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Introduction to Stochastic Finance: Random Variables and Arbitrage Theory

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Summary. Using the Mizar system [1], [5], we start to show, that the Call-Option, the Put-Option and the Straddle (more generally defined as in the literature) are random variables ([4], p. 15), see (Def. 1) and (Def. 2). Next we construct and prove the simple random variables ([2], p. 14) in (Def. 8).

In the third section, we introduce the definition of arbitrage opportunity, see (Def. 12). Next we show, that this definition can be characterized in a different way (Lemma 1.3. in [4], p. 5), see (17). In our formalization for Lemma 1.3 we make the assumption that φ is a sequence of real numbers (there are only finitely many valued of interest, the values of φ in R^d). For the definition of almost sure with probability 1 see p. 6 in [2]. Last we introduce the risk-neutral probability (Definition 1.4, p. 6 in [4]), here see (Def. 16).

We give an example in real world: Suppose you have some assets like bonds (riskless assets). Then we can fix our price for these bonds with x for today and $x \cdot (1 + r)$ for tomorrow, r is the interest rate. So we simply assume, that in every possible market evolution of tomorrow we have a determinated value. Then every probability measure of Ω_{fut1} is a risk-neutral measure, see (21). This example shows the existence of some risk-neutral measure. If you find more than one of them, you can determine – with an additional conidition to the probability measures – whether a market model is arbitrage free or not (see Theorem 1.6. in [4], p. 6.)

A short graph for (21):

Suppose we have a portfolio with many (in this example infinitely many) assets. For asset d we have the price $\pi(d)$ for today, and the price $\pi(d) \cdot (1 + r)$ for tomorrow with some interest rate $r > 0$.

Let G be a sequence of random variables on Ω_{fut1}, Borel sets. So you have many functions $f_k : \{1, 2, 3, 4\} \to R$ with $G(k) = f_k$ and f_k is a random variable of Ω_{fut1}, Borel sets. For every f_k we have $f_k(w) = \pi(k)\cdot(1+r)$ for $w \in \{1, 2, 3, 4\}$.

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Today	*Tomorrow*
only one scenario	$\begin{cases} w_{21} = \{1,2\}, \\ w_{22} = \{3,4\}, \end{cases}$
for all $d \in \mathbb{N}$ holds $\pi(d)$	$\begin{cases} f_d(w) = G(d)(w) = \pi(d) \cdot (1+r), \\ w \in w_{21} \text{ or } w \in w_{22}, \\ r > 0 \text{ is the interest rate.} \end{cases}$

Here, every probability measure of Ω_{fut1} is a risk-neutral measure.

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1. Put-Option, Call-Option and Straddle are Random Variables

From now on Ω denotes a non empty set and F denotes a σ-field of subsets of Ω.

Now we state the propositions:

(1) $]0, +\infty[$ is an element of the Borel sets.

(2) Let us consider a random variable R of F and the Borel sets, an element K of $\mathbb{R}$, and a function g from Ω into $\mathbb{R}$. Suppose $g = \chi_{(R-(\Omega \longmapsto K))^{-1}([0,+\infty[),\Omega}$. Then Call-Option$(R, K) = g \cdot (R - (\Omega \longmapsto K))$.

(3) Let us consider a random variable R of F and the Borel sets, and a real number K. Then $(\Omega \longmapsto K) - R$ is a random variable of F and the Borel sets.

(4) Let us consider an element A of F. Then $\chi_{A,\Omega}$ is a random variable of F and the Borel sets.

(5) $\chi_{\Omega,\Omega}$ is random variable on F and the Borel sets. The theorem is a consequence of (4).

(6) Let us consider random variables f, R of F and the Borel sets, and a real number K. Then $(f - R)^{-1}([0, +\infty[)$ is an element of F. The theorem is a consequence of (1).

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. Let us note that the functor Call-Option(R, K) yields a random variable of F and the Borel sets. The functor Put-Option(R, K) yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 1) for every element w of Ω, if $((\Omega \longmapsto K) - R)(w) \geqslant 0$, then $it(w) = ((\Omega \longmapsto K) - R)(w)$ and if $((\Omega \longmapsto K) - R)(w) < 0$, then $it(w) = 0$.

Now we state the proposition:

(7) Let us consider a random variable R of F and the Borel sets, a real number K, and a function g from Ω into $\mathbb{R}$. Suppose $g = \chi_{((\Omega \longmapsto K)-R)^{-1}([0,+\infty[),\Omega}$. Then Put-Option$(R, K) = g \cdot ((\Omega \longmapsto K) - R)$.

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. Note that the functor Put-Option(R, K) yields a random variable of F and the Borel sets.

2. Simple Random Variables

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. The functor Straddle(R, K) yielding a random variable of F and the Borel sets is defined by the term

(Def. 2) Put-Option(R, K) + Call-Option(R, K).

Now we state the proposition:

(8) Let us consider a random variable R of F and the Borel sets, a real number K, and an element w of Ω. Then $(\text{Straddle}(R, K))(w) = |(R - (\Omega \longmapsto K))(w)|$.

Let us consider Ω and F. The functors: the set of constants F and the set of χ_F yielding sets are defined by terms

(Def. 3) $\{f$, where f is a function from Ω into $\mathbb{R}$: f is random variable on F and the Borel sets and constant$\}$,

(Def. 4) $\{\chi_{A,\Omega}$, where A is an element of F : $\chi_{A,\Omega}$ is random variable on F and the Borel sets$\}$,

respectively. Let X be a set. We say that X is F-random membered if and only if

(Def. 5) for every object x such that $x \in X$ there exists a function f from Ω into $\mathbb{R}$ such that $f = x$ and f is random variable on F and the Borel sets.

Observe that the set of constants F is non empty and the set of χ_F is non empty and the set of constants F is F-random membered and the set of χ_F is F-random membered and there exists a set which is F-random membered and non empty.

Let D be an F-random membered, non empty set, C_1 be a sequence of D, and n be a natural number. The change type of C_1 and n yielding a random variable of F and the Borel sets is defined by the term

(Def. 6) $C_1(n)$.

Let C_2 be a sequence of D and w be an element of Ω. The change all types of C_2 and w yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 7) for every natural number n, $it(n) =$ (the change type of C_2 and n)(w).

Let D_1, D_2 be F-random membered, non empty sets, C_1 be a sequence of D_1, C_2 be a sequence of D_2, and n be a natural number. The simple $\mathcal{RV}$ of C_1, C_2 and n yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 8) for every element w of Ω, $it(w) = (\sum_{\alpha=0}^{\kappa}$((the change all types of C_2 and w) $\cdot$ (the change all types of C_1 and w))$(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Observe that the simple $\mathcal{RV}$ of C_1, C_2 and n yields a random variable of F and the Borel sets.

3. Arbitrage Theory: Definition and Alternative Representation

From now on φ denotes a sequence of real numbers and π denotes a price function.

Let us consider Ω and F. Let q be a natural number and G be a sequence of the set of random variables on F and the Borel sets. The change element to functions G and q yielding a real-valued random variable on F is defined by the term

(Def. 9) $G(q)$.

Let us consider φ. Let n be a natural number. The functors: the first $\mathcal{AO}$-set of φ, Ω, F, G and n and the second $\mathcal{AO}$-set of φ, Ω, F, G and n yielding elements of F are defined by terms

(Def. 10) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and n)$^{-1}([0, +\infty[)$,

(Def. 11) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and n)$^{-1}(]0, +\infty[)$,

respectively. Let P be a probability on F and π be a price function. We say that there exists an $\mathcal{AO}$ w.r.t. P, G, π and n if and only if

(Def. 12) there exists a sequence φ of real numbers such that the buy portfolio extension of φ, π, and $n \leqslant 0$ and P(the first $\mathcal{AO}$-set of φ, Ω, F, G and n) $= 1$ and P(the second $\mathcal{AO}$-set of φ, Ω, F, G and n) > 0.

Let r be a real number. The first $\mathcal{RV}$ of r yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 13) $\{1, 2, 3, 4\} \longmapsto r$.

Let π be a price function and d be a natural number. The first $\mathcal{RV}$ of π, r and d yielding an element of the set of random variables on Ω_{fut1} and the Borel sets is defined by the term

(Def. 14) the first $\mathcal{RV}$ of $\pi(d) \cdot (1 + r)$.

Now we state the propositions:

(9) There exists a sequence G of the set of random variables on Ω_{now} and the Borel sets such that

(i) $G(0) = \{1, 2, 3, 4\} \longmapsto 1$, and

(ii) $G(1) = \{1, 2, 3, 4\} \longmapsto 5$, and

(iii) for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \longmapsto 0$.

PROOF: Define $\mathcal{U}$(natural number) $= (\$_1 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (\$_1 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. Consider f being a sequence of the set of random variables on Ω_{now} and the Borel sets such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$. $f(0) = (0 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (0 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. $f(1) = (1 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (1 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. For every natural number k such that $k > 1$ holds $f(k) = \{1, 2, 3, 4\} \longmapsto 0$. $\square$

(10) Let us consider a probability P on Ω_{now}, and a sequence G of the set of random variables on Ω_{now} and the Borel sets. Suppose $G(0) = \{1, 2, 3, 4\} \longmapsto 1$ and $G(1) = \{1, 2, 3, 4\} \longmapsto 5$ and for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \longmapsto 0$. Then there exists a price function π such that there exists an $\mathcal{AO}$ w.r.t. P, G, π and 1.
PROOF: Set $\Omega = \{1, 2, 3, 4\}$. Set $F = \Omega_{now}$. $P(\Omega) = 1$ and $P(\emptyset) = 0$. Define $\mathcal{U}$(element of $\mathbb{N}$) $= (\$_1 = 0 \rightarrow 1, (\$_1 = 1 \rightarrow 1, 0))(\in \mathbb{R})$. Consider f being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$. f is a price function. Reconsider $\pi = f$ as a price function. Define $\mathcal{U}$(element of $\mathbb{N}$) $= (\$_1 = 0 \rightarrow -1, (\$_1 = 1 \rightarrow 1, 0))(\in \mathbb{R})$. Consider φ being a sequence of real numbers such that for every element k of $\mathbb{N}$, $\varphi(k) = \mathcal{U}(k)$. P(the first $\mathcal{AO}$-set of φ, Ω, F, G and 1) $= 1$ and P(the second $\mathcal{AO}$-set of φ, Ω, F, G and 1) > 0 and the buy portfolio extension of φ, π, and $1 \leqslant 0$ by [7, (9)]. $\square$

(11) Let us consider a natural number n, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of n, φ, F, G and $w \geqslant 0\} =$ (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1, a real number r, and a sequence G of the set of random variables on F and the Borel sets.

