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Finite Product of Semiring of Sets

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Summary. We formalize that the image of a semiring of sets [17] by an injective function is a semiring of sets. We offer a non-trivial example of a semiring of sets in a topological space [21]. Finally, we show that the finite product of a semiring of sets is also a semiring of sets [21] and that the finite product of a classical semiring of sets [8] is a classical semiring of sets. In this case, we use here the notation from the book of Aliprantis and Border [1].

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The notation and terminology used in this paper have been introduced in the following articles: [9], [2], [3], [4], [22], [7], [15], [23], [10], [11], [6], [12], [20], [26], [27], [19], [14], [16], [25], [18], and [13].

1. Preliminaries

From now on X_1, X_2, X_3, X_4 denote sets.

Now we state the propositions:

(1) (i) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$, and

(ii) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$, and

(iii) $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$.

(2) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = (X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2)$.

(3) $(X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$.

(4) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$. The theorem is a consequence of (2) and (3).

(5) $\bigcup\{X_1, X_2, X_3\} = (X_1 \cup X_2) \cup X_3$.

2. The Direct Image of a Semiring of Sets by an Injective Function

Now we state the proposition:

(6) Let us consider sets T, S, a function f from T into S, and a family G of subsets of T. Then $f^{\circ}G = \{f^{\circ}A$, where A is a subset of $T : A \in G\}$.

Let T, S be sets, f be a function from T into S, and G be a finite family of subsets of T. Let us note that $f^{\circ}G$ is finite.

Let f be a function and A be a countable set. Let us note that $f^{\circ}A$ is countable.

The scheme *FraenkelCountable* deals with a set $\mathcal{A}$ and a set $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) $\{\mathcal{F}(w)$, where w is an element of $\mathcal{A} : w \in \mathcal{X}\}$ is countable

provided

- $\mathcal{X}$ is countable.

Let T, S be sets, f be a function from T into S, and G be a countable family of subsets of T. Let us note that $f^{\circ}G$ is countable.

Let X, Y be sets, S be a family of subsets of X with the empty element, and f be a function from X into Y. One can verify that $f^{\circ}S$ has the empty element.

Now we state the propositions:

(7) Let us consider sets X, Y, a function f from X into Y, and families S_1, S_2 of subsets of X. If $S_1 \subseteq S_2$, then $f^{\circ}S_1 \subseteq f^{\circ}S_2$. The theorem is a consequence of (6).

(8) Let us consider sets X, Y, a $\cap$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^{\circ}S$ is a $\cap$-closed family of subsets of Y.

(9) Let us consider non empty sets X, Y, a $\cap_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^{\circ}S$ is a $\cap_{fp}$-closed family of subsets of Y.

(10) Let us consider non empty sets X, Y, a $\setminus^{\subseteq}_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one and $f^{\circ}S$ is not empty. Then $f^{\circ}S$ is a $\setminus^{\subseteq}_{fp}$-closed family of subsets of Y.

Proof: Reconsider $f_1 = f^\circ S$ as a family of subsets of Y. f_1 is $\setminus^{\subseteq}_{fp}$-closed by [10, (64), (87)], [11, (103)], [26, (123)]. $\square$

(11) Let us consider non empty sets X, Y, a $\setminus_{fp}$-closed family S of subsets of X, and a function f from X into Y. Suppose f is one-to-one. Then $f^\circ S$ is a $\setminus_{fp}$-closed family of subsets of Y.

(12) Let us consider non empty sets X, Y, a semiring S of sets of X, and a function f from X into Y. If f is one-to-one, then $f^\circ S$ is a semiring of sets of Y.

3. The Set of Set Differences of All Elements of a Semiring of Sets

Now we state the proposition:

(13) Let us consider a 1-element finite sequence X. Suppose $X(1)$ is not empty. Then there exists a function I from $X(1)$ into $\prod X$ such that

(i) I is one-to-one and onto, and

(ii) for every object x such that $x \in X(1)$ holds $I(x) = \langle x \rangle$.

Let X be a set. Observe that 2^X_* is $\cap$-closed and there exists a $\cap$-closed family of subsets of X which has the empty element and there exists a $\cap$-closed family of subsets of X with the empty element which is $\cup$-closed.

Let X, Y be non empty sets. Let us observe that $X \setminus\!\!\setminus Y$ is non empty.

Now we state the proposition:

(14) Let us consider a set X, and a family S of subsets of X with the empty element. Then $S \setminus\!\!\setminus S =$ the set of all $A \setminus B$ where A, B are elements of S.

