

The Basic Existence Theorem of Riemann-Stieltjes Integral

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Summary. In this article, the basic existence theorem of Riemann-Stieltjes integral is formalized. This theorem states that if f is a continuous function and ρ is a function of bounded variation in a closed interval of real line, f is Riemann-Stieltjes integrable with respect to ρ . In the first section, basic properties of real finite sequences are formalized as preliminaries. In the second section, we formalized the existence theorem of the Riemann-Stieltjes integral. These formalizations are based on [15], [12], [10], and [11].

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1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider a real number E , a finite sequence q of elements of \mathbb{R} , and a finite sequence S of elements of \mathbb{R} . Suppose $\text{len } S = \text{len } q$ and for every natural number i such that $i \in \text{dom } S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$. Then $\sum S = \sum q \cdot E$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence q of elements of \mathbb{R} for every finite sequence S of elements of \mathbb{R} such that $\$1 = \text{len } S$

and $\text{len } S = \text{len } q$ and for every natural number i such that $i \in \text{dom } S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$ holds $\sum S = \sum q \cdot E$. $\mathcal{P}[0]$ by [7, (72)]. For every natural number i , $\mathcal{P}[i]$ from [1, Sch. 2]. \square

- (2) Let us consider finite sequences x, y of elements of \mathbb{R} . Suppose $\text{len } x = \text{len } y$ and for every element i of \mathbb{N} such that $i \in \text{dom } x$ there exists a real number v such that $v = x(i)$ and $y(i) = |v|$. Then $|\sum x| \leq \sum y$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequences x, y of elements of \mathbb{R} such that $\$1 = \text{len } x$ and $\text{len } x = \text{len } y$ and for every element i of \mathbb{N} such that $i \in \text{dom } x$ there exists a real number v such that $v = x(i)$ and $y(i) = |v|$ holds $|\sum x| \leq \sum y$. $\mathcal{P}[0]$ by [7, (72)], [3, (44)]. For every natural number i , $\mathcal{P}[i]$ from [1, Sch. 2]. \square

- (3) Let us consider finite sequences p, q of elements of \mathbb{R} . Suppose $\text{len } p = \text{len } q$ and for every natural number j such that $j \in \text{dom } p$ holds $|p(j)| \leq q(j)$. Then $|\sum p| \leq \sum q$.

PROOF: Define $\mathcal{P}[\text{natural number, set}] \equiv$ there exists a real number v such that $v = p(\$1)$ and $\$2 = |v|$. For every natural number i such that $i \in \text{Seg len } p$ there exists an element x of \mathbb{R} such that $\mathcal{P}[i, x]$. Consider u being a finite sequence of elements of \mathbb{R} such that $\text{dom } u = \text{Seg len } p$ and for every natural number i such that $i \in \text{Seg len } p$ holds $\mathcal{P}[i, u(i)]$ from [2, Sch. 5]. For every element i of \mathbb{N} such that $i \in \text{dom } p$ there exists a real number v such that $v = p(i)$ and $u(i) = |v|$. $|\sum p| \leq \sum u$. \square

- (4) Let us consider a natural number n , and an object a . Then $\text{len}(n \mapsto a) = n$.

- (5) Let us consider a finite sequence p , and an object a . Then $p = \text{len } p \mapsto a$ if and only if for every object k such that $k \in \text{dom } p$ holds $p(k) = a$.

PROOF: If $p = \text{len } p \mapsto a$, then for every object k such that $k \in \text{dom } p$ holds $p(k) = a$ by [4, (57)]. \square

- (6) Let us consider a finite sequence p of elements of \mathbb{R} , a natural number i , and a real number r . Suppose $i \in \text{dom } p$ and $p(i) = r$ and for every natural number k such that $k \in \text{dom } p$ and $k \neq i$ holds $p(k) = 0$. Then $\sum p = r$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence p of elements of \mathbb{R} for every natural number i for every real number r such that $\text{len } p = \$1$ and $i \in \text{dom } p$ and $p(i) = r$ and for every natural number k such that $k \in \text{dom } p$ and $k \neq i$ holds $p(k) = 0$ holds $\sum p = r$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by [4, (19), (16)], [18, (25)], [17, (7)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

- (7) Let us consider finite sequences p, q of elements of \mathbb{R} . Suppose $\text{len } p \leq$

len q and for every natural number i such that $i \in \text{dom } q$ holds if $i \leq \text{len } p$, then $q(i) = p(i)$ and if $\text{len } p < i$, then $q(i) = 0$. Then $\sum q = \sum p$.

