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Construction of Measure from Semialgebra of Sets[1]

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Summary. In our previous article [22], we showed complete additivity as a condition for extension of a measure. However, this condition premised the existence of a σ-field and the measure on it. In general, the existence of the measure on σ-field is not obvious. On the other hand, the proof of existence of a measure on a semialgebra is easier than in the case of a σ-field. Therefore, in this article we define a measure (`pre-measure`) on a semialgebra and extend it to a measure on a σ-field. Furthermore, we give a σ-measure as an extension of the measure on a σ-field. We follow [24], [10], and [31].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [19], [11], [5], [12], [17], [32], [13], [14], [26], [6], [7], [22], [20], [18], [21], [3], [4], [15], [27], [28], [35], [36], [30], [29], [23], [34], [8], [9], [25], and [16].

1. Joining Finite Sequences

Now we state the propositions:

(1) Let us consider a binary relation K. If $\operatorname{rng} K$ is empty-membered, then $\bigcup \operatorname{rng} K = \emptyset$.

(2) Let us consider a function K. Then $\operatorname{rng} K$ is empty-membered if and only if for every object x, $K(x) = \emptyset$.

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Let D be a set, F be a set of finite sequences of D, f be a finite sequence of elements of F, and n be a natural number. Note that the functor $f(n)$ yields a finite sequence of elements of D. Let Y be a set of finite sequences of D and F be a finite sequence of elements of Y. The functor $\mathrm{Length}\,F$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by

(Def. 1) $\mathrm{dom}\,it = \mathrm{dom}\,F$ and for every natural number n such that $n \in \mathrm{dom}\,it$ holds $it(n) = \mathrm{len}(F(n))$.

Now we state the propositions:

(3) Let us consider a set D, a set Y of finite sequences of D, and a finite sequence F of elements of Y. Suppose for every natural number n such that $n \in \mathrm{dom}\,F$ holds $F(n) = \varepsilon_D$. Then $\sum \mathrm{Length}\,F = 0$.

(4) Let us consider a set D, a set Y of finite sequences of D, a finite sequence F of elements of Y, and a natural number k. Suppose $k < \mathrm{len}\,F$. Then $\mathrm{Length}(F{\restriction}(k+1)) = \mathrm{Length}(F{\restriction}k) \frown \langle \mathrm{len}(F(k+1)) \rangle$.

(5) Let us consider a set D, a set Y of finite sequences of D, a finite sequence F of elements of Y, and a natural number n. Suppose $1 \leqslant n \leqslant \sum \mathrm{Length}\,F$. Then there exist natural numbers k, m such that

(i) $1 \leqslant m \leqslant \mathrm{len}(F(k+1))$, and

(ii) $k < \mathrm{len}\,F$, and

(iii) $m + \sum \mathrm{Length}(F{\restriction}k) = n$, and

(iv) $n \leqslant \sum \mathrm{Length}(F{\restriction}(k+1))$.

The theorem is a consequence of (4).

(6) Let us consider a set D, a set Y of finite sequences of D, and finite sequences F_1, F_2 of elements of Y. Then $\mathrm{Length}(F_1 \frown F_2) = \mathrm{Length}\,F_1 \frown \mathrm{Length}\,F_2$.

(7) Let us consider a set D, a set Y of finite sequences of D, a finite sequence F of elements of Y, and natural numbers k_1, k_2. Suppose $k_1 \leqslant k_2$. Then $\sum \mathrm{Length}(F{\restriction}k_1) \leqslant \sum \mathrm{Length}(F{\restriction}k_2)$. The theorem is a consequence of (6).

(8) Let us consider a set D, a set Y of finite sequences of D, a finite sequence F of elements of Y, and natural numbers m_1, m_2, k_1, k_2. Suppose $1 \leqslant m_1$ and $1 \leqslant m_2$ and $m_1 + \sum \mathrm{Length}(F{\restriction}k_1) = m_2 + \sum \mathrm{Length}(F{\restriction}k_2)$ and $m_1 + \sum \mathrm{Length}(F{\restriction}k_1) \leqslant \sum \mathrm{Length}(F{\restriction}(k_1+1))$ and $m_2 + \sum \mathrm{Length}(F{\restriction}k_2) \leqslant \sum \mathrm{Length}(F{\restriction}(k_2+1))$. Then

(i) $m_1 = m_2$, and

(ii) $k_1 = k_2$.

The theorem is a consequence of (7).

Let D be a non empty set, Y be a set of finite sequences of D, and F be a finite sequence of elements of Y. The functor joinedFinSeq F yielding a finite sequence of elements of D is defined by

(Def. 2) len $it = \sum \mathrm{Length}\, F$ and for every natural number n such that $n \in \mathrm{dom}\, it$ there exist natural numbers k, m such that $1 \leqslant m \leqslant \mathrm{len}(F(k+1))$ and $k < \mathrm{len}\, F$ and $m + \sum \mathrm{Length}(F{\restriction}k) = n$ and $n \leqslant \sum \mathrm{Length}(F{\restriction}(k+1))$ and $it(n) = F(k+1)(m)$.