Let X be a set and S be a family of subsets of X with the empty element. The functor semidiff S yielding a family of subsets of X is defined by the term

(Def. 1) $S \setminus\!\!\setminus S$.

Now we state the proposition:

(15) Let us consider a set X, a family S of subsets of X with the empty element, and an object x. Suppose $x \in$ semidiff S. Then there exist elements A, B of S such that $x = A \setminus B$. The theorem is a consequence of (14).

Let X be a set and S be a family of subsets of X with the empty element. Observe that semidiff S has the empty element.

Let S be a $\cap$-closed, $\cup$-closed family of subsets of X with the empty element. Note that semidiff S is $\cap$-closed and $\setminus_{fp}$-closed.

Now we state the proposition:

(16) Let us consider a set X, and a $\cap$-closed, $\cup$-closed family S of subsets of X with the empty element. Then semidiff S is a semiring of sets of X.

4. The Collection of All Locally Closed Sets $LC(X,\tau)$ of a Topological Space (X,τ)

Let T be a non empty topological space. The functor $\mathrm{LC}(T)$ yielding a family of subsets of Ω_T is defined by the term

(Def. 2) $\{A \cap B$, where A, B are subsets of $T : A$ is open and B is closed$\}$.

Let us note that $\mathrm{LC}(T)$ is $\cap$-closed and $\setminus_{fp}$-closed and has the empty element.

(17) Let us consider a non empty topological space T. Then $\mathrm{LC}(T)$ is a semiring of sets of Ω_T.

5. The Finite Product of Semirings of Sets

Let n be a natural number. Note that there exists an n-element finite sequence which is non-empty.

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A semiring family of X is an n-element finite sequence and is defined by

(Def. 3) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $it(i)$ is a semiring of sets of $X(i)$.

In the sequel n denotes a non zero natural number and X denotes a non-empty, n-element finite sequence. Now we state the propositions:

(18) Let us consider a semiring family S of X. Then $\mathrm{dom}\, S = \mathrm{dom}\, X$.

(19) Let us consider a semiring family S of X, and a natural number i. If $i \in \mathrm{Seg}\, n$, then $\bigcup(S(i)) \subseteq X(i)$.

(20) Let us consider a function f, and an n-element finite sequence X. If $f \in \prod X$, then f is an n-element finite sequence.

Let n be a non zero natural number and X be an n-element finite sequence. The functor SemiringProduct X yielding a set is defined by

(Def. 4) for every object f, $f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod X$.

Now we state the propositions:

(21) Let us consider an n-element finite sequence X. Then SemiringProduct $X \subseteq 2^{(\bigcup \bigcup X)^{\mathrm{dom}\, X}}$.

(22) Let us consider a semiring family S of X. Then SemiringProduct S is a family of subsets of $\prod X$.

Proof: Reconsider $S_1 =$ SemiringProduct S as a subset of $2^{(\bigcup \bigcup S)^{\mathrm{dom}\, S}}$. $S_1 \subseteq 2^{\prod X}$ by [3, (9)], (18), [7, (89)], (19). □

(23) Let us consider a non-empty, 1-element finite sequence X. Then $\prod X =$ the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (13).

One can check that $\prod\langle\emptyset\rangle$ is empty. Now we state the propositions:

(24) Let us consider a non empty set x. Then $\prod\langle x \rangle =$ the set of all $\langle y \rangle$ where y is an element of x. The theorem is a consequence of (23).

(25) Let us consider a non-empty, 1-element finite sequence X, and a semiring family S of X. Then SemiringProduct $S =$ the set of all $\prod\langle s \rangle$ where s is an element of $S(1)$. PROOF: S is non-empty by (18), [7, (3)]. $\prod S =$ the set of all $\langle s \rangle$ where s is an element of $S(1)$. □

Let us consider sets x, y. Now we state the propositions:

(26) $\prod\langle x \rangle \cap \prod\langle y \rangle = \prod\langle x \cap y \rangle$. The theorem is a consequence of (24).

(27) $\prod\langle x \rangle \setminus \prod\langle y \rangle = \prod\langle x \setminus y \rangle$. The theorem is a consequence of (24).

Let us consider a non-empty, 1-element finite sequence X and a semiring family S of X. Now we state the propositions:

(28) the set of all $\prod\langle s \rangle$ where s is an element of $S(1)$ is a semiring of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (24), (26), and (27).

(29) SemiringProduct S is a semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (28).

(30) Let us consider sets X_1, X_2, a semiring S_1 of sets of X_1, and a semiring S_2 of sets of X_2. Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1, s_2 is an element of S_2 is a semiring of sets of $X_1 \times X_2$.