(12) Suppose $d_1 = d - 1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\} =$ ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)))^{-1}([0, +\infty[)$.
PROOF: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\}$. Set $S_2 =$ ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and

$d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}([0, +\infty[)$. For every object x, $x \in S_1$ iff $x \in S_2$. $\square$

(13) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}([0, +\infty[)$ is an event of F.

(14) Let us consider a natural number d, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of d, φ, F, G and $w > 0\}$ = (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $d)^{-1}(]0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1, a real number r, and a sequence G of the set of random variables on F and the Borel sets.

(15) Suppose $d_1 = d - 1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\}$ = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0, +\infty[)$.
PROOF: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\}$. Set S_2 = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0, +\infty[)$. For every object x, $x \in S_1$ iff $x \in S_2$. $\square$

(16) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0, +\infty[)$ is an event of F.

(17) Let us consider a price function π, and natural numbers d, d_1. Suppose $d > 0$ and $d_1 = d - 1$. Let us consider a probability P on F, and a real number r. Suppose $r > -1$. Let us consider a sequence G of the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = \Omega \longmapsto 1 + r$. Then there exists an $\mathcal{AO}$ w.r.t. P, G, π and d if and only if there exists a sequence φ_1 of real numbers such that P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}([0, +\infty[)) = 1$ and P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}(]0, +\infty[)) > 0$.
PROOF: If there exists an $\mathcal{AO}$ w.r.t. P, G, π and d, then there exists a sequence φ_1 of real numbers such that P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}([0, +\infty[)) = 1$ and P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}(]0, +\infty[)) > 0$. Define $\mathcal{U}$(natural number) $= (\$_1 = 0 \rightarrow -$(the buy portfolio of φ_1, π, and $d), \varphi_1(\$_1))(\in \mathbb{R})$. Consider φ being a se-

quence of real numbers such that for every element n of $\mathbb{N}$, $\varphi(n) = \mathcal{U}(n)$. For every natural number n, if $n = 0$, then $\varphi(n) = -$(the buy portfolio of φ_1, π, and d) and if $n > 0$, then $\varphi(n) = \varphi_1(n)$. The buy portfolio extension of φ, π, and $d = 0$. P(the first $\mathcal{AO}$-set of φ, Ω, F, G and $d) = 1$. P(the second $\mathcal{AO}$-set of φ, Ω, F, G and $d) > 0$. □

4. Risk-Neutral Probability Measure

Let us consider Ω and F. Let R be a real-valued random variable on F and r be a real number. The r-discounted value of R yielding a real-valued random variable on F is defined by the term

(Def. 15) $R \cdot \frac{1}{1+r}$.

Let π be a price function and G be a sequence of the set of random variables on F and the Borel sets. We say that there exists a risk neutral measure w.r.t. G, π and r if and only if

(Def. 16) there exists a probability P on F such that for every natural number d, $\pi(d) = E_P\{$the r-discounted value of (the change element to functions G and d)$\}$.

From now on P denotes a probability on Ω_{fut1}.

Now we state the propositions:

(18) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π, and a natural number d. Then there exists a real-valued random variable f on Ω_{fut1} such that

(i) $f = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$, and

(ii) f is integrable on P2M(P), and

(iii) f is simple function in Ω_{fut1}.

Proof: Set $\Omega_2 = \{1, 2, 3, 4\}$. Define $\mathcal{U}$(element of Ω_2) $= \pi(d) \cdot (1+r) (\in \mathbb{R})$. Consider f being a function from Ω_2 into $\mathbb{R}$ such that for every element d of Ω_2, $f(d) = \mathcal{U}(d)$. Set $g = \Omega_2 \longmapsto \pi(d) \cdot (1 + r) (\in \mathbb{R})$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$. f is integrable on P2M(P) by [6, (9), (3)], [3, (12)]. □

(19) Let us consider a natural number n, and a real number r. Suppose $r > 0$. Let us consider a price function π, a natural number d, and a real-valued random variable R on Ω_{fut1}. Suppose $R = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and R is integrable on P2M(P) and R is simple function in Ω_{fut1}. Then $\pi(d) = E_P\{$the r-discounted value of $R\}$.

Proof: Set $F = \Omega_{fut1}$. $\overline{\mathbb{R}}(R) = R$ and R is non-negative. Set $m = \pi(d) \cdot (1 + r)$. for every object x such that $x \in \operatorname{dom} \overline{\mathbb{R}}(R)$ holds $(\overline{\mathbb{R}}(R))(x) = m$ and $\operatorname{dom} \overline{\mathbb{R}}(R) \in F$ and $0 \leqslant m$. □

(20) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π. Then there exists a sequence G of the set of random variables on Ω_{fut1} and the Borel sets such that for every natural number d, $G(d) = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on P2M(P) and the change element to functions G and d is simple function in Ω_{fut1}.

Proof: Define $\mathcal{U}$(natural number) = the first $\mathcal{RV}$ of π, r and $\$_1$. Consider g being a sequence of the set of random variables on Ω_{fut1} and the Borel sets such that for every element d of $\mathbb{N}$, $g(d) = \mathcal{U}(d)$. There exists a real-valued random variable R on Ω_{fut1} such that $R = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)(\in \mathbb{R})$ and R is integrable on P2M(P) and R is simple function in Ω_{fut1}. □

(21) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π, and a sequence G of the set of random variables on Ω_{fut1} and the Borel sets. Suppose for every natural number d, $G(d) = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on P2M(P) and the change element to functions G and d is simple function in Ω_{fut1}. Then

(i) there exists a risk neutral measure w.r.t. G, π and r, and

(ii) for every natural number s, $\pi(s) = E_P\{$the r-discounted value of (the change element to functions G and s)$\}$.

The theorem is a consequence of (19).

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Kleene Algebra of Partial Predicates

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Summary. We show that the set of all partial predicates over a set D together with the disjunction, conjunction, and negation operations, defined in accordance with the truth tables of S.C. Kleene's strong logic of indeterminacy [17], forms a Kleene algebra. A Kleene algebra is a De Morgan algebra [3] (also called quasi-Boolean algebra) which satisfies the condition $x \wedge \neg x \leqslant y \vee \neg y$ (sometimes called the normality axiom). We use the formalization of De Morgan algebras from [8].

The term "Kleene algebra" was introduced by A. Monteiro and D. Brignole in [3]. A similar notion of a "normal i-lattice" had been previously studied by J.A. Kalman [16]. More details about the origin of this notion and its relation to other notions can be found in [24, 4, 1, 2]. It should be noted that there is a different widely known class of algebras, also called Kleene algebras [22, 6], which generalize the algebra of regular expressions, however, the term "Kleene algebra" used in this paper does not refer to them.

Algebras of partial predicates naturally arise in computability theory in the study on partial recursive predicates. They were studied in connection with non-classical logics [17, 5, 18, 32, 29, 30]. A partial predicate also corresponds to the notion of a partial set [26] on a given domain, which represents a (partial) property which for any given element of this domain may hold, not hold, or neither hold nor not hold. The field of all partial sets on a given domain is an algebra with generalized operations of union, intersection, complement, and three constants (0, 1, n which is the fixed point of complement) which can be generalized to an equational class of algebras called DMF-algebras (De Morgan algebras with a single fixed point of involution) [25]. In [27] partial sets and DMF-algebras were considered as a basis for unification of set-theoretic and linguistic approaches to probability.

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Partial predicates over classes of mathematical models of data were used for formalizing semantics of computer programs in the composition-nominative approach to program formalization [31, 28, 33, 15], for formalizing extensions of the Floyd-Hoare logic [7, 9] which allow reasoning about properties of programs in the case of partial pre- and postconditions [23, 20, 19, 21], for formalizing dynamical models with partial behaviors in the context of the mathematical systems theory [11, 13, 14, 12, 10].

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1. Partial Predicates

From now on x denotes an object and D denotes a set.

Let us consider D. The functor $\mathrm{Pr}(D)$ yielding a set is defined by the term

(Def. 1) $D \dot{\to} Boolean$.

Observe that $\mathrm{Pr}(D)$ is non empty and functional.

A partial predicate of D is a partial function from D to *Boolean*. From now on p denotes a partial predicate of D.

Now we state the propositions:

(1) If $x \in \mathrm{Pr}(D)$, then x is a partial predicate of D.

(2) $p \in \mathrm{Pr}(D)$.

(3) If $x \in \operatorname{dom} p$, then $p(x) = true$ or $p(x) = false$.

Let us consider D. The functor $\mathrm{PPneg}(D)$ yielding a function from $\mathrm{Pr}(D)$ into $\mathrm{Pr}(D)$ is defined by

(Def. 2) for every partial predicate p of D, $\operatorname{dom}(it(p)) = \operatorname{dom} p$ and for every object d, if $d \in \operatorname{dom} p$ and $p(d) = true$, then $it(p)(d) = false$ and if $d \in \operatorname{dom} p$ and $p(d) = false$, then $it(p)(d) = true$.

Let us consider p. The functor $\neg p$ yielding a partial predicate of D is defined by the term

(Def. 3) $(\mathrm{PPneg}(D))(p)$.

Let us note that the functor is involutive.

Now we state the propositions:

(4) If $x \in \operatorname{dom} p$ and $(\neg p)(x) = false$, then $p(x) = true$. The theorem is a consequence of (3).

(5) If $x \in \operatorname{dom} p$ and $(\neg p)(x) = true$, then $p(x) = false$. The theorem is a consequence of (3).

(6) If $x \in \operatorname{dom} \neg p$ and $(\neg p)(x) = \mathit{false}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$. The theorem is a consequence of (3).

(7) If $x \in \operatorname{dom} \neg p$ and $(\neg p)(x) = \mathit{true}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$. The theorem is a consequence of (3).

In the sequel D denotes a non empty set and p, q, r denote partial predicates of D.

Let us consider D. The functor $\mathrm{PPdisj}(D)$ yielding a function from $\mathrm{Pr}(D) \times \mathrm{Pr}(D)$ into $\mathrm{Pr}(D)$ is defined by

(Def. 4) for every partial predicates p, q of D, $\operatorname{dom} \mathit{it}(p, q) = \{d$, where d is an element of $D : d \in \operatorname{dom} p$ and $p(d) = \mathit{true}$ or $d \in \operatorname{dom} q$ and $q(d) = \mathit{true}$ or $d \in \operatorname{dom} p$ and $p(d) = \mathit{false}$ and $d \in \operatorname{dom} q$ and $q(d) = \mathit{false}\}$ and for every object d, if $d \in \operatorname{dom} p$ and $p(d) = \mathit{true}$ or $d \in \operatorname{dom} q$ and $q(d) = \mathit{true}$, then $\mathit{it}(p, q)(d) = \mathit{true}$ and if $d \in \operatorname{dom} p$ and $p(d) = \mathit{false}$ and $d \in \operatorname{dom} q$ and $q(d) = \mathit{false}$, then $\mathit{it}(p, q)(d) = \mathit{false}$.