Let D be a set, Y be a set of finite sequences of D and s be a sequence of Y. The functor Length s yielding a sequence of $\mathbb{N}$ is defined by

(Def. 3) for every natural number n, $it(n) = \mathrm{len}(s(n))$.

Let s be a sequence of $\mathbb{N}$. One can check that the functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ yields a sequence of $\mathbb{N}$. Let D be a non empty set. Let us note that there exists a set of finite sequences of D which is non empty and has a non-empty element.

Let us consider a non empty set D, a non empty set Y of finite sequences of D with a non-empty element, a non-empty sequence s of Y, and a natural number n. Now we state the propositions:

(9) (i) $\mathrm{len}(s(n)) \geqslant 1$, and

(ii) $n < (\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(n) < (\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(n+1)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 < (\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k, $\mathrm{len}(s(k)) \geqslant 1$ by [5, (20)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(10) There exist natural numbers k, m such that

(i) $m \in \mathrm{dom}(s(k))$, and

(ii) $(\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(k) - \mathrm{len}(s(k)) + m - 1 = n$.

The theorem is a consequence of (9).

(11) Let us consider a non empty set D, a non empty set Y of finite sequences of D with a non-empty element, and a non-empty sequence s of Y. Then $(\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}$ is increasing.

(12) Let us consider a non empty set D, a non empty set Y of finite sequences of D with a non-empty element, a non-empty sequence s of Y, and natural numbers m_1, m_2, k_1, k_2. Suppose $m_1 \in \mathrm{dom}(s(k_1))$ and $m_2 \in \mathrm{dom}(s(k_2))$ and $(\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(k_1) - \mathrm{len}(s(k_1)) + m_1 = (\sum_{\alpha=0}^{\kappa} (\mathrm{Length}\, s)(\alpha))_{\kappa \in \mathbb{N}}(k_2) - \mathrm{len}(s(k_2)) + m_2$. Then

(i) $m_1 = m_2$, and

(ii) $k_1 = k_2$.

The theorem is a consequence of (11).

(13) Let us consider a non empty set D, a set Y of finite sequences of D with a non-empty element, and a non-empty sequence s of Y. Then there exists an increasing sequence N of $\mathbb{N}$ such that for every natural number k, $N(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(k) - 1$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{natural number}] \equiv \$_2 = (\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(\$_1) - 1$. For every element k of $\mathbb{N}$, there exists an element n of $\mathbb{N}$ such that $\mathcal{P}[k, n]$ by (9), [3, (20)]. Consider N being a function from $\mathbb{N}$ into $\mathbb{N}$ such that for every element k of $\mathbb{N}$, $\mathcal{P}[k, N(k)]$ from [14, Sch. 3]. For every natural number k, $N(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(k) - 1$. For every natural number n, $N(n) < N(n+1)$. $\square$

Let D be a non empty set, Y be a set of finite sequences of D with a non-empty element, and s be a non-empty sequence of Y. The functor joinedSeq s yielding a sequence of D is defined by

(Def. 4) for every natural number n, there exist natural numbers k, m such that $m \in \mathrm{dom}(s(k))$ and $(\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(k) - \mathrm{len}(s(k)) + m - 1 = n$ and $it(n) = s(k)(m)$.

Now we state the propositions:

(14) Let us consider a non empty set D, a set Y of finite sequences of D with a non-empty element, a non-empty sequence s of Y, and a sequence s_1 of D. Suppose for every natural number n, $s_1(n) = (\mathrm{joinedSeq}\, s)((\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(n) - 1)$. Then s_1 is a subsequence of joinedSeq s.

PROOF: Consider N being an increasing sequence of $\mathbb{N}$ such that for every natural number n, $N(n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(n) - 1$. For every element n of $\mathbb{N}$, $s_1(n) = (\mathrm{joinedSeq}\, s \cdot N)(n)$ by [14, (15)]. $\square$

(15) Let us consider a non empty set D, a set Y of finite sequences of D with a non-empty element, a non-empty sequence s of Y, and natural numbers k, m. Suppose $m \in \mathrm{dom}(s(k))$. Then there exists a natural number n such that

(i) $n = (\sum_{\alpha=0}^{\kappa}(\mathrm{Length}\, s)(\alpha))_{\kappa\in\mathbb{N}}(k) - \mathrm{len}(s(k)) + m - 1$, and

(ii) $(\mathrm{joinedSeq}\, s)(n) = s(k)(m)$.

The theorem is a consequence of (12).

Let us consider a non empty set D, a set Y of finite sequences of D, and a finite sequence F of elements of Y. Now we state the propositions:

(16) Suppose for every natural numbers n, m such that $n \neq m$ holds $\bigcup \mathrm{rng}(F(n))$ misses $\bigcup \mathrm{rng}(F(m))$ and for every natural number n, $F(n)$ is disjoint valued. Then joinedFinSeq F is disjoint valued.

(17) $\operatorname{rng} \operatorname{joinedFinSeq} F = \bigcup\{\operatorname{rng}(F(n))$, where n is a natural number : $n \in \operatorname{dom} F\}$. The theorem is a consequence of (4), (7), and (8).