PROOF: Consider i_1 being a natural number such that $i_1 = \text{len } q - \text{len } p$. Set $x = i_1 \mapsto (0 \text{ qua real number})$. $q = p \hat{\ } x$ by (4), [18, (25)], [16, (13)], [4, (57)]. \square

(8) Let us consider real numbers a, b, c, d . If $b \leq c$, then $[a, b] \cap [c, d] \subseteq [b, b]$.

(9) Let us consider a real number a , a subset A of \mathbb{R} , and a real-valued function ϱ . If $A \subseteq [a, a]$, then $\text{vol}(A, \varrho) = 0$.

(10) Let us consider a non empty, increasing finite sequence s of elements of \mathbb{R} , and a natural number m . Suppose $m \in \text{dom } s$. Then $s \upharpoonright m$ is a non empty, increasing finite sequence of elements of \mathbb{R} .

PROOF: Set $H = s \upharpoonright m$. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } H$ and $e_1 < e_2$ holds $H(e_1) < H(e_2)$ by [19, (57)], [5, (47)]. \square

(11) Let us consider non empty, increasing finite sequences s, t of elements of \mathbb{R} . Suppose $s(\text{len } s) < t(1)$. Then $s \hat{\ } t$ is a non empty, increasing finite sequence of elements of \mathbb{R} .

PROOF: Set $H = s \hat{\ } t$. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } H$ and $e_1 < e_2$ holds $H(e_1) < H(e_2)$ by [18, (25)], [2, (25), (3)]. \square

(12) Let us consider a non empty, increasing finite sequence s of elements of \mathbb{R} , and a real number a . Suppose $s(\text{len } s) < a$. Then $s \hat{\ } \langle a \rangle$ is a non empty, increasing finite sequence of elements of \mathbb{R} . The theorem is a consequence of (11).

(13) Let us consider a finite sequence T of elements of \mathbb{R} , and natural numbers n, m . Suppose $n + 1 < m \leq \text{len } T$. Then there exists a finite sequence T_1 of elements of \mathbb{R} such that

(i) $\text{len } T_1 = m - (n + 1)$, and

(ii) $\text{rng } T_1 \subseteq \text{rng } T$, and

(iii) for every natural number i such that $i \in \text{dom } T_1$ holds $T_1(i) = T(i + n)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = T(\$_1 + n)$. Reconsider $m_1 = m - (n + 1)$ as a natural number. Consider p being a finite sequence such that $\text{len } p = m_1$ and for every natural number k such that $k \in \text{dom } p$ holds $p(k) = \mathcal{F}(k)$ from [2, Sch. 2]. $\text{rng } p \subseteq \text{rng } T$ by [18, (25)], [5, (3)]. \square

(14) Let us consider a non empty, increasing finite sequence T of elements of \mathbb{R} , and natural numbers n, m . Suppose $n + 1 < m \leq \text{len } T$. Then there exists a non empty, increasing finite sequence T_1 of elements of \mathbb{R} such that

- (i) $\text{len } T_1 = m - (n + 1)$, and
- (ii) $\text{rng } T_1 \subseteq \text{rng } T$, and
- (iii) for every natural number i such that $i \in \text{dom } T_1$ holds $T_1(i) = T(i + n)$.