Let us consider p and q. The functor $p \vee q$ yielding a partial predicate of D is defined by the term

(Def. 5) $(\mathrm{PPdisj}(D))(p, q)$.

Observe that the functor is commutative and idempotent.

Now we state the propositions:

(8) Suppose $x \in \operatorname{dom}(p \vee q)$. Then

(i) $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$, or

(ii) $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$, or

(iii) $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$.

(9) If $x \in \operatorname{dom} p$ and $x \in \operatorname{dom} q$ and $(p \vee q)(x) = \mathit{true}$, then $p(x) = \mathit{true}$ or $q(x) = \mathit{true}$. The theorem is a consequence of (3).

(10) If $x \in \operatorname{dom}(p \vee q)$ and $(p \vee q)(x) = \mathit{true}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$ or $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$. The theorem is a consequence of (8) and (9).

(11) If $x \in \operatorname{dom} p$ and $(p \vee q)(x) = \mathit{false}$, then $p(x) = \mathit{false}$. The theorem is a consequence of (3).

(12) If $x \in \operatorname{dom} q$ and $(p \vee q)(x) = \mathit{false}$, then $q(x) = \mathit{false}$. The theorem is a consequence of (3).

(13) If $x \in \operatorname{dom}(p \vee q)$ and $(p \vee q)(x) = \mathit{false}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$. The theorem is a consequence of (8) and (12).

(14) Associativity law:
$p \vee (q \vee r) = (p \vee q) \vee r$. The theorem is a consequence of (8) and (11).

(15) $(p \vee q) \vee (p \vee r) = (p \vee q) \vee r$. The theorem is a consequence of (14).

Let us consider D, p, and q. The functor $p \wedge q$ yielding a partial predicate of D is defined by the term

(Def. 6) $\neg(\neg p \vee \neg q)$.

Observe that the functor is commutative and idempotent. The functor $p \Rightarrow q$ yielding a partial predicate of D is defined by the term

(Def. 7) $\neg p \vee q$.

Now we state the propositions:

(16) $\mathrm{dom}(p \wedge q) = \{d$, where d is an element of $D : d \in \mathrm{dom}\, p$ and $p(d) = \mathit{false}$ or $d \in \mathrm{dom}\, q$ and $q(d) = \mathit{false}$ or $d \in \mathrm{dom}\, p$ and $p(d) = \mathit{true}$ and $d \in \mathrm{dom}\, q$ and $q(d) = \mathit{true}\}$. The theorem is a consequence of (5) and (4).

(17) Suppose $x \in \mathrm{dom}(p \wedge q)$. Then

(i) $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$, or

(ii) $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$, or

(iii) $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$.

The theorem is a consequence of (16).

(18) If $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$, then $(p \wedge q)(x) = \mathit{true}$.

(19) If $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$, then $(p \wedge q)(x) = \mathit{false}$.

(20) If $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$, then $(p \wedge q)(x) = \mathit{false}$.

(21) If $x \in \mathrm{dom}\, p$ and $(p \wedge q)(x) = \mathit{true}$, then $p(x) = \mathit{true}$.

(22) If $x \in \mathrm{dom}\, q$ and $(p \wedge q)(x) = \mathit{true}$, then $q(x) = \mathit{true}$.

(23) If $x \in \mathrm{dom}(p \wedge q)$ and $(p \wedge q)(x) = \mathit{true}$, then $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$. The theorem is a consequence of (17) and (19).

(24) If $x \in \mathrm{dom}\, p$ and $x \in \mathrm{dom}\, q$ and $(p \wedge q)(x) = \mathit{false}$, then $p(x) = \mathit{false}$ or $q(x) = \mathit{false}$. The theorem is a consequence of (18) and (3).

(25) If $x \in \mathrm{dom}(p \wedge q)$ and $(p \wedge q)(x) = \mathit{false}$, then $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$ or $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$. The theorem is a consequence of (17) and (24).

(26) Associativity law:
$p \wedge (q \wedge r) = (p \wedge q) \wedge r$.

(27) $(p \wedge q) \wedge (p \wedge r) = (p \wedge q) \wedge r$.

(28) Meet-absorbing law:
$(p \wedge q) \vee q = q$. The theorem is a consequence of (16), (8), (17), (19), and (3).

(29) Join-absorbing law:
$p \wedge (p \vee q) = p$. The theorem is a consequence of (16), (17), (8), (3), (19), and (18).

(30) Distributivity law:
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. The theorem is a consequence of (16), (17), (19), (13), (10), (18), (8), (23), and (25).

(31) $\mathrm{dom}(p \Rightarrow q) = \{d$, where d is an element of $D : d \in \mathrm{dom}\, p$ and $p(d) =$ *false* or $d \in \mathrm{dom}\, q$ and $q(d) =$ *true* or $d \in \mathrm{dom}\, p$ and $p(d) =$ *true* and $d \in \mathrm{dom}\, q$ and $q(d) =$ *false*$\}$. The theorem is a consequence of (5) and (4).

(32) Suppose $x \in \mathrm{dom}(p \Rightarrow q)$. Then

(i) $x \in \mathrm{dom}\, p$ and $p(x) =$ *false*, or

(ii) $x \in \mathrm{dom}\, q$ and $q(x) =$ *true*, or

(iii) $x \in \mathrm{dom}\, p$ and $p(x) =$ *true* and $x \in \mathrm{dom}\, q$ and $q(x) =$ *false*.

The theorem is a consequence of (31).

(33) If $x \in \mathrm{dom}\, p$ and $p(x) =$ *false*, then $(p \Rightarrow q)(x) =$ *true*.

(34) If $x \in \mathrm{dom}\, q$ and $q(x) =$ *true*, then $(p \Rightarrow q)(x) =$ *true*.

(35) If $x \in \mathrm{dom}\, p$ and $p(x) =$ *true* and $x \in \mathrm{dom}\, q$ and $q(x) =$ *false*, then $(p \Rightarrow q)(x) =$ *false*.

(36) If $x \in \mathrm{dom}\, p$ and $x \in \mathrm{dom}\, q$ and $(p \Rightarrow q)(x) =$ *true*, then $p(x) =$ *false* or $q(x) =$ *true*. The theorem is a consequence of (35) and (3).

(37) If $x \in \mathrm{dom}\, p$ and $(p \Rightarrow q)(x) =$ *false*, then $p(x) =$ *true*.

(38) If $x \in \mathrm{dom}\, q$ and $(p \Rightarrow q)(x) =$ *false*, then $q(x) =$ *false*.

(39) If $x \in \mathrm{dom}(p \Rightarrow q)$ and $(p \Rightarrow q)(x) =$ *false*, then $x \in \mathrm{dom}\, p$ and $p(x) =$ *true* and $x \in \mathrm{dom}\, q$ and $q(x) =$ *false*. The theorem is a consequence of (32) and (33).

(40) If $x \in \mathrm{dom}(p \Rightarrow q)$ and $(p \Rightarrow q)(x) =$ *true*, then $x \in \mathrm{dom}\, p$ and $p(x) =$ *false* or $x \in \mathrm{dom}\, q$ and $q(x) =$ *true*. The theorem is a consequence of (32) and (35).

(41) $(p \Rightarrow r) \wedge (q \Rightarrow r) = (p \vee q) \Rightarrow r$. The theorem is a consequence of (30).

(42) $(p \Rightarrow r) \vee (q \Rightarrow r) = (p \wedge q) \Rightarrow r$. The theorem is a consequence of (15) and (14).

Let D be a set. The functor $\mathrm{true}_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 8) $D \longmapsto$ *true*.

Let D be a set. The functor $\mathrm{false}_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 9) $D \longmapsto$ *false*.

Let us consider a set D. Now we state the propositions:

(43) $\neg\,\mathrm{false}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(44) $\neg\,\mathrm{true}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$. The theorem is a consequence of (43).

Now we state the propositions:

(45) $p \vee \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(46) $\mathrm{true}_{\mathrm{PP}}(D) \vee p = \mathrm{true}_{\mathrm{PP}}(D)$.

(47) $p \wedge \mathrm{false}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$.

(48) $\mathrm{false}_{\mathrm{PP}}(D) \wedge p = \mathrm{false}_{\mathrm{PP}}(D)$.

(49) $p \vee \neg p = \mathrm{true}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, p$. The theorem is a consequence of (8) and (3).

(50) $\neg p \vee p = \mathrm{true}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, p$.

(51) $p \wedge \neg p = \mathrm{false}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, p$. The theorem is a consequence of (16), (17), (3), and (19).

(52) $\neg p \wedge p = \mathrm{false}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, p$.

(53) $\mathrm{false}_{\mathrm{PP}}(D) \Rightarrow p = \mathrm{true}_{\mathrm{PP}}(D)$. The theorem is a consequence of (43) and (45).

(54) $p \Rightarrow \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(55) $\mathrm{false}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, p \vee \mathrm{true}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, q = \mathrm{true}_{\mathrm{PP}}(D){\restriction}\,\mathrm{dom}\, q$.

Let D be a set. The functor $\bot_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 10) $\emptyset$.

Now we state the propositions:

(56) Let us consider a set D. Then $\neg\bot_{\mathrm{PP}}(D) = \bot_{\mathrm{PP}}(D)$.

(57) $\bot_{\mathrm{PP}}(D) \vee \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(58) $\bot_{\mathrm{PP}}(D) \wedge \mathrm{false}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$.

(59) $\bot_{\mathrm{PP}}(D) \Rightarrow \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$. The theorem is a consequence of (56) and (57).

2. Algebra of Partial Connectives with (strong) Kleene Logical Connectives

Let us consider D. The functors: $\bigwedge_D$ and $\bigvee_D$ yielding binary operations on $\mathrm{Pr}(D)$ are defined by conditions

(Def. 11) for every partial predicates p, q of D, $\bigwedge_D(p, q) = p \wedge q$,

(Def. 12) for every partial predicates p, q of D, $\bigvee_D(p, q) = p \vee q$,

respectively.The functor $\bar{\cdot}_D$ yielding a unary operation on $\mathrm{Pr}(D)$ is defined by

(Def. 13) for every partial predicate p of D, $it(p) = \neg p$.

The functor PartPredLatt(D) yielding a strict ortholattice structure is defined by the term

(Def. 14) $\langle \mathrm{Pr}(D), \bigvee_D, \bigwedge_D, \bar{\cdot}_D \rangle$.

Let D be a non empty set, f, g be binary operations on D, and h be a unary operation on D. One can verify that $\langle D, f, g, h\rangle$ is non empty.

Let us consider D. Let us note that PartPredLatt(D) is non empty and constituted functions and there exists a lattice structure which is non empty and constituted functions and there exists an ortholattice structure which is strict, non empty, and constituted functions.