2. Extended Real-Valued Matrix

Let x be an extended real number. One can check that the functor $\langle x \rangle$ yields a finite sequence of elements of $\overline{\mathbb{R}}$. Let e be a finite sequence of elements of $\overline{\mathbb{R}}^*$. The functor $\sum e$ yielding a finite sequence of elements of $\overline{\mathbb{R}}$ is defined by

(Def. 5) $\operatorname{len} it = \operatorname{len} e$ and for every natural number k such that $k \in \operatorname{dom} it$ holds $it(k) = \sum(e(k))$.

Let M be a matrix over $\overline{\mathbb{R}}$. The functor $\operatorname{SumAll} M$ yielding an element of $\overline{\mathbb{R}}$ is defined by the term

(Def. 6) $\sum\sum M$.

Now we state the propositions:

(18) Let us consider a matrix M over $\overline{\mathbb{R}}$. Then

(i) $\operatorname{len} \sum M = \operatorname{len} M$, and

(ii) for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $(\sum M)(i) = \sum \operatorname{Line}(M, i)$.

(19) Let us consider a finite sequence F of elements of $\overline{\mathbb{R}}$. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \neq -\infty$. Then $\sum F \neq -\infty$.

Proof: Consider f being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0$ and for every natural number i such that $i < \operatorname{len} F$ holds $f(i+1) = f(i) + F(i+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $f(\$_1) \neq -\infty$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$ by [3, (13), (11)], [33, (25)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. $\square$

(20) Let us consider finite sequences F, G, H of elements of $\overline{\mathbb{R}}$. Suppose $-\infty \notin \operatorname{rng} F$ and $-\infty \notin \operatorname{rng} G$ and $\operatorname{dom} F = \operatorname{dom} G$ and $H = F + G$. Then $\sum H = \sum F + \sum G$.

Proof: Consider h being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum H = h(\operatorname{len} H)$ and $h(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} H$ holds $h(i+1) = h(i) + H(i+1)$. Consider f being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} F$ holds $f(i+1) = f(i) + F(i+1)$. Consider g being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum G = g(\operatorname{len} G)$ and $g(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} G$ holds $g(i+1) = g(i) + G(i+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} H$, then $h(\$_1) = f(\$_1) + g(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13), (11)], [33, (25)], [13, (3)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(21) Let us consider an extended real number r, and a finite sequence F of elements of $\overline{\mathbb{R}}$. Then $\sum(F \frown \langle r \rangle) = \sum F + r$.
PROOF: Consider f being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum(F \frown \langle r \rangle) = f(\mathrm{len}(F \frown \langle r \rangle))$ and $f(0) = 0$ and for every natural number i such that $i < \mathrm{len}(F \frown \langle r \rangle)$ holds $f(i+1) = f(i) + (F \frown \langle r \rangle)(i+1)$. Consider g being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum F = g(\mathrm{len}\, F)$ and $g(0) = 0$ and for every natural number i such that $i < \mathrm{len}\, F$ holds $g(i+1) = g(i) + F(i+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, F$, then $f(\$_1) = g(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)], [5, (64)], [3, (11)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(22) Let us consider an extended real number r, and a natural number i. If r is real, then $\sum(i \mapsto r) = i \cdot r$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \sum(\$_1 \mapsto r) = \$_1 \cdot r$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [12, (60)], (21). For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(23) Let us consider a matrix M over $\overline{\mathbb{R}}$. If $\mathrm{len}\, M = 0$, then $\mathrm{SumAll}\, M = 0$.

(24) Let us consider a natural number m, and a matrix M over $\overline{\mathbb{R}}$ of dimension $m \times 0$. Then $\mathrm{SumAll}\, M = 0$. The theorem is a consequence of (23) and (22).

(25) Let us consider natural numbers n, m, k, a matrix M_1 over $\overline{\mathbb{R}}$ of dimension $n \times k$, and a matrix M_2 over $\overline{\mathbb{R}}$ of dimension $m \times k$. Then $\sum(M_1 \frown M_2) = \sum M_1 \frown \sum M_2$.

Let us consider matrices M_1, M_2 over $\overline{\mathbb{R}}$. Now we state the propositions:

(26) Suppose for every natural number i such that $i \in \mathrm{dom}\, M_1$ holds $-\infty \notin \mathrm{rng}(M_1(i))$ and for every natural number i such that $i \in \mathrm{dom}\, M_2$ holds $-\infty \notin \mathrm{rng}(M_2(i))$. Then $\sum M_1 + \sum M_2 = \sum(M_1 \frown M_2)$. The theorem is a consequence of (19).

(27) Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and for every natural number i such that $i \in \mathrm{dom}\, M_1$ holds $-\infty \notin \mathrm{rng}(M_1(i))$ and for every natural number i such that $i \in \mathrm{dom}\, M_2$ holds $-\infty \notin \mathrm{rng}(M_2(i))$. Then $\mathrm{SumAll}\, M_1 + \mathrm{SumAll}\, M_2 = \mathrm{SumAll}(M_1 \frown M_2)$. The theorem is a consequence of (19), (26), and (20).