PROOF: Consider p being a finite sequence of elements of \mathbb{R} such that $\text{len } p = m - (n + 1)$ and $\text{rng } p \subseteq \text{rng } T$ and for every natural number i such that $i \in \text{dom } p$ holds $p(i) = T(i + n)$. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } p$ and $e_1 < e_2$ holds $p(e_1) < p(e_2)$ by [18, (25)]. \square

- (15) Let us consider a finite sequence p of elements of \mathbb{R} , and natural numbers n, m . Suppose $n + 1 < m \leq \text{len } p$. Then there exists a finite sequence p_1 of elements of \mathbb{R} such that

- (i) $\text{len } p_1 = m - (n + 1) - 1$, and
- (ii) $\text{rng } p_1 \subseteq \text{rng } p$, and
- (iii) for every natural number i such that $i \in \text{dom } p_1$ holds $p_1(i) = p(i + n + 1)$.

The theorem is a consequence of (13).

2. EXISTENCE OF RIEMANN-STIELTJES INTEGRAL FOR CONTINUOUS FUNCTIONS

Now we state the propositions:

- (16) Let us consider a non empty, closed interval subset A of \mathbb{R} , a partition T of A , a real-valued function ϱ , a non empty, closed interval subset B of \mathbb{R} , a non empty, increasing finite sequence S_0 of elements of \mathbb{R} , and a finite sequence S_1 of elements of \mathbb{R} .

Suppose $B \subseteq A$ and $\inf B = \inf A$ and there exists a partition S of B such that $S = S_0$ and $\text{len } S_1 = \text{len } S$ and for every natural number j such that $j \in \text{dom } S$ there exists a finite sequence p of elements of \mathbb{R} such that $S_1(j) = \sum p$ and $\text{len } p = \text{len } T$ and for every natural number i such that $i \in \text{dom } T$ holds $p(i) = |\text{vol}(\text{divset}(T, i) \cap \text{divset}(S, j), \varrho)|$.

Then there exists a partition H of B and there exists a var-volume F of ϱ and H such that $\sum S_1 = \sum F$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non empty, closed interval subset B of \mathbb{R} for every non empty, increasing finite sequence S_0 of elements of \mathbb{R} for every finite sequence S_1 of elements of \mathbb{R} such that $B \subseteq A$ and $\inf B = \inf A$ and $\text{len } S_0 = \mathcal{S}_1$ and there exists a partition S of B such that $S = S_0$ and $\text{len } S_1 = \text{len } S$ and for every natural number j such

that $j \in \text{dom } S$ there exists a finite sequence p of elements of \mathbb{R} such that $S_1(j) = \sum p$ and $\text{len } p = \text{len } T$ and for every natural number i such that $i \in \text{dom } T$ holds $p(i) = |\text{vol}(\text{divset}(T, i) \cap \text{divset}(S, j), \varrho)|$ there exists a partition H of B and there exists a var-volume F of ϱ and H such that $\sum S_1 = \sum F$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [18, (29)], [1, (14)], [18, (25)], [2, (40)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

(17) Let us consider a non empty, closed interval subset A of \mathbb{R} , a function ϱ from A into \mathbb{R} , and partitions T, S of A . Suppose ϱ is bounded-variation. Then there exists a finite sequence S_1 of elements of \mathbb{R} such that

- (i) $\text{len } S_1 = \text{len } S$, and
- (ii) $\sum S_1 \leq \text{TotalVD}(\varrho)$, and
- (iii) for every natural number j such that $j \in \text{dom } S$ there exists a finite sequence p of elements of \mathbb{R} such that $S_1(j) = \sum p$ and $\text{len } p = \text{len } T$ and for every natural number i such that $i \in \text{dom } T$ holds $p(i) = |\text{vol}(\text{divset}(T, i) \cap \text{divset}(S, j), \varrho)|$.

PROOF: Define $\mathcal{P}[\text{natural number, object}] \equiv$ there exists a finite sequence p of elements of \mathbb{R} such that $\$2 = \sum p$ and $\text{len } p = \text{len } T$ and for every natural number i such that $i \in \text{dom } T$ holds $p(i) = |\text{vol}(\text{divset}(T, i) \cap \text{divset}(S, \$1), \varrho)|$. For every natural number j such that $j \in \text{Seg len } S$ there exists an element x of \mathbb{R} such that $\mathcal{P}[j, x]$. Consider S_1 being a finite sequence of elements of \mathbb{R} such that $\text{dom } S_1 = \text{Seg len } S$ and for every natural number j such that $j \in \text{Seg len } S$ holds $\mathcal{P}[j, S_1(j)]$ from [2, Sch. 5]. Consider H being a partition of A , F being a var-volume of ϱ and H such that $\sum S_1 = \sum F$. \square

(18) Let us consider a non empty, closed interval subset A of \mathbb{R} , a function ϱ from A into \mathbb{R} , and a partial function u from \mathbb{R} to \mathbb{R} .