Let us consider D. One can verify that PartPredLatt(D) is lattice-like and PartPredLatt(D) is bounded and PartPredLatt(D) is de Morgan and join-idempotent and has idempotent element.

Now we state the propositions:

(60) $\top_{\mathrm{PartPredLatt}(D)} = \mathrm{true}_{\mathrm{PP}}(D)$.

(61) $\bot_{\mathrm{PartPredLatt}(D)} = \mathrm{false}_{\mathrm{PP}}(D)$.

Let L be a non empty, constituted functions lattice structure and a, b be elements of L. We say that a is a partial complement of b if and only if

(Def. 15) $a \sqcup b = \top_L \restriction \mathrm{dom}\, b$ and $b \sqcup a = \top_L \restriction \mathrm{dom}\, b$ and $a \sqcap b = \bot_L \restriction \mathrm{dom}\, b$ and $b \sqcap a = \bot_L \restriction \mathrm{dom}\, b$.

We say that L is partially complemented if and only if

(Def. 16) for every element b of L, there exists an element a of L such that a is a partial complement of b.

Let L be a constituted functions, non empty lattice structure. We say that L is partially Boolean if and only if

(Def. 17) L is bounded, partially complemented, and distributive.

One can verify that every constituted functions, non empty lattice structure which is partially Boolean is also bounded, partially complemented, and distributive and every constituted functions, non empty lattice structure which is bounded, partially complemented, and distributive is also partially Boolean.

Now we state the proposition:

(62) Let us consider elements a, b of PartPredLatt(D). If $a = p$ and $b = \neg p$, then b is a partial complement of a. The theorem is a consequence of (60), (49), (61), and (51).

Let us consider D. Note that PartPredLatt(D) is partially Boolean.

Now we state the proposition:

(63) Distributivity law:
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

Let L be a non empty ortholattice structure. We say that L is Kleene if and only if

(Def. 18) for every elements x, y of L, $x \sqcap x^{\mathrm{c}} \sqsubseteq y \sqcup y^{\mathrm{c}}$.

Let us observe that every meet-absorbing, join-absorbing, meet-commutative, non empty ortholattice structure which is Boolean and well-complemented is also Kleene.

Let us consider D. Observe that PartPredLatt(D) is Kleene and there exists a non empty, constituted functions lattice structure which is partially Boolean, join-idempotent, and lattice-like and there exists a non empty ortholattice structure which is Kleene, de Morgan, join-idempotent, lattice-like, and strict and has idempotent element and there exists a non empty, constituted functions ortholattice structure which is partially Boolean, Kleene, de Morgan, join-idempotent, lattice-like, and strict and has idempotent element.

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Klein-Beltrami Model. Part I

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Summary. Tim Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [3], [4], [14], [5].

With the Mizar system [2], [7] we use some ideas are taken from Tim Makarios' MSc thesis [13] for the formalization of some definitions (like the absolute) and lemmas necessary for the verification of the independence of the parallel postulate. This work can be also treated as further development of Tarski's geometry in the formal setting [6]. Note that the model presented here, may also be called "Beltrami-Klein Model", "Klein disk model", and the "Cayley-Klein model" [1].

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Keywords: Tarski's geometry axioms; foundations of geometry; Klein-Beltrami model

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1. Preliminaries

From now on a, b, c, d, e, f denote real numbers, g denotes a positive real number, x, y denote complex numbers, S, T denote elements of $\mathcal{R}^2$, and u, v, w denote elements of $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the propositions:

(1) Let us consider elements P_1, P_2, P_3 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose u is not zero and v is not zero and w is not zero and P_1 = the direction of u and P_2 = the direction of v and P_3 = the direction of w. Then P_1, P_2 and P_3 are collinear if and only if $\langle|u, v, w|\rangle = 0$.

[1] https://www.isa-afp.org/entries/Tarskis_Geometry.html
[2] https://github.com/jrh13/hol-light/blob/master/100/independence.ml

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(2) If ($a \neq 0$ or $b \neq 0$) and $a \cdot d = b \cdot c$, then there exists e such that $c = e \cdot a$ and $d = e \cdot b$.

(3) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $(c \cdot a)^{\mathbf{2}} + (c \cdot b)^{\mathbf{2}} = 1$, then $c = 1$ or $c = -1$.

(4) $a \cdot u + (-a) \cdot u = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(5) If $0 \leqslant a$ and $c < 0$ and $\Delta(a, b, c) = 0$, then $a = 0$.
Proof: $0 \leqslant b^{\mathbf{2}}$. □

(6) $\sum({}^2(T - S)) = \sum({}^2(S - T))$.

(7) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $c^{\mathbf{2}} + d^{\mathbf{2}} = 1$ and $c \cdot a + d \cdot b = 1$, then $b \cdot c = a \cdot d$.

(8) If $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $a = 0$, then $b = 1$ or $b = -1$.

(9) $0 \leqslant a^{\mathbf{2}}$.

(10) If $(a \cdot b)^{\mathbf{2}} + b^{\mathbf{2}} = 1$, then $b = \frac{1}{\sqrt{1+a^{\mathbf{2}}}}$ or $b = \frac{-1}{\sqrt{1+a^{\mathbf{2}}}}$.

(11) If $a \neq 0$ and $b^{\mathbf{2}} = 1 + a \cdot a$, then $a \cdot \frac{1}{b} \cdot a \cdot \frac{-1}{b} + \frac{1}{b} \cdot \frac{-1}{b} = -1$.
Proof: $b \neq 0$. □

(12) $a^{\mathbf{2}} \cdot \frac{1}{b^{\mathbf{2}}} = (\frac{a}{b})^{\mathbf{2}}$.

(13) $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ if and only if $[a, b] \in \mathrm{circle}(0, 0, 1)$.

(14) $a^{\mathbf{2}} + b^{\mathbf{2}} = g^{\mathbf{2}}$ if and only if $[a, b] \in \mathrm{circle}(0, 0, g)$.

(15) If $a \neq 0$ and $-1 < a < 1$ and $b = \frac{2+\sqrt{\Delta(a \cdot a, -2, 1)}}{2 \cdot a \cdot a}$, then $(1 + a \cdot a) \cdot b \cdot b - 2 \cdot b + 1 - b \cdot b = 0$.
Proof: $0 \leqslant 1 - a^{\mathbf{2}}$. $\Delta(a \cdot a, -2, 1) \geqslant 0$. □

(16) Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$ and $-1 < c < 1$. Then there exists d and there exists e and there exists f such that $e = d \cdot c \cdot a + (1 - d) \cdot (-b)$ and $f = d \cdot c \cdot b + (1 - d) \cdot a$ and $e^{\mathbf{2}} + f^{\mathbf{2}} = d^{\mathbf{2}}$.

(17) If $a^{\mathbf{2}} + b^{\mathbf{2}} < 1$ and $c^{\mathbf{2}} + d^{\mathbf{2}} = 1$, then $(\frac{a+c}{2})^{\mathbf{2}} + (\frac{b+d}{2})^{\mathbf{2}} < 1$.

(18) If $|S|^{\mathbf{2}} \leqslant 1$, then $0 \leqslant \Delta(\sum({}^2(T - S)), b, \sum({}^2 S) - 1)$.

(19) If $a^{\mathbf{2}} + b^{\mathbf{2}}$ is negative, then $a = 0$ and $b = 0$.

(20) If $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$, then $\langle |u, v, w| \rangle = 0$.

(21) (i) $a \cdot |(u, v)| = |(a \cdot u, v)|$, and

(ii) $a \cdot |(u, v)| = |(u, a \cdot v)|$.

In the sequel a, b, c denote elements of $\mathbb{R}_{\mathrm{F}}$ and M, N denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

Now we state the propositions:

(22) If $M = \mathrm{symmetric3}(0, 0, 0, 0, 0, 0)$, then $\mathrm{Det}\, M = 0_{\mathbb{R}_{\mathrm{F}}}$.

(23) Suppose $N = \langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$. Then

(i) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,1} = a \cdot (M_{1,1}) \cdot (M_{1,1}) + b \cdot (M_{2,1}) \cdot (M_{2,1}) + c \cdot (M_{3,1}) \cdot (M_{3,1})$, and

(ii) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,2} = a \cdot (M_{1,1}) \cdot (M_{1,2}) + b \cdot (M_{2,1}) \cdot (M_{2,2}) + c \cdot (M_{3,1}) \cdot (M_{3,2})$, and

(iii) $M^{\mathrm{T}} \cdot (N \cdot M)_{1,3} = a \cdot (M_{1,1}) \cdot (M_{1,3}) + b \cdot (M_{2,1}) \cdot (M_{2,3}) + c \cdot (M_{3,1}) \cdot (M_{3,3})$, and

(iv) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,1} = a \cdot (M_{1,2}) \cdot (M_{1,1}) + b \cdot (M_{2,2}) \cdot (M_{2,1}) + c \cdot (M_{3,2}) \cdot (M_{3,1})$, and

(v) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,2} = a \cdot (M_{1,2}) \cdot (M_{1,2}) + b \cdot (M_{2,2}) \cdot (M_{2,2}) + c \cdot (M_{3,2}) \cdot (M_{3,2})$, and

(vi) $M^{\mathrm{T}} \cdot (N \cdot M)_{2,3} = a \cdot (M_{1,2}) \cdot (M_{1,3}) + b \cdot (M_{2,2}) \cdot (M_{2,3}) + c \cdot (M_{3,2}) \cdot (M_{3,3})$, and

(vii) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,1} = a \cdot (M_{1,3}) \cdot (M_{1,1}) + b \cdot (M_{2,3}) \cdot (M_{2,1}) + c \cdot (M_{3,3}) \cdot (M_{3,1})$, and

(viii) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,2} = a \cdot (M_{1,3}) \cdot (M_{1,2}) + b \cdot (M_{2,3}) \cdot (M_{2,2}) + c \cdot (M_{3,3}) \cdot (M_{3,2})$, and

(ix) $M^{\mathrm{T}} \cdot (N \cdot M)_{3,3} = a \cdot (M_{1,3}) \cdot (M_{1,3}) + b \cdot (M_{2,3}) \cdot (M_{2,3}) + c \cdot (M_{3,3}) \cdot (M_{3,3})$.

(24) Let us consider natural numbers m, n, a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension m, and a matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times n$. Suppose $m > 0$. Then $M \cdot N$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times n$.

In the sequel D denotes a non empty set, d_1, d_2, d_3 denote elements of D, A denotes a matrix over D of dimension 1×3, and B denotes a matrix over D of dimension 3×1.

Now we state the propositions:

(25) Let us consider a square matrix M over D of dimension 1. Then $M^{\mathrm{T}} = M$.

(26) A^{T} is 3,1-size.

(27) $\langle\langle d_1, d_2, d_3 \rangle\rangle$ is a matrix over D of dimension 1×3.