Now we state the propositions:

(28) Let us consider a finite sequence p of elements of $\overline{\mathbb{R}}$. Suppose $-\infty \notin \mathrm{rng}\, p$. Then $\mathrm{SumAll}\langle p \rangle = \mathrm{SumAll}\langle p \rangle^{\mathrm{T}}$.
PROOF: Define $x[\text{finite sequence of elements of } \overline{\mathbb{R}}] \equiv$ if $-\infty \notin \mathrm{rng}\, \$_1$, then $\mathrm{SumAll}\langle \$_1 \rangle = \mathrm{SumAll}\langle \$_1 \rangle^{\mathrm{T}}$. For every finite sequence p of elements of $\overline{\mathbb{R}}$ and for every element x of $\overline{\mathbb{R}}$ such that $x[p]$ holds $x[p \frown \langle x \rangle]$ by [5, (31),

(38), (6)]. $x[\varepsilon_{\overline{\mathbb{R}}}]$. For every finite sequence p of elements of $\overline{\mathbb{R}}$, $x[p]$ from [12, Sch. 2]. $\square$

(29) Let us consider an extended real number p, and a matrix M over $\overline{\mathbb{R}}$. Suppose for every natural number i such that $i \in \operatorname{dom} M$ holds $p \notin \operatorname{rng}(M(i))$. Let us consider a natural number j. If $j \in \operatorname{dom} M^{\mathrm{T}}$, then $p \notin \operatorname{rng}(M^{\mathrm{T}}(j))$.

(30) Let us consider a matrix M over $\overline{\mathbb{R}}$. Suppose for every natural number i such that $i \in \operatorname{dom} M$ holds $-\infty \notin \operatorname{rng}(M(i))$. Then $\operatorname{SumAll} M = \operatorname{SumAll} M^{\mathrm{T}}$.

Proof: Define $x[\text{natural number}] \equiv$ for every matrix M over $\overline{\mathbb{R}}$ such that $\operatorname{len} M = \$_1$ and for every natural number i such that $i \in \operatorname{dom} M$ holds $-\infty \notin \operatorname{rng}(M(i))$ holds $\operatorname{SumAll} M = \operatorname{SumAll} M^{\mathrm{T}}$. For every natural number n such that $x[n]$ holds $x[n+1]$ by [3, (11)], [33, (25)], [5, (40)], (28). $x[0]$. For every natural number n, $x[n]$ from [3, Sch. 2]. $\square$

3. Definition of Pre-Measure

Let x be an object. Let us observe that $\langle x \rangle$ is disjoint valued.

Now we state the proposition:

(31) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, a finite sequence F of elements of S, and an element G of S. Then there exists a disjoint valued finite sequence H of elements of S such that $G \setminus \bigcup F = \bigcup H$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence f of elements of S such that $\operatorname{len} f = \$_1$ there exists a disjoint valued finite sequence H of elements of S such that $G \setminus \bigcup f = \bigcup H$. For every finite sequence f of elements of S such that $\operatorname{len} f = 0$ there exists a disjoint valued finite sequence H of elements of S such that $G \setminus \bigcup f = \bigcup H$ by [16, (2)], [5, (38)], [16, (25)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [3, (11)], [5, (59)], [33, (55)], [5, (36), (38)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. $\square$

Let X be a set and P be a semi-diff-closed, $\cap$-closed family of subsets of X with the empty element. Let us note that there exists a sequence of P which is disjoint valued.

Let P be a non empty family of subsets of X. Note that there exists a function from P into $\overline{\mathbb{R}}$ which is non-negative, additive, and zeroed.

Let P be a family of subsets of X with the empty element. One can check that there exists a function from $\mathbb{N}$ into P which is disjoint valued.

A pre-measure of P is a non-negative, zeroed function from P into $\overline{\mathbb{R}}$ and is defined by

(Def. 7) for every disjoint valued finite sequence F of elements of P such that $\bigcup F \in P$ holds $it(\bigcup F) = \sum(it \cdot F)$ and for every disjoint valued function K from $\mathbb{N}$ into P such that $\bigcup K \in P$ holds $it(\bigcup K) \leqslant \overline{\sum}(it \cdot K)$.

Now we state the propositions:

(32) Let us consider a set X with the empty element, and a finite sequence F of elements of X. Then there exists a function G from $\mathbb{N}$ into X such that

(i) for every natural number i, $F(i) = G(i)$, and

(ii) $\bigcup F = \bigcup G$.

PROOF: Define $\mathcal{P}[\text{element of } \mathbb{N}, \text{set}] \equiv$ if $\$_1 \in \operatorname{dom} F$, then $F(\$_1) = \$_2$ and if $\$_1 \notin \operatorname{dom} F$, then $\$_2 = \emptyset$. For every element i of $\mathbb{N}$, there exists an element y of X such that $\mathcal{P}[i, y]$ by [13, (3)]. Consider G being a function from $\mathbb{N}$ into X such that for every element i of $\mathbb{N}$, $\mathcal{P}[i, G(i)]$ from [14, Sch. 3]. $\square$

(33) Let us consider a non empty set X, a finite sequence F of elements of X, and a function G from $\mathbb{N}$ into X. Suppose for every natural number i, $F(i) = G(i)$. Then F is disjoint valued if and only if G is disjoint valued.