(31) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a semiring family S_3 of X_3, and a semiring family S_1 of X_1. Suppose SemiringProduct S_3 is a semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a semiring of sets of $\prod X_1$. Let us consider a family S_4 of subsets of $\prod X_3 \times \prod X_1$. Suppose $S_4 =$ the set of all $s_1 \times s_2$ where s_1 is an element of SemiringProduct S_3, s_2 is an element of SemiringProduct S_1. Then there exists a function I from $\prod X_3 \times \prod X_1$ into $\prod(X_3 \frown X_1)$ such that

   (i) I is one-to-one and onto, and

   (ii) for every finite sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \frown y$, and

   (iii) $I^{\circ} S_4 =$ SemiringProduct$(S_3 \frown S_1)$.

PROOF: $\bigcup(S_1(1)) \subseteq X_1(1)$. Consider I being a function from $\prod X_3 \times \prod X_1$ into $\prod(X_3 \frown X_1)$ such that I is one-to-one and I is onto and for every finite sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \frown y$. $I^\circ S_4 = \mathrm{SemiringProduct}(S_3 \frown S_1)$ by (25), (20), [7, (89)], [24, (153)]. □

(32) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a semiring family S_3 of X_3, and a semiring family S_1 of X_1. Suppose SemiringProduct S_3 is a semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a semiring of sets of $\prod X_1$. Then $\mathrm{SemiringProduct}(S_3 \frown S_1)$ is a semiring of sets of $\prod(X_3 \frown X_1)$. The theorem is a consequence of (30), (31), (9), and (10).

(33) Let us consider a semiring family S of X. Then SemiringProduct S is a semiring of sets of $\prod X$. Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence X for every semiring family S of X, SemiringProduct S is a semiring of sets of $\prod X$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$ from [5, Sch. 10]. □

Let n be a non zero natural number, X be a non-empty, n-element finite sequence, and S be a semiring family of X. We say that S is $\cap$-closed yielding if and only if

(Def. 5) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $S(i)$ is $\cap$-closed.

Note that there exists a semiring family of X which is $\cap$-closed yielding.

6. The Finite Product of Classical Semirings of Sets

Let X be a set. Note that there exists a semiring of sets of X which is $\cap$-closed.

Let us consider a non-empty, 1-element finite sequence X and a $\cap$-closed yielding semiring family S of X. Now we state the propositions:

(34) the set of all $\prod\langle s \rangle$ where s is an element of $S(1)$ is a $\cap$-closed semiring of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (26) and (28).

(35) SemiringProduct S is a $\cap$-closed semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (34).

Now we state the propositions:

(36) Let us consider sets X_1, X_2, a $\cap$-closed semiring S_1 of sets of X_1, and a $\cap$-closed semiring S_2 of sets of X_2. Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1, s_2 is an element of S_2 is a $\cap$-closed semiring of sets of $X_1 \times X_2$.

(37) Let us consider a non-empty, n-element finite sequence X_3, a non-empty, 1-element finite sequence X_1, a $\cap$-closed yielding semiring family S_3 of X_3, and a $\cap$-closed yielding semiring family S_1 of X_1. Suppose SemiringProduct

S_3 is a $\cap$-closed semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a $\cap$-closed semiring of sets of $\prod X_1$. Then SemiringProduct$(S_3 \frown S_1)$ is a $\cap$-closed semiring of sets of $\prod(X_3 \frown X_1)$. The theorem is a consequence of (30), (31), (36), (8), and (10).

Let us consider n and X. Let S be a $\cap$-closed yielding semiring family of X. One can check that SemiringProduct S is $\cap$-closed.

(38) Let us consider a $\cap$-closed yielding semiring family S of X. Then SemiringProduct S is a $\cap$-closed semiring of sets of $\prod X$.

7. Measurable Rectangle

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A classical semiring family of X is an n-element finite sequence and is defined by

(Def. 6) for every natural number i such that $i \in \operatorname{Seg} n$ holds $it(i)$ is a semi-diff-closed, $\cap$-closed family of subsets of $X(i)$ with the empty element.

Let X be an n-element finite sequence. We introduce MeasurableRectangle X as a synonym of SemiringProduct X. Now we state the propositions:

(39) Every classical semiring family of X is a $\cap$-closed yielding semiring family of X.

(40) Let us consider a classical semiring family S of X. Then MeasurableRectangle S is a semi-diff-closed, $\cap$-closed family of subsets of $\prod X$ with the empty element. The theorem is a consequence of (39) and (33).

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