider $f$ being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number $n$, $f(n) = r$. For every natural number $n$, $f(n) \leqslant s(n)$. □

Now we state the proposition:

(88) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) for every natural numbers $i_1, i_2, j$ such that $i_1 \leqslant i_2$ holds (the partial sums in the first coordinate of $f)(i_1, j) \leqslant$ (the partial sums in the first coordinate of $f)(i_2, j)$, and

(ii) for every natural numbers $i, j_1, j_2$ such that $j_1 \leqslant j_2$ holds (the partial sums in the second coordinate of $f)(i, j_1) \leqslant$ (the partial sums in the second coordinate of $f)(i, j_2)$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers $i$, $j$, $k$. Now we state the propositions:

(89) Suppose for every element $m$ of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the second coordinate of $f)(i, k) \leqslant$ (the partial sums in the second coordinate of $f)(j, k)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(i, \$_1) \leqslant$ (the partial sums in the second coordinate of $f)(j, \$_1)$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [26, (7)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(90) Suppose for every element $n$ of $\mathbb{N}$, $\text{curry}(f, n)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the first coordinate of $f)(k, i) \leqslant$ (the partial sums in the first coordinate of $f)(k, j)$. The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence $s$ of extended reals. Now we state the propositions:

(91) Suppose for every element $m$ of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}'(f, m)$. Then

(i) the lim in the second coordinate of the partial sums in the second coordinate of $f$ is non-decreasing, and

(ii) $\sum s = \lim$(the lim in the second coordinate of the partial sums in the second coordinate of $f$).

PROOF: $\sum s \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of $f$). For every natural numbers $n$, $m$, $f(n,m) \leqslant s(m)$ by [26, (37)], [6, (3)]. For every natural numbers $n$, $m$ such that $m \leqslant n$ holds (the lim in the second coordinate of the partial sums in the second coordinate of $f)(m) \leqslant$ (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37)], (89), [26, (38)]. For every natural number $n$, (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n) \leqslant \sum s$ by [26, (37)], [4, (39)], (87), [26, (41)]. $\limsup$(the lim in the second coordinate of the partial sums in the second coordinate of $f) \leqslant \sum s$. □

(92) Suppose for every element $m$ of $\mathbb{N}$, curry$(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}(f, m)$. Then

(i) the lim in the first coordinate of the partial sums in the first coordinate of $f$ is non-decreasing, and

(ii) $\sum s = \lim$(the lim in the first coordinate of the partial sums in the first coordinate of $f$).

The theorem is a consequence of (32), (91), (33), and (40).

## 5. Pringsheim Sense Convergence for Extended Real-Valued Double Sequences

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(93) If $f$ is P-convergent to $+\infty$, then $f$ is not P-convergent to $-\infty$ and $f$ is not P-convergent to a finite limit.

(94) If $f$ is P-convergent to $-\infty$, then $f$ is not P-convergent to $+\infty$ and $f$ is not P-convergent to a finite limit.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $f$ is P-convergent if and only if

(Def. 17) $f$ is P-convergent to a finite limit or P-convergent to $+\infty$ or P-convergent to $-\infty$.

Assume $f$ is P-convergent. The functor P-lim $f$ yielding an extended real is defined by

(Def. 18) there exists a real number $p$ such that $it = p$ and for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|f(n,m) - it| < e$

and $f$ is P-convergent to a finite limit or $it = +\infty$ and $f$ is P-convergent to $+\infty$ or $it = -\infty$ and $f$ is P-convergent to $-\infty$.

Now we state the propositions:

(95) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number $r$. Suppose for every natural numbers $n$, $m$, $f(n, m) = r$. Then

(i) $f$ is P-convergent to a finite limit, and

(ii) P-lim $f = r$.

(96) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f(n_1, m_1) \leqslant f(n_2, m_2)$. Then

(i) $f$ is P-convergent, and

(ii) P-lim $f = \sup \operatorname{rng} f$.

(97) Let us consider functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n$, $m$, $f_1(n, m) \leqslant f_2(n, m)$. Then $\sup \operatorname{rng} f_1 \leqslant \sup \operatorname{rng} f_2$.

(98) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $f(n, m) \leqslant \sup \operatorname{rng} f$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an extended real number $K$. Now we state the propositions:

(99) If for every natural numbers $n$, $m$, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f \leqslant K$.

(100) If $K \neq +\infty$ and for every natural numbers $n$, $m$, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f < +\infty$.

Now we state the propositions:

(101) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\sup \operatorname{rng} f \neq +\infty$ if and only if there exists a real number $K$ such that $0 < K$ and for every natural numbers $n$, $m$, $f(n, m) \leqslant K$.

(102) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an extended real $c$. Suppose for every natural numbers $n$, $m$, $f(n, m) = c$. Then

(i) $f$ is P-convergent, and

(ii) P-lim $f = c$, and

(iii) P-lim $f = \sup \operatorname{rng} f$.

(103) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$ and for every natural numbers $n$, $m$, $f_1(n, m) + f_2(n, m) = f(n, m)$. Then

(i) $f$ is P-convergent, and

(ii) P-lim $f = \sup \operatorname{rng} f$, and

(iii) P-lim $f =$ P-lim $f_1 +$ P-lim $f_2$, and

(iv) $\sup \operatorname{rng} f = \sup \operatorname{rng} f_1 + \sup \operatorname{rng} f_2$.

The theorem is a consequence of (96) and (101).

Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number $c$. Now we state the propositions:

(104) Suppose $0 \leqslant c$ and for every natural numbers $n$, $m$, $f_2(n, m) = c \cdot f_1(n, m)$. Then

(i) $\sup \operatorname{rng} f_2 = c \cdot \sup \operatorname{rng} f_1$, and

(ii) $f_2$ is without $-\infty$.

The theorem is a consequence of (102) and (101).

(105) Suppose $0 \leqslant c$ and for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers $n$, $m$, $f_2(n, m) = c \cdot f_1(n, m)$. Then

(i) for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$, and

(ii) $f_2$ is without $-\infty$ and P-convergent, and

(iii) P-lim $f_2 = \sup \operatorname{rng} f_2$, and

(iv) P-lim $f_2 = c \cdot$ P-lim $f_1$.

The theorem is a consequence of (96) and (104).

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