(34) Let us consider a finite sequence F of elements of $\overline{\mathbb{R}}$, and a sequence G of extended reals. Suppose for every natural number i, $F(i) = G(i)$. Then F is non-negative if and only if G is non-negative.

Let us observe that there exists a finite sequence of elements of $\overline{\mathbb{R}}$ which is non-negative and there exists a finite sequence of elements of $\overline{\mathbb{R}}$ which is without $-\infty$ and there exists a finite sequence of elements of $\overline{\mathbb{R}}$ which is non-positive and there exists a finite sequence of elements of $\overline{\mathbb{R}}$ which is without $+\infty$ and every finite sequence of elements of $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every finite sequence of elements of $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$.

Let X, Y be non empty sets, F be a without $-\infty$ function from Y into $\overline{\mathbb{R}}$, and G be a function from X into Y. One can check that $F \cdot G$ is without $-\infty$ as a function from X into $\overline{\mathbb{R}}$.

Let F be a non-negative function from Y into $\overline{\mathbb{R}}$. One can check that $F \cdot G$ is non-negative as a function from X into $\overline{\mathbb{R}}$.

Now we state the propositions:

(35) Let us consider an extended real number a. Then $\sum\langle a\rangle = a$.

(36) Let us consider a finite sequence F of elements of $\overline{\mathbb{R}}$, and a natural number k. Then

(i) if F is without $-\infty$, then $F{\restriction}k$ is without $-\infty$, and

(ii) if F is without $+\infty$, then $F{\restriction}k$ is without $+\infty$.

(37) Let us consider a without $-\infty$ finite sequence F of elements of $\overline{\mathbb{R}}$, and a sequence G of extended reals. Suppose for every natural number i, $F(i) = G(i)$. Let us consider a natural number i. Then $\sum(F{\restriction}i) = (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}(i)$. The theorem is a consequence of (36) and (35).

(38) Let us consider a without $-\infty$ finite sequence F of elements of $\overline{\mathbb{R}}$, and a sequence G of extended reals. Suppose for every natural number i, $F(i) = G(i)$. Then

(i) G is summable, and

(ii) $\sum F = \sum G$.

PROOF: $\sum(F{\restriction}\operatorname{len} F) = (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}(\operatorname{len} F)$. Define $\mathcal{P}$[natural number] $\equiv \sum F = (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}(\operatorname{len} F + \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (11), (19)], [33, (25)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(39) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, a disjoint valued finite sequence F of elements of S, and a non empty, preboolean family R of subsets of X. Suppose $S \subseteq R$ and $\bigcup F \in R$. Let us consider a natural number i. Then $\bigcup(F{\restriction}i) \in R$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \bigcup(F{\restriction}\$_1) \in R$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [3, (12)], [5, (58)], [3, (13)], [5, (82), (17)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(40) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, a pre-measure P of S, and disjoint valued finite sequences F_1, F_2 of elements of S. Suppose $\bigcup F_1 \in S$ and $\bigcup F_1 = \bigcup F_2$. Then $P(\bigcup F_1) = P(\bigcup F_2)$.

(41) Let us consider a non empty, $\cap$-closed set S, and finite sequences F_1, F_2 of elements of S. Then there exists a matrix M over S of dimension $\operatorname{len} F_1 \times \operatorname{len} F_2$ such that for every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = F_1(i) \cap F_2(j)$.
PROOF: Define $\mathcal{P}$[natural number, natural number, set] $\equiv \$_3 = F_1(\$_1) \cap F_2(\$_2)$. For every natural numbers i, j such that $\langle i, j\rangle \in \operatorname{Seg} \operatorname{len} F_1 \times \operatorname{Seg} \operatorname{len} F_2$ there exists an element K of S such that $\mathcal{P}[i, j, K]$ by [16, (87)], [13, (3)]. Consider M being a matrix over S of dimension $\operatorname{len} F_1 \times \operatorname{len} F_2$ such that for every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $\mathcal{P}[i, j, M_{i,j}]$. □

Let us consider a set X, a $\cap$-closed family S of subsets of X with the empty element, non empty, disjoint valued finite sequences F_1, F_2 of elements of S, a non-negative, zeroed function P from S into $\overline{\mathbb{R}}$, and a matrix M over $\overline{\mathbb{R}}$ of dimension $\operatorname{len} F_1 \times \operatorname{len} F_2$.

Let us assume that $\bigcup F_1 = \bigcup F_2$ and for every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = P(F_1(i) \cap F_2(j))$ and for every disjoint valued finite sequence F of elements of S such that $\bigcup F \in S$ holds $P(\bigcup F) = \sum(P \cdot F)$. Now we state the propositions:

(42) (i) for every natural number i such that $i \leqslant \operatorname{len}(P \cdot F_1)$ holds $(P \cdot F_1)(i) = (\sum M)(i)$, and

(ii) $\sum(P \cdot F_1) = \operatorname{SumAll} M$.