(28) $\langle\langle d_1 \rangle, \langle d_2 \rangle, \langle d_3 \rangle\rangle$ is a matrix over D of dimension 3×1.

(29) $A = \langle\langle A_{1,1}, A_{1,2}, A_{1,3} \rangle\rangle$.

Proof: Reconsider $B = \langle\langle A_{1,1}, A_{1,2}, A_{1,3} \rangle\rangle$ as a matrix over D of dimension 1×3. For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of A holds $A_{i,j} = B_{i,j}$. □

(30) $B = \langle\langle B_{1,1} \rangle, \langle B_{2,1} \rangle, \langle B_{3,1} \rangle\rangle$.

Proof: Reconsider $C = \langle\langle B_{1,1} \rangle, \langle B_{2,1} \rangle, \langle B_{3,1} \rangle\rangle$ as a matrix over D of dimension 3×1. For every natural numbers i, j such that $\langle i, j \rangle \in$ the indices of B holds $B_{i,j} = C_{i,j}$. □

(31) $A^{\mathrm{T}} = \langle\langle A_{1,1} \rangle, \langle A_{1,2} \rangle, \langle A_{1,3} \rangle\rangle$. The theorem is a consequence of (26) and (30).

(32) There exists d_1 and there exists d_2 and there exists d_3 such that $A = \langle\langle d_1, d_2, d_3 \rangle\rangle$. The theorem is a consequence of (29).

(33) Let us consider a finite sequence p of elements of $\mathbb{R}^1$. If $\operatorname{len} p = 3$, then $\text{ColVec2Mx}(\text{M2F}(p)) = p$. The theorem is a consequence of (30).

(34) Let us consider a square matrix M over $\mathbb{R}_{\rm F}$ of dimension 3, a square matrix M_1 over $\mathbb{R}$ of dimension 3, an element v of $\mathcal{E}^3_{\rm T}$, a finite sequence u_1 of elements of $\mathbb{R}_{\rm F}$, a finite sequence u_2 of elements of $\mathbb{R}$, and a finite sequence p of elements of $\mathbb{R}^1$. Suppose $p = M \cdot u_1$ and $v = \text{M2F}(p)$ and $\operatorname{len} u_1 = 3$ and $u_1 = u_2$ and $M_1 = M$. Then $v = M_1 \cdot u_2$.

(35) Let us consider a square matrix N over $\mathbb{R}$ of dimension 3, and a finite sequence u_1 of elements of $\mathbb{R}$. If $u_1 = 0_{\mathcal{E}^3_{\rm T}}$, then $N \cdot u_1 = 0_{\mathcal{E}^3_{\rm T}}$.

(36) Let us consider a square matrix N over $\mathbb{R}$ of dimension 3, a finite sequence u_1 of elements of $\mathbb{R}$, and an element u of $\mathcal{E}^3_{\rm T}$. Suppose N is invertible and $u = u_1$ and u is not zero. Then $N \cdot u_1 \neq 0_{\mathcal{E}^3_{\rm T}}$. The theorem is a consequence of (35).

(37) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, a square matrix N_2 over $\mathbb{R}$ of dimension 3, elements P, Q of the projective space over $\mathcal{E}^3_{\rm T}$, non zero elements u, v of $\mathcal{E}^3_{\rm T}$, and finite sequences v_1, u_2 of elements of $\mathbb{R}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $u = u_2$ and $v = v_1$ and $N = N_2$ and $N_2 \cdot u_2 = v_1$. Then (the homography of $N)(P) = Q$. The theorem is a consequence of (34).

(38) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, a square matrix N_2 over $\mathbb{R}$ of dimension 3, elements P, Q of the projective space over $\mathcal{E}^3_{\rm T}$, non zero elements u, v of $\mathcal{E}^3_{\rm T}$, finite sequences v_1, u_2 of elements of $\mathbb{R}$, and a non zero real number a. Suppose $P =$ the direction of u and $Q =$ the direction of v and $u = u_2$ and $v = v_1$ and $N = N_2$ and $N_2 \cdot u_2 = a \cdot v_1$. Then (the homography of $N)(P) = Q$. The theorem is a consequence of (34) and (36).

Let us consider a finite sequence p of elements of $\mathbb{R}$ and a square matrix M over $\mathbb{R}$ of dimension 3. Now we state the propositions:

(39) If $\operatorname{len} p = 3$, then $|(a \cdot p, M \cdot (b \cdot p))| = a \cdot b \cdot |(p, M \cdot p)|$.

(40) If $\operatorname{len} p = 3$, then $\text{SumAll QuadraticForm}(a \cdot p, M, b \cdot p) = a \cdot b \cdot (\text{SumAll QuadraticForm}(p, M, p))$. The theorem is a consequence of (39).

(41) Let us consider real numbers a, b. Then $[a, b, 1]$ is not zero.

(42) Let us consider an element P of $\mathcal{E}^2_{\rm T}$, an element Q of $\mathcal{E}^2_{\rm T}$, and a real number r. Then $P \in \text{Sphere}(Q, r)$ if and only if $P \in \text{circle}(Q(1), Q(2), r)$.

In the sequel u, v denote non zero elements of $\mathcal{E}^3_{\rm T}$.

(43) If the direction of u = the direction of v and $u(3) = v(3)$ and $v(3) \neq 0$, then $u = v$.

The functor Dir101 yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 1) the direction of $[1, 0, 1]$.

The functor Dirm101 yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 2) the direction of $[-1, 0, 1]$.

The functor Dir011 yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 3) the direction of $[0, 1, 1]$.

Now we state the propositions:

(44) (i) Dir101, Dirm101 and Dir011 are not collinear, and

(ii) Dir101, Dirm101 and Dir010 are not collinear, and

(iii) Dir101, Dir011 and Dir010 are not collinear, and

(iv) Dirm101, Dir011 and Dir010 are not collinear.

PROOF: Dir101, Dirm101 and Dir011 are not collinear. Dir101, Dirm101 and Dir010 are not collinear. Dir101, Dir011 and Dir010 are not collinear. Dirm101, Dir011 and Dir010 are not collinear. □

(45) symmetric3$(1, 1, 1, 0, 0, 0) = I^{3\times 3}_{\mathbb{R}_{\rm F}}$.

(46) Let us consider elements r, a, b, c, d, e, f, g, h, i of $\mathbb{R}_{\rm F}$, and a square matrix M over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Then $r \cdot M = \langle\langle r \cdot a, r \cdot b, r \cdot c\rangle, \langle r \cdot d, r \cdot e, r \cdot f\rangle, \langle r \cdot g, r \cdot h, r \cdot i\rangle\rangle$.

(47) Let us consider a real number a, and an element r of $\mathbb{R}_{\rm F}$. Suppose $r = a \cdot a$. Then (symmetric3$(a, a, -a, 0, 0, 0)$) $\cdot$ (symmetric3$(a, a, -a, 0, 0, 0)$) $= r \cdot (I^{3\times 3}_{\mathbb{R}_{\rm F}})$. The theorem is a consequence of (46).

Let us consider a non zero real number a. Now we state the propositions:

(48) (symmetric3$(a, a, -a, 0, 0, 0)$) $\cdot$ (symmetric3$(\frac{1}{a}, \frac{1}{a}, -\frac{1}{a}, 0, 0, 0)$) $= I^{3\times 3}_{\mathbb{R}_{\rm F}}$.

(49) (symmetric3$(\frac{1}{a}, \frac{1}{a}, -\frac{1}{a}, 0, 0, 0)$)$\cdot$(symmetric3$(a, a, -a, 0, 0, 0)$) $= I^{3\times 3}_{\mathbb{R}_{\rm F}}$. The theorem is a consequence of (48).

(50) (symmetric3$(1, 1, -1, 0, 0, 0)$) $\cdot$ (symmetric3$(1, 1, -1, 0, 0, 0)$) $= I^{3\times 3}_{\mathbb{R}_{\rm F}}$. The theorem is a consequence of (48).

(51) Let us consider a non zero real number a, and a square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. If N = symmetric3$(a, a, -a, 0, 0, 0)$, then N is invertible. The theorem is a consequence of (48) and (49).

(52) (i) symmetric3$(1, 1, -1, 0, 0, 0)$ is an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3, and

(ii) $(\text{symmetric3}(1, 1, -1, 0, 0, 0))^{\smile} = \text{symmetric3}(1, 1, -1, 0, 0, 0)$.

The theorem is a consequence of (50).

(53) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, a square matrix N_1 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and square matrices M, N_2 over $\mathbb{R}$ of dimension 3. Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $N_1 = M$ and $N_2 = (\mathbb{R}_{\mathrm{F}} \rightarrow \mathbb{R})N^{\smile}$. Then $N^{\mathrm{T}} \cdot N_1 \cdot N = ((\mathbb{R}_{\mathrm{F}} \rightarrow \mathbb{R})((\mathbb{R} \rightarrow \mathbb{R}_{\mathrm{F}})N_2{}^{\mathrm{T}})^{\smile}) \cdot M \cdot ((\mathbb{R}_{\mathrm{F}} \rightarrow \mathbb{R})((\mathbb{R} \rightarrow \mathbb{R}_{\mathrm{F}})N_2)^{\smile})$.

PROOF: $(\mathbb{R}_{\mathrm{F}} \rightarrow \mathbb{R})((\mathbb{R} \rightarrow \mathbb{R}_{\mathrm{F}})N_2{}^{\mathrm{T}})^{\smile} = N^{\mathrm{T}}$ by [15, (13), (16)]. □

(54) Let us consider a natural number n, an element a of $\mathbb{R}_{\mathrm{F}}$, a real number r, a square matrix A over $\mathbb{R}_{\mathrm{F}}$ of dimension n, and a square matrix r_1 over $\mathbb{R}$ of dimension n. If $a = r$ and $A = r_1$, then $a \cdot A = r \cdot r_1$.

(55) Let us consider a natural number n, an element a of $\mathbb{R}_{\mathrm{F}}$, and square matrices A, B over $\mathbb{R}_{\mathrm{F}}$ of dimension n. If $n > 0$, then $(a \cdot A) \cdot B = a \cdot (A \cdot B)$. The theorem is a consequence of (54).

(56) $\text{symmetric3}(a, a, -a, 0, 0, 0) = a \cdot (\text{symmetric3}(1, 1, -1, 0, 0, 0))$. The theorem is a consequence of (46).

(57) If $M = \text{symmetric3}(a, a, -a, 0, 0, 0)$, then $M \cdot M \cdot M = a \cdot a \cdot a \cdot (\text{symmetric3}(1, 1, -1, 0, 0, 0))$. The theorem is a consequence of (47), (55), and (56).