Suppose ϱ is bounded-variation and $\text{dom } u = A$ and $u \upharpoonright A$ is uniformly continuous. Let us consider a division sequence T of A , and a middle volume sequence S of ϱ, u and T . Suppose δ_T is convergent and $\lim \delta_T = 0$. Then $\text{middle-sum}(S)$ is convergent.

PROOF: For every division sequence T of A and for every middle volume sequence S of ϱ, u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds $\text{middle-sum}(S)$ is convergent by [14, (6)], [9, (9)], [8, (87)], [6, (5)]. \square

(19) Let us consider a non empty, closed interval subset A of \mathbb{R} , a function ϱ from A into \mathbb{R} , a partial function u from \mathbb{R} to \mathbb{R} , division sequences T_0, T, T_1 of A , a middle volume sequence S_0 of ϱ, u and T_0 , and a middle volume sequence S of ϱ, u and T .

Suppose for every natural number i , $T_1(2 \cdot i) = T_0(i)$ and $T_1(2 \cdot i + 1) = T(i)$. Then there exists a middle volume sequence S_1 of ϱ , u and T_1 such that for every natural number i , $S_1(2 \cdot i) = S_0(i)$ and $S_1(2 \cdot i + 1) = S(i)$.

PROOF: Reconsider $S_3 = S_0$, $S_2 = S$ as a sequence of \mathbb{R}^* . Define \mathcal{F} (natural number) = $S_3(\$_1)$ ($\in \mathbb{R}^*$). Define \mathcal{G} (natural number) = $S_2(\$_1)$ ($\in \mathbb{R}^*$). Consider S_1 being a sequence of \mathbb{R}^* such that for every natural number n , $S_1(2 \cdot n) = \mathcal{F}(n)$ and $S_1(2 \cdot n + 1) = \mathcal{G}(n)$ from [13, Sch. 1]. For every element i of \mathbb{N} , $S_1(i)$ is a middle volume of ϱ , u and $T_1(i)$ by [13, (14)], [6, (5)]. \square

- (20) Let us consider sequences S_1 , S_2 , S_3 of real numbers. Suppose S_3 is convergent and for every natural number i , $S_3(2 \cdot i) = S_1(i)$ and $S_3(2 \cdot i + 1) = S_2(i)$. Then

- (i) S_1 is convergent, and
- (ii) $\lim S_1 = \lim S_3$, and
- (iii) S_2 is convergent, and
- (iv) $\lim S_2 = \lim S_3$.

PROOF: For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leq i$ holds $|S_1(i) - \lim S_3| < r$ by [13, (14)], [1, (11)]. For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leq i$ holds $|S_2(i) - \lim S_3| < r$ by [13, (14)], [1, (11)]. \square

- (21) Let us consider a non empty, closed interval subset A of \mathbb{R} , a function ϱ from A into \mathbb{R} , and a continuous partial function u from \mathbb{R} to \mathbb{R} .

Suppose ϱ is bounded-variation and $\text{dom } u = A$. Then u is Riemann-Stieltjes integrable with ϱ .

PROOF: Consider T_0 being a division sequence of A such that δ_{T_0} is convergent and $\lim \delta_{T_0} = 0$. Set $S_0 =$ the middle volume sequence of ϱ , u and T_0 . Set $I = \lim \text{middle-sum}(S_0)$. For every division sequence T of A and for every middle volume sequence S of ϱ , u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds $\text{middle-sum}(S)$ is convergent and $\lim \text{middle-sum}(S) = I$ by (18), [13, (15)], (19), [13, (16)]. \square

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