Proof: Consider K being a matrix over S of dimension $\operatorname{len} F_1 \times \operatorname{len} F_2$ such that for every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of K holds $K_{i,j} = F_1(i) \cap F_2(j)$. For every natural number i such that $i \leqslant \operatorname{len}(P \cdot F_1)$ holds $(P \cdot F_1)(i) = (\sum M)(i)$ by [33, (24)], [3, (14)], [33, (25)], [13, (11), (3)]. Consider Q being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum(P \cdot F_1) = Q(\operatorname{len}(P \cdot F_1))$ and $Q(0) = 0$ and for every natural number i such that $i < \operatorname{len}(P \cdot F_1)$ holds $Q(i+1) = Q(i) + (P \cdot F_1)(i+1)$. Consider L being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\operatorname{SumAll} M = L(\operatorname{len} \sum M)$ and $L(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} \sum M$ holds $L(i + 1) = L(i) + (\sum M)(i + 1)$. Define $\mathcal{R}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len}(P \cdot F_1)$, then $Q(\$_1) = L(\$_1)$. For every natural number i such that $\mathcal{R}[i]$ holds $\mathcal{R}[i+1]$ by [3, (13)]. For every natural number i, $\mathcal{R}[i]$ from [3, Sch. 2]. $\square$

(43) (i) for every natural number i such that $i \leqslant \operatorname{len}(P \cdot F_2)$ holds $(P \cdot F_2)(i) = (\sum M^{\mathrm{T}})(i)$, and

(ii) $\sum(P \cdot F_2) = \operatorname{SumAll} M^{\mathrm{T}}$.

Proof: Consider K being a matrix over S of dimension $\operatorname{len} F_1 \times \operatorname{len} F_2$ such that for every natural numbers i, j such that $\langle i, j\rangle \in$ the indices of K holds $K_{i,j} = F_1(i) \cap F_2(j)$. For every natural number i such that $i \leqslant \operatorname{len}(P \cdot F_2)$ holds $(P \cdot F_2)(i) = (\sum M^{\mathrm{T}})(i)$ by [33, (24)], [3, (14)], [33, (25)], [13, (11), (3)]. Consider Q being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum(P \cdot F_2) = Q(\operatorname{len}(P \cdot F_2))$ and $Q(0) = 0$ and for every natural number i such that $i < \operatorname{len}(P \cdot F_2)$ holds $Q(i+1) = Q(i) + (P \cdot F_2)(i+1)$. Consider L being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\operatorname{SumAll} M^{\mathrm{T}} = L(\operatorname{len} \sum M^{\mathrm{T}})$ and $L(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} \sum M^{\mathrm{T}}$ holds $L(i + 1) = L(i) + (\sum M^{\mathrm{T}})(i + 1)$. Define $\mathcal{R}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len}(P \cdot F_2)$, then $Q(\$_1) = L(\$_1)$. For every natural number i such that $\mathcal{R}[i]$ holds $\mathcal{R}[i+1]$ by [3, (13)]. For every natural number i, $\mathcal{R}[i]$ from [3, Sch. 2]. $\square$

(44) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, a pre-measure P of S, and a set A. Suppose $A \in$ the ring generated by S. Let us consider disjoint valued finite sequences F_1,

F_2 of elements of S. If $A = \bigcup F_1$ and $A = \bigcup F_2$, then $\sum(P \cdot F_1) = \sum(P \cdot F_2)$. The theorem is a consequence of (42), (43), and (30).

(45) Let us consider finite sequences f_1, f_2. Suppose f_1 is disjoint valued and f_2 is disjoint valued and $\bigcup \operatorname{rng} f_1$ misses $\bigcup \operatorname{rng} f_2$. Then $f_1 \frown f_2$ is disjoint valued.

(46) Let us consider a set X, a semi-diff-closed family P of subsets of X with the empty element, a pre-measure M of P, and sets A, B. If A, B, $A \setminus B \in P$ and $B \subseteq A$, then $M(A) \geqslant M(B)$. The theorem is a consequence of (45).

(47) Let us consider non empty sets Y, S, a partial function F from Y to S, and a function M from S into $\overline{\mathbb{R}}$. If M is non-negative, then $M \cdot F$ is non-negative.

(48) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, and a pre-measure P of S. Then there exists a non-negative, additive, zeroed function M from the ring generated by S into $\overline{\mathbb{R}}$ such that for every set A such that $A \in$ the ring generated by S for every disjoint valued finite sequence F of elements of S such that $A = \bigcup F$ holds $M(A) = \sum(P \cdot F)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every disjoint valued finite sequence F of elements of S such that $\$_1 = \bigcup F$ holds $\$_2 = \sum(P \cdot F)$. For every object A such that $A \in$ the ring generated by S there exists an object p such that $p \in \overline{\mathbb{R}}$ and $\mathcal{P}[A, p]$ by [23, (18)], (44). Consider M being a function from the ring generated by S into $\overline{\mathbb{R}}$ such that for every object A such that $A \in$ the ring generated by S holds $\mathcal{P}[A, M(A)]$ from [14, Sch. 1]. For every element A of the ring generated by S, $0 \leqslant M(A)$ by [23, (18)], [3, (11)], [33, (25)], [13, (12)]. For every elements A, B of the ring generated by S such that A misses B and $A \cup B \in$ the ring generated by S holds $M(A \cup B) = M(A) + M(B)$ by [23, (18)], (45), [5, (31)], [16, (78)]. $\square$

(49) Let us consider sets X, Y, and functions F, G from $\mathbb{N}$ into 2^X. Suppose for every natural number i, $G(i) = F(i) \cap Y$ and $\bigcup F = Y$. Then $\bigcup G = \bigcup F$.