Let us consider a natural number n, a real number a, a square matrix M over $\mathbb{R}$ of dimension n, and a finite sequence x of elements of $\mathbb{R}$. Now we state the propositions:

(58) If $n > 0$ and $\operatorname{len} x = n$, then $M \cdot (a \cdot x) = (a \cdot M) \cdot x$.

(59) If $n > 0$ and $\operatorname{len} x = n$, then $a \cdot (M \cdot x) = (a \cdot M) \cdot x$. The theorem is a consequence of (58).

(60) Let us consider a natural number n, and a square matrix N over $\mathbb{R}$ of dimension n. Suppose N is invertible. Then

(i) N^{T} is invertible, and

(ii) $\operatorname{Inv} N^{\mathrm{T}} = (\operatorname{Inv} N)^{\mathrm{T}}$.

(61) Let us consider a non zero real number r, and matrices N, O, M over $\mathbb{R}$ of dimension 3×3. Suppose N is invertible and $M = r \cdot O$ and $M = N^{\mathrm{T}} \cdot O \cdot N$. Then $(\operatorname{Inv} N)^{\mathrm{T}} \cdot O \cdot (\operatorname{Inv} N) = \frac{1}{r} \cdot O$. The theorem is a consequence of (60).

(62) Let us consider a real number r, square matrices M, N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and square matrices M_1, N_2 over $\mathbb{R}$ of dimension 3. Suppose $M_1 = M$ and $N_2 = N$ and N is symmetric and $M_1 = r \cdot N_2$. Then M is symmetric.

Let us consider a real number r and square matrices O, M over $\mathbb{R}$ of dimension 3. Now we state the propositions:

(63) Suppose $O = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r \cdot O$. Then

(i) $O \cdot M = r \cdot (1_{\mathbb{R}}\,\text{matrix}(3))$, and

(ii) $M \cdot O = r \cdot (1_{\mathbb{R}}\,\text{matrix}(3))$.

The theorem is a consequence of (50).

(64) If $O = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r \cdot O$, then $(M^{\mathrm{T}} \cdot O)^{\mathrm{T}} \cdot O \cdot (M^{\mathrm{T}} \cdot O) = r^{\mathbf{2}} \cdot O$.

PROOF: Reconsider $M_1 = M$ as a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. M_1 is symmetric. $r \cdot (1_{\mathbb{R}}\,\text{matrix}(3)) \cdot O \cdot (r \cdot (1_{\mathbb{R}}\,\text{matrix}(3))) = r^{\mathbf{2}} \cdot O$. □

(65) Let us consider square matrices O, N over $\mathbb{R}$ of dimension 3. Then $(N^{\mathrm{T}} \cdot O)^{\mathrm{T}} \cdot O \cdot (N^{\mathrm{T}} \cdot O) = (O^{\mathrm{T}} \cdot (N \cdot O \cdot N^{\mathrm{T}})) \cdot O$.

(66) Let us consider square matrices N_2, M_1 over $\mathbb{R}$ of dimension 3, and finite sequences p_1, p_2, p_3 of elements of $\mathbb{R}$. Suppose $p_1 = \langle 1, 0, 0 \rangle$ and $p_2 = \langle 0, 1, 0 \rangle$ and $p_3 = \langle 0, 0, 1 \rangle$ and $N_2 \cdot p_1 = M_1 \cdot p_1$ and $N_2 \cdot p_2 = M_1 \cdot p_2$ and $N_2 \cdot p_3 = M_1 \cdot p_3$. Then $N_2 = M_1$.

(67) Let us consider a non zero real number a, and an element u of $\mathcal{E}^3_{\mathrm{T}}$. If $a \cdot u = 0_{\mathcal{E}^3_{\mathrm{T}}}$, then u is zero.

(68) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b. Suppose ($a \neq 0$ or $b \neq 0$) and $a \cdot u + b \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. Then u and v are proportional.

PROOF: Reconsider $a_1 = a \cdot u$, $b_1 = b \cdot v$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Consider c being a real number such that $c \neq 0$ and $a_1 = c \cdot b_1$. $a \neq 0$ and $b \neq 0$ by [11, (3), (1)]. □

(69) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and points P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and (the homography of N)(P) $= Q$ and (the homography of N)(Q) $= P$ and P, Q and R are collinear. Then (the homography of N)((the homography of N)(R)) $= R$.

PROOF: Consider u_1, v_1 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_4 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_1 being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u_1 and u_1 is not zero and $u_1 = u_4$ and $p_1 = N \cdot u_4$ and $v_1 = \text{M2F}(p_1)$ and v_1 is not zero and (the homography of N)(P) $=$ the direction of v_1. Consider u_2, v_2 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_5 being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_2 being a finite sequence of elements of $\mathbb{R}^1$ such that $Q =$ the direction of u_2 and u_2 is not zero and $u_2 = u_5$ and $p_2 = N \cdot u_5$ and $v_2 = \text{M2F}(p_2)$ and v_2 is not zero and (the homography of N)(Q) $=$ the direction of v_2. Consider u_3 being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_3 is not zero and $R =$ the direction of u_3. Consider l_1 being a real number such that $l_1 \neq 0$ and $v_2 = l_1 \cdot u_1$. Consider l_2 being a real number such that $l_2 \neq 0$ and $v_1 = l_2 \cdot u_2$. $\langle |u_1, u_2, u_3| \rangle = 0$. Consider a, b, c being real numbers such that $a \cdot u_1 + b \cdot u_2 + c \cdot u_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). $c \neq 0$. (The homography of $N \cdot N$)(R) $= R$. □

(70) Let us consider a natural number n, elements u, v of $\mathcal{E}^n_{\mathrm{T}}$, and real numbers a, b. If $(1-a)\cdot u+a\cdot v = (1-b)\cdot v+b\cdot u$, then $(1-(a+b))\cdot u = (1-(a+b))\cdot v$.
PROOF: Reconsider $r_1 = u$, $r_2 = v$ as an element of $\mathcal{R}^n$. $(1-a)\cdot r_1 + a \cdot r_2 - a \cdot r_2 = (1-a)\cdot r_1$. $(1-b)\cdot r_2 - a\cdot r_2 + b \cdot r_1 - b\cdot r_1 = (1-b)\cdot r_2 - a\cdot r_2$. □

(71) The projective space over $\mathcal{E}^3_{\mathrm{T}}$ is proper.

The real projective plane yielding a non empty, proper projective plane defined in terms of collinearity is defined by the term

(Def. 4) the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

From now on P, Q, R denote points of Inc-ProjSp(the real projective plane), L denotes a line of Inc-ProjSp(the real projective plane), and p, q, r denote points of the real projective plane.

Now we state the propositions:

(72) Let us consider an element L of L(the real projective plane). Then there exists p and there exists q such that $p \neq q$ and $L = \mathrm{Line}(p, q)$.

(73) There exists p and there exists q such that $p \neq q$ and $L = \mathrm{Line}(p, q)$.

(74) If $R = r$ and $L = \mathrm{Line}(p, q)$, then R lies on L iff p, q and r are collinear.

(75) Inc-ProjSp(the real projective plane) is an incidence projective plane.
PROOF: Inc-ProjSp(the real projective plane) is 2-dimensional. □

(76) Let us consider lines L_1, L_2 of the real projective plane. Then L_1 meets L_2. The theorem is a consequence of (75).

In the sequel u, v, w denote non zero elements of $\mathcal{E}^3_{\mathrm{T}}$.

(77) Suppose $p =$ the direction of u and $q =$ the direction of v and $R =$ the direction of w and $L = \mathrm{Line}(p, q)$. Then R lies on L if and only if $\langle |u, v, w| \rangle = 0$. The theorem is a consequence of (74).

(78) Let us consider elements p, q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $p \neq q$ and $p =$ the direction of u and $q =$ the direction of v. Then $u \times v$ is not zero.

Let p, q be points of the real projective plane. Assume $p \neq q$. The functor $\mathrm{L2P}(p, q)$ yielding a point of the real projective plane is defined by

(Def. 5) there exist non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$ such that $p =$ the direction of u and $q =$ the direction of v and *it* $=$ the direction of $u \times v$.

Now we state the propositions:

(79) Let us consider points p, q of the real projective plane. Suppose $p \neq q$. Then

(i) $\mathrm{L2P}(q,p) = \mathrm{L2P}(p,q)$, and

(ii) $p \neq \mathrm{L2P}(p,q)$.

Proof: Consider u_1, v_1 being non zero elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $p =$ the direction of u_1 and $q =$ the direction of v_1 and $\mathrm{L2P}(p,q) =$ the direction of $u_1 \times v_1$. Consider u_2, v_2 being non zero elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $q =$ the direction of u_2 and $p =$ the direction of v_2 and $\mathrm{L2P}(q,p) =$ the direction of $u_2 \times v_2$. Consider a being a real number such that $a \neq 0$ and $u_1 = a \cdot v_2$. Consider b being a real number such that $b \neq 0$ and $v_1 = b \cdot u_2$. $a \cdot v_2 \times b \cdot u_2 = (-a \cdot b) \cdot (u_2 \times v_2)$. $u_1 \times v_1$ is not zero. $u_2 \times v_2$ is not zero. $p \neq \mathrm{L2P}(p,q)$. □

(80) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Then dom(the homography of N) = the projective points over $\mathcal{E}^3_{\mathrm{T}}$.

2. Absolute

Let a, b, c, d, e, f be real numbers. The interior of the conic for a, b, c, d, e and f yielding a subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 6) $\{P$, where P is a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$: for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $P =$ the direction of u holds $\mathrm{qfconic}(a,b,c,d,e,f,u)$ is negative$\}$.

Now we state the proposition:

(81) Let us consider real numbers a, b, c, d, e, f, and non zero elements u_1, u_2 of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of $u_1 =$ the direction of u_2 and $\mathrm{qfconic}(a,b,c,d,e,f,u_1)$ is negative. Then $\mathrm{qfconic}(a,b,c,d,e,f,u_2)$ is negative.

The absolute yielding a non empty subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 7) $\mathrm{conic}(1,1,-1,0,0,0)$.

Now we state the proposition:

(82) Let us consider a square matrix O over $\mathbb{R}$ of dimension 3, an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a finite sequence p of elements of $\mathbb{R}$. Suppose $O = \mathrm{symmetric3}(1,1,-1,0,0,0)$ and $P =$ the direction of u and $u = p$. Then $P \in$ the absolute if and only if $\mathrm{SumAll\,QuadraticForm}(p,O,p) = 0$. The theorem is a consequence of (40).

Let us consider an element P of the absolute. Now we state the propositions:

(83) If $P =$ the direction of u, then $u(3) \neq 0$.

Proof: Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P = Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and

$Q =$ the direction of u holds qfconic$(1, 1, -1, 0, 0, 0, u) = 0$. $u(3) \neq 0$ by [8, (3), (4)]. □

(84) If $P =$ the direction of u and $u(3) = 1$, then $[u(1), u(2)] \in$ circle$(0, 0, 1)$. The theorem is a consequence of (13).