(50) Let us consider a set X, a semi-diff-closed, $\cap$-closed family S of subsets of X with the empty element, and a pre-measure P of S. Then there exists a function M from the ring generated by S into $\overline{\mathbb{R}}$ such that

(i) $M(\emptyset) = 0$, and

(ii) for every disjoint valued finite sequence K of elements of S such that $\bigcup K \in$ the ring generated by S holds $M(\bigcup K) = \sum(P \cdot K)$.

The theorem is a consequence of (48).

(51) Let us consider sets X, Z, a semi-diff-closed, $\cap$-closed family P of subsets of X with the empty element, and a disjoint valued function K from $\mathbb{N}$ into the ring generated by P. Suppose $Z = \{\langle n, F \rangle$, where n is a natural number, F is a disjoint valued finite sequence of elements of $P : \bigcup F = K(n)$ and if $K(n) = \emptyset$, then $F = \langle \emptyset \rangle\}$. Then

(i) $\pi_2(Z)$ is a set of finite sequences of P, and

(ii) for every object x, $x \in \operatorname{rng} K$ iff there exists a finite sequence F of elements of P such that $F \in \pi_2(Z)$ and $\bigcup F = x$, and

(iii) $\pi_2(Z)$ has non empty elements.

(52) Let us consider a set X, a semi-diff-closed, $\cap$-closed family P of subsets of X with the empty element, and a disjoint valued function K from $\mathbb{N}$ into the ring generated by P. Suppose $\operatorname{rng} K$ has a non-empty element. Then there exists a non empty set Y of finite sequences of P such that

(i) $Y = \{F$, where F is a disjoint valued finite sequence of elements of $P : \bigcup F \in \operatorname{rng} K$ and $F \neq \emptyset\}$, and

(ii) Y has non empty elements.

4. Pre-Measure on Semialgebra and Construction of Measure

Now we state the propositions:

(53) Let us consider sets X, Z, a semialgebra P of sets of X, and a disjoint valued function K from $\mathbb{N}$ into the field generated by P. Suppose $Z = \{\langle n, F \rangle$, where n is a natural number, F is a disjoint valued finite sequence of elements of $P : \bigcup F = K(n)$ and if $K(n) = \emptyset$, then $F = \langle \emptyset \rangle\}$. Then

(i) $\pi_2(Z)$ is a set of finite sequences of P, and

(ii) for every object x, $x \in \operatorname{rng} K$ iff there exists a finite sequence F of elements of P such that $F \in \pi_2(Z)$ and $\bigcup F = x$, and

(iii) $\pi_2(Z)$ has non empty elements.

(54) Let us consider a set X, a semialgebra S of sets of X, a pre-measure P of S, a set A, and disjoint valued finite sequences F_1, F_2 of elements of S. If $A = \bigcup F_1$ and $A = \bigcup F_2$, then $\sum(P \cdot F_1) = \sum(P \cdot F_2)$. The theorem is a consequence of (42), (43), and (30).

(55) Let us consider a set X, a semialgebra S of sets of X, and a pre-measure P of S. Then there exists a measure M on the field generated by S such that for every set A such that $A \in$ the field generated by S for every disjoint valued finite sequence F of elements of S such that $A = \bigcup F$ holds $M(A) = \sum(P \cdot F)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every disjoint valued finite sequence F of elements of S such that $\$_1 = \bigcup F$ holds $\$_2 = \sum(P \cdot F)$. For every object A such that $A \in$ the field generated by S there exists an object p such that $p \in \overline{\mathbb{R}}$ and $\mathcal{P}[A, p]$ by [23, (22)], (54). Consider M being a function from the field generated by S into $\overline{\mathbb{R}}$ such that for every object A such that $A \in$ the field generated by S holds $\mathcal{P}[A, M(A)]$ from [14, Sch. 1]. For every element A of the field generated by S, $0 \leqslant M(A)$ by [23, (22)], [3, (11)], [33, (25)], [13, (12)]. For every elements A, B of the field generated by S such that A misses B holds $M(A \cup B) = M(A) + M(B)$ by [23, (22)], (45), [5, (31)], [16, (78)]. □

(56) Let us consider a sequence F of extended reals, a natural number n, and an extended real number a. Suppose for every natural number k, $F(k) = a$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = a \cdot (n + 1)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(\$_1) = a \cdot (\$_1 + 1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(57) Let us consider a non empty set X, a sequence F of X, and a natural number n. Then $\operatorname{rng}(F{\restriction}\mathbb{Z}_{n+1}) = \operatorname{rng}(F{\restriction}\mathbb{Z}_n) \cup \{F(n)\}$.