(85) Let us consider a point P of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of u and $u(3) = 1$ and $[u(1), u(2)] \in$ circle$(0, 0, 1)$. Then P is an element of the absolute.

(86) Let us consider a point P of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element u of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of u and $u(3) = 1$. Then $[u(1), u(2)] \in$ circle$(0, 0, 1)$ if and only if P is an element of the absolute.

Let P be an element of the absolute. The absolute to unit circle of P yielding an element of circle$(0, 0, 1)$ is defined by

(Def. 8) there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of u and $u(3) = 1$ and *it* $= [u(1), u(2)]$.

Now we state the proposition:

(87) The carrier of TopUnitCircle 2 = circle$(0, 0, 1)$.
PROOF: The carrier of TopUnitCircle 2 $\subseteq$ circle$(0, 0, 1)$. circle$(0, 0, 1) \subseteq$ the carrier of TopUnitCircle 2 by [9, (52)], [10, (9)]. □

Let u be a non zero element of $\mathcal{E}^2_{\rm T}$. Assume $u \in$ circle$(0, 0, 1)$. The unit circle to absolute of u yielding an element of the absolute is defined by the term

(Def. 9) the direction of $[u(1), u(2), 1]$.

Now we state the proposition:

(88) Let us consider an element u of $\mathcal{E}^3_{\rm T}$. Suppose qfconic$(1, 1, -1, 0, 0, 0, u) = 0$ and $u(3) = 1$. Then $[u(1), u(2)] \in$ Sphere$(0_{\mathcal{E}^2_{\rm T}}, 1)$. The theorem is a consequence of (13).

Let us consider an element P of the absolute. Now we state the propositions:

(89) There exists u such that

(i) $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} = 1$, and

(ii) $u(3) = 1$, and

(iii) $P =$ the direction of u.

The theorem is a consequence of (83), (84), and (14).

(90) There exists an element Q of the absolute such that $P \neq Q$.
PROOF: Consider Q being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that $P = Q$ and for every element u of $\mathcal{E}^3_{\rm T}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(1, 1, -1, 0, 0, 0, u) = 0$. Consider u being an element of $\mathcal{E}^3_{\rm T}$ such that u is not zero and the direction of $u = P$. $u(3) \neq 0$. $[u(1), u(2), -u(3)]$ is not zero. Reconsider $v = [u(1), u(2), -u(3)]$ as

a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $R =$ the direction of v as an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $R \neq P$. For every element w of $\mathcal{E}^3_{\mathrm{T}}$ such that w is not zero and $R =$ the direction of w holds qfconic$(1, 1, -1, 0, 0, 0, w) = 0$. □

(91) Let us consider real numbers a, b. Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = 1$. Then $[-b, a, 0]$ is not zero.

(92) Let us consider elements P, Q, R of the absolute. If P, Q, R are mutually different, then P, Q and R are not collinear.

PROOF: Consider u_{12} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{12} is not zero and $P =$ the direction of u_{12}. Consider u_{16} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{16} is not zero and $Q =$ the direction of u_{16}. Consider u_{20} being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that u_{20} is not zero and $R =$ the direction of u_{20}. Reconsider $u_{13} = (u_{12})_{\mathbf{1}}$, $u_{14} = (u_{12})_{\mathbf{2}}$, $u_{15} = (u_{12})_{\mathbf{3}}$, $u_{17} = (u_{16})_{\mathbf{1}}$, $u_{18} = (u_{16})_{\mathbf{2}}$, $u_{19} = (u_{16})_{\mathbf{3}}$, $u_{21} = (u_{20})_{\mathbf{1}}$, $u_{22} = (u_{20})_{\mathbf{2}}$, $u_{23} = (u_{20})_{\mathbf{3}}$ as a real number. $u_{12}(3) \neq 0$ and $u_{16}(3) \neq 0$ and $u_{20}(3) \neq 0$. Reconsider $v_5 = \frac{u_{13}}{u_{15}}$, $v_6 = \frac{u_{14}}{u_{15}}$, $v_8 = \frac{u_{17}}{u_{19}}$, $v_9 = \frac{u_{18}}{u_{19}}$, $v_{11} = \frac{u_{21}}{u_{23}}$, $v_{12} = \frac{u_{22}}{u_{23}}$ as a real number. Reconsider $v_4 = [v_5, v_6, 1]$, $v_7 = [v_8, v_9, 1]$, $v_{10} = [v_{11}, v_{12}, 1]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $P =$ the direction of v_4 and $Q =$ the direction of v_7 and $R =$ the direction of v_{10}. Consider t_1, t_2, t_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of t_1 and $Q =$ the direction of t_2 and $R =$ the direction of t_3 and t_1 is not zero and t_2 is not zero and t_3 is not zero and there exist real numbers a_1, b_1, c_1 such that $a_1 \cdot t_1 + b_1 \cdot t_2 + c_1 \cdot t_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a_1 \neq 0$ or $b_1 \neq 0$ or $c_1 \neq 0$). Consider a_1, b_1, c_1 being real numbers such that $a_1 \cdot t_1 + b_1 \cdot t_2 + c_1 \cdot t_3 = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and $a_1 \neq 0$ or $b_1 \neq 0$ or $c_1 \neq 0$. Consider l_1 being a real number such that $l_1 \neq 0$ and $t_1 = l_1 \cdot v_4$. Consider l_2 being a real number such that $l_2 \neq 0$ and $t_2 = l_2 \cdot v_7$. Consider l_3 being a real number such that $l_3 \neq 0$ and $t_3 = l_3 \cdot v_{10}$. Reconsider $A = [(v_4)_{\mathbf{1}}, (v_4)_{\mathbf{2}}]$, $B = [(v_7)_{\mathbf{1}}, (v_7)_{\mathbf{2}}]$, $C = [(v_{10})_{\mathbf{1}}, (v_{10})_{\mathbf{2}}]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $A \neq B$. $A \neq C$. $B \neq C$. $A \in$ Sphere$(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. qfconic$(1, 1, -1, 0, 0, 0, v_7) = 0$. $B \in$ Sphere$(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. $C \in$ Sphere$(0_{\mathcal{E}^2_{\mathrm{T}}}, 1)$. □

(93) Let us consider a non zero real number r, and invertible square matrices O, N, M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $O =$ symmetric3$(1, 1, -1, 0, 0, 0)$ and $M =$ symmetric3$(r, r, -r, 0, 0, 0)$ and
$M = N^{\mathrm{T}} \cdot O \cdot N$ and (the homography of M)$^\circ$(the absolute) = the absolute. Then (the homography of N)$^\circ$(the absolute) = the absolute.

PROOF: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute.
The absolute $\subseteq$ (the homography of N)$^\circ$(the absolute) by [12, (50)]. □

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Klein-Beltrami Model. Part II

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Summary. Tim Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) have shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2, 3, 15, 4].

With the Mizar system [1], [10] we use some ideas are taken from Tim Makarios' MSc thesis [12] for formalized some definitions (like the tangent) and lemmas necessary for the verification of the independence of the parallel postulate. This work can be also treated as a further development of Tarski's geometry in the formal setting [9].

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Keywords: Tarski's geometry axioms; foundations of geometry; Klein-Beltrami model

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1. Beltrami-Cayley-Klein Disk Model

The BK-model yielding a non empty subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 1) the interior of the conic for 1, 1, -1, 0, 0 and 0.

Now we state the propositions:

(1) The BK-model misses the absolute.

(2) Let us consider an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $P \in$ the BK-model. Then $u(3) \neq 0$.

[1]`https://www.isa-afp.org/entries/Tarskis_Geometry.html`
[2]`https://github.com/jrh13/hol-light/blob/master/100/independence.ml`

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Let P be an element of the BK-model. The functor BK-to-REAL2(P) yielding an element of the inside of circle(0,0,1) is defined by

(Def. 2) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and $it = [u(1), u(2)]$.

Let Q be an element of the inside of circle(0,0,1). The functor REAL2-to-BK(Q) yielding an element of the BK-model is defined by

(Def. 3) there exists an element P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = Q$ and $it =$ the direction of $[(P)_{\mathbf{1}}, (P)_{\mathbf{2}}, 1]$.

Now we state the propositions:

(3) Let us consider an element P of the BK-model.
Then REAL2-to-BK(BK-to-REAL2(P)) $= P$.
PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Consider Q being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q =$ BK-to-REAL2(P) and REAL2-to-BK(BK-to-REAL2(P)) $=$ the direction of $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$. $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$ and u are proportional. □

(4) Let us consider elements P, Q of the BK-model. Then $P = Q$ if and only if BK-to-REAL2(P) $=$ BK-to-REAL2(Q).

(5) Let us consider an element Q of the inside of circle(0,0,1).
Then BK-to-REAL2(REAL2-to-BK(Q)) $= Q$.

(6) Let us consider elements P, Q of the BK-model, and elements P_1, P_2, P_3 of the absolute. Suppose $P \neq Q$ and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, Q and P_3 are collinear. Then

(i) $P_3 = P_1$, or

(ii) $P_3 = P_2$.

PROOF: $P_3 = P_1$ or $P_3 = P_2$. □

(7) Let us consider an element P of the BK-model, an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and $Q =$ the direction of v and $v(3) = 1$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.
PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot \sum^2(T-S)}$ as a real number. $s \neq t$. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that

$\{e_1\} = \text{HalfLine}(s, t) \cap \text{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. □

(8) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

(i) $P \neq Q$, and

(ii) $Q \in L$.

(9) Let us consider real numbers a, b, c, d, e, and elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = [a, b, e]$ and $v = [c, d, 0]$ and $w = [a + c, b + d, e]$. Then $\langle|u, v, w|\rangle = 0$.

(10) Let us consider real numbers a, b, and a non zero real number c. Then $[a, b, c]$ is a non zero element of $\mathcal{E}^3_{\mathrm{T}}$.

(11) Let us consider elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b, c, d, e. Suppose $u = [a, b, c]$ and $v = [d, e, 0]$ and u and v are proportional. Then $c = 0$.

(12) Let us consider elements P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then

(i) $R \neq P$, and

(ii) $R \neq Q$.

(13) Let us consider a line L of Inc-ProjSp(the real projective plane), and elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P \neq Q$ and P, $Q \in L$, then $L = \text{Line}(P, Q)$.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane), elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in L$ and $P =$ the direction of u and $Q =$ the direction of v and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$. Then

(i) $P \neq Q$, and

(ii) the direction of $[(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}] \in L$.

PROOF: $P \neq Q$. Reconsider $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $\langle|u, v, w|\rangle = 0$. □

(15) Let us consider elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then $\langle|u, v, w|\rangle = 0$.