(58) Let us consider a set X, a field S of subsets of X, a measure M on S, a sequence F of separated subsets of S, and a natural number n. Then

(i) $\bigcup \operatorname{rng}(F{\restriction}\mathbb{Z}_{n+1}) \in S$, and

(ii) $(\sum_{\alpha=0}^{\kappa}(M \cdot F)(\alpha))_{\kappa \in \mathbb{N}}(n) = M(\bigcup \operatorname{rng}(F{\restriction}\mathbb{Z}_{n+1}))$.

PROOF: $\operatorname{rng}(F{\restriction}\mathbb{Z}_{0+1}) = \operatorname{rng}(F{\restriction}\mathbb{Z}_0) \cup \{F(0)\}$. Define $\mathcal{R}[\text{natural number}] \equiv \bigcup \operatorname{rng}(F{\restriction}\mathbb{Z}_{\$_1+1}) \in S$. For every natural number k such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$ by (57), [16, (78), (25)], [27, (3)]. For every natural number k, $\mathcal{R}[k]$ from [3, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(M \cdot F)(\alpha))_{\kappa \in \mathbb{N}}(\$_1) = M(\bigcup \operatorname{rng}(F{\restriction}\mathbb{Z}_{\$_1+1}))$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by [14, (15)], [35, (57)], [3, (44)], [13, (47)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(59) Let us consider a set X, a semialgebra S of sets of X, a pre-measure P of S, and a measure M on the field generated by S. Suppose for every set A such that $A \in$ the field generated by S for every disjoint valued finite sequence F of elements of S such that $A = \bigcup F$ holds $M(A) = \sum(P \cdot F)$. Then M is completely-additive. The theorem is a consequence of (53), (15), (13), (58), and (1).

Let X be a set, S be a semialgebra of sets of X, and P be a pre-measure of S.

An induced measure of S and P is a measure on the field generated by S and is defined by

(Def. 8) for every set A such that $A \in$ the field generated by S for every disjoint valued finite sequence F of elements of S such that $A = \bigcup F$ holds $it(A) = \sum(P \cdot F)$.

Now we state the propositions:

(60) Let us consider a set X, a semialgebra S of sets of X, and a pre-measure P of S. Then every induced measure of S and P is completely-additive. The theorem is a consequence of (59).

(61) Let us consider a non empty set X, a semialgebra S of sets of X, a pre-measure P of S, and an induced measure M of S and P. Then σ-Meas(the Caratheodory measure determined by M)$\restriction\sigma$(the field generated by S) is a σ-measure on σ(the field generated by S). The theorem is a consequence of (60).

Let X be a non empty set, S be a semialgebra of sets of X, P be a pre-measure of S, and M be an induced measure of S and P.

An induced σ-measure of S and M is a σ-measure on σ(the field generated by S) and is defined by

(Def. 9) $it = \sigma$-Meas(the Caratheodory measure determined by M)$\restriction\sigma$(the field generated by S).

Now we state the proposition:

(62) Let us consider a non empty set X, a semialgebra S of sets of X, a pre-measure P of S, and an induced measure m of S and P. Then every induced σ-measure of S and m is an extension of m. The theorem is a consequence of (60).

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.

[7] Józef Białas. Properties of Caratheodor's measure. *Formalized Mathematics*, 3(**1**):67–70, 1992.

[8] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.

[9] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[10] V.I. Bogachev. *Measure Theory*, volume 1. Springer, 2006.

[11] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[12] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[13] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[14] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[15] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[16] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[17] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[18] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006. doi:10.2478/v10037-006-0008-x.

[19] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.

[20] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.

[21] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(**2**):167–175, 2008. doi:10.2478/v10037-008-0023-1.

[22] Noboru Endou, Hiroyuki Okazaki, and Yasunari Shidama. Hopf extension theorem of measure. *Formalized Mathematics*, 17(**2**):157–162, 2009. doi:10.2478/v10037-009-0018-6.

[23] Noboru Endou, Kazuhisa Nakasho, and Yasunari Shidama. σ-ring and σ-algebra of sets. *Formalized Mathematics*, 23(**1**):51–57, 2015. doi:10.2478/forma-2015-0004.

[24] P. R. Halmos. *Measure Theory.* Springer-Verlag, 1974.

[25] Andrzej Kondracki. The Chinese Remainder Theorem. *Formalized Mathematics*, 6(**4**): 573–577, 1997.

[26] Robert Milewski. Associated matrix of linear map. *Formalized Mathematics*, 5(**3**):339–345, 1996.

[27] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.

[28] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(**4**):745–749, 1990.

[29] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[30] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.

[31] M.M. Rao. *Measure Theory and Integration.* CRC Press, 2nd edition, 2004.

[32] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1 (**1**):187–190, 1990.

[33] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[34] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[35] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[36] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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