(16) Let us consider a line L of Inc-ProjSp(the real projective plane), an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$.

Suppose $P =$ the direction of u and $P \in L$ and $u(3) \neq 0$. Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and there exists a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of v and $Q \in L$ and $P \neq Q$ and $v(3) \neq 0$. The theorem is a consequence of (15).

(17) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Suppose $P \in L$. Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

- (i) $P \neq Q$, and
- (ii) $Q \in L$, and
- (iii) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of u holds $u(3) \neq 0$.

The theorem is a consequence of (16).

(18) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and a non zero real number k. Suppose $u = k \cdot v$. Then the direction of $u =$ the direction of v.

(19) Let us consider an element P of the BK-model, and an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Reconsider $L = \mathrm{Line}(P, Q)$ as a line of Inc-ProjSp(the real projective plane). Consider R being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P \neq R$ and $R \in L$ and for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $R =$ the direction of u holds $u(3) \neq 0$. Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2$(P) = [u(1), u(2)]$. Consider v' being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that v' is not zero and the direction of $v' = R$. Reconsider $k = \frac{1}{(v')_{\mathbf{3}}}$ as a non zero real number. $k \cdot v'$ is not zero. Reconsider $v = k \cdot v'$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. the direction of $v = R$ and $v(3) = 1$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot \sum({}^2(T-S)}$ as a real number. $s \neq t$. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} = \mathrm{HalfLine}(s, t) \cap \mathrm{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. □

(20) Let us consider elements P, Q of the BK-model. Suppose $P \neq Q$. Then there exist elements P_1, P_2 of the absolute such that

- (i) $P_1 \neq P_2$, and
- (ii) P, Q and P_1 are collinear, and

(iii) P, Q and P_2 are collinear.

PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\rm T}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2$(P) = [u(1), u(2)]$. Consider v being a non zero element of $\mathcal{E}^3_{\rm T}$ such that the direction of $v = Q$ and $v(3) = 1$ and BK-to-REAL2$(Q) = [v(1), v(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\rm T}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2\cdot|(t-s,s-[a,b])|+\sqrt{\Delta(\sum(^2(T-S)),2\cdot|(t-s,s-[a,b])|,\sum(^2(S-X))-r^{\mathbf{2}})}}{2\cdot(\sum(^2(T-S)))}$ as a real number. Consider e_1 being a point of $\mathcal{E}^2_{\rm T}$ such that $\{e_1\} =$ HalfLine$(s,t) \cap$ circle(a,b,r) and $e_1 = (1-w_1)\cdot s + w_1 \cdot t$. Reconsider $w_2 = \frac{-2\cdot|(s-t,t-[a,b])|+\sqrt{\Delta(\sum(^2(S-T)),2\cdot|(s-t,t-[a,b])|,\sum(^2(T-X))-r^{\mathbf{2}})}}{2\cdot(\sum(^2(S-T)))}$ as a real number. Consider e_2 being a point of $\mathcal{E}^2_{\rm T}$ such that $\{e_2\} =$ HalfLine$(t,s) \cap$ circle(a,b,r) and $e_2 = (1-w_2)\cdot t + w_2 \cdot s$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\rm T}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Reconsider $P_1 =$ the direction of e_3 as a point of the projective space over $\mathcal{E}^3_{\rm T}$. $1\cdot e_3 + (-(1-w_1))\cdot u + (-w_1)\cdot v = 0_{\mathcal{E}^3_{\rm T}}$. Reconsider $g = [(e_2)_{\mathbf{1}}, (e_2)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\rm T}$. Reconsider $e_4 = g$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Reconsider $P_2 =$ the direction of e_4 as a point of the projective space over $\mathcal{E}^3_{\rm T}$. $1\cdot e_4 + (-(1-w_2))\cdot v + (-w_2)\cdot u = 0_{\mathcal{E}^3_{\rm T}}$. $P_1 \neq P_2$. □

(21) Let us consider elements P, Q, R of the real projective plane, non zero elements u, v, w of $\mathcal{E}^3_{\rm T}$, and real numbers a, b, c, d. Suppose $P \in$ the BK-model and $Q \in$ the absolute and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$. Then

(i) $R \in$ the BK-model, and

(ii) $R \neq P$, and

(iii) P, R and Q are collinear.

PROOF: Reconsider $P_6 = P$ as an element of the BK-model. Consider u_2 being a non zero element of $\mathcal{E}^3_{\rm T}$ such that the direction of $u_2 = P_6$ and $u_2(3) = 1$ and BK-to-REAL2$(P_6) = [u_2(1), u_2(2)]$. Consider p being a point of $\mathcal{E}^2_{\rm T}$ such that $[v(1), v(2)] = p$ and $|p - [0,0]| = 1$. Reconsider $R_1 = [w(1), w(2)]$ as an element of $\mathcal{E}^2_{\rm T}$. $|R_1 - [0,0]|^{\mathbf{2}} < 1$. Consider P_1 being an element of $\mathcal{E}^2_{\rm T}$ such that $P_1 = R_1$ and REAL2-to-BK$(R_1) =$ the direction of $[(P_1)_{\mathbf{1}}, (P_1)_{\mathbf{2}}, 1]$. $P \neq R$ by [13, (29)]. □

(22) Let us consider elements P, Q of the real projective plane. Suppose $P \in$ the absolute and $Q \in$ the BK-model. Then there exists an element R of the real projective plane such that

(i) $R \in$ the BK-model, and

(ii) $Q \neq R$, and

(iii) R, Q and P are collinear.

The theorem is a consequence of (21).

(23) Let us consider a line L of Inc-ProjSp(the real projective plane), points p, q of Inc-ProjSp(the real projective plane), and elements P, Q of the real projective plane. Suppose $p = P$ and $q = Q$ and $P \in$ the BK-model and $Q \in$ the absolute and q lies on L and p lies on L. Then there exist points p_1, p_2 of Inc-ProjSp(the real projective plane) and there exist elements P_1, P_2 of the real projective plane such that $p_1 = P_1$ and $p_2 = P_2$ and $P_1 \neq P_2$ and P_1, $P_2 \in$ the absolute and p_1 lies on L and p_2 lies on L. The theorem is a consequence of (1), (22), and (20).

(24) Let us consider an element P of the BK-model, and an element Q of the absolute. Then there exists an element R of the absolute such that

(i) $Q \neq R$, and

(ii) Q, P and R are collinear.

The theorem is a consequence of (1) and (23).

(25) Let us consider an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $u(3) = 1$. Then $(u(1))^{\mathbf{2}} + (u(2))^{\mathbf{2}} < 1$.

(26) Let us consider elements P_1, P_2 of the absolute, an element Q of the BK-model, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of $u = P_1$ and the direction of $v = P_2$ and the direction of $w = Q$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 1$ and $v(1) = -u(1)$ and $v(2) = -u(2)$ and P_1, Q and P_2 are collinear. Then there exists a real number a such that

(i) $-1 < a < 1$, and

(ii) $w(1) = a \cdot u(1)$, and

(iii) $w(2) = a \cdot u(2)$.

The theorem is a consequence of (25).

2. Tangent

Let P be an element of the absolute. The functor PoleInfty(P) yielding an element of the real projective plane is defined by

(Def. 4) there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of u and $u(3) = 1$ and $(u(1))^{\mathbf{2}} + (u(2))^{\mathbf{2}} = 1$ and $it =$ the direction of $[-u(2), u(1), 0]$.

Now we state the propositions:

(27) Let us consider an element P of the absolute. Then $P \neq$ PoleInfty(P).

(28) Let us consider elements P_1, P_2 of the absolute. Suppose PoleInfty(P_1) = PoleInfty(P_2). Then

(i) $P_1 = P_2$, or

(ii) there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$.

Proof: Consider u_1 being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u_1 and $u_1(3) = 1$ and $u_1(1)^{\mathbf{2}} + u_1(2)^{\mathbf{2}} = 1$ and PoleInfty (P_1) = the direction of $[-u_1(2), u_1(1), 0]$. Consider u_2 being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P_2 =$ the direction of u_2 and $u_2(3) = 1$ and $(u_2(1))^{\mathbf{2}} + (u_2(2))^{\mathbf{2}} = 1$ and PoleInfty(P_2) = the direction of $[-u_2(2), u_2(1), 0]$. Reconsider $w_1 = [-u_1(2), u_1(1), 0]$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Reconsider $w_2 = [-u_2(2), u_2(1), 0]$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Consider a being a real number such that $a \neq 0$ and $w_1 = a \cdot w_2$. If $a = 1$, then $P_1 = P_2$. If $a = -1$, then there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$. □

Let P be an element of the absolute. The functor tangent(P) yielding a line of the real projective plane is defined by

(Def. 5) there exists an element p of the real projective plane such that $p = P$ and $it =$ Line(p, PoleInfty(P)).

Let us consider an element P of the absolute. Now we state the propositions:

(29) $P \in$ tangent(P).

(30) tangent(P) $\cap$ (the absolute) = $\{P\}$.

Proof: $\{P\} \subseteq$ tangent(P) $\cap$ (the absolute). tangent(P) $\cap$ (the absolute) $\subseteq \{P\}$. □

(31) Let us consider elements P_1, P_2 of the absolute. If tangent(P_1) = tangent(P_2), then $P_1 = P_2$. The theorem is a consequence of (30).

(32) Let us consider elements P, Q of the absolute. Then there exists an element R of the real projective plane such that

(i) $R \in \text{tangent}(P)$, and

(ii) $R \in \text{tangent}(Q)$.

(33) Let us consider elements P_1, P_2 of the absolute. Suppose $P_1 \neq P_2$. Then there exists an element P of the real projective plane such that $\text{tangent}(P_1) \cap \text{tangent}(P_2) = \{P\}$. The theorem is a consequence of (31).

(34) Let us consider a square matrix M over $\mathbb{R}$ of dimension 3, an element P of the absolute, an element Q of the real projective plane, non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and finite sequences f_3, f_7 of elements of $\mathbb{R}$. Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $P =$ the direction of u and $Q =$ the direction of v and $u = f_3$ and $v = f_7$ and $Q \in \text{tangent}(P)$. Then $\text{SumAll}\,\text{QuadraticForm}(f_7, M, f_3) = 0$.

PROOF: Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider w being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of w and $w(3) = 1$ and $(w(1))^{\mathbf{2}} + (w(2))^{\mathbf{2}} = 1$ and $\text{PoleInfty}(P) =$ the direction of $[-w(2), w(1), 0]$. Consider a_1 being a real number such that $a_1 \neq 0$ and $w = a_1 \cdot u$. $w(1) = a_1 \cdot ((u)_{\mathbf{1}})$ and $w(2) = a_1 \cdot ((u)_{\mathbf{2}})$ and $w(3) = a_1 \cdot ((u)_{\mathbf{3}})$. $\text{len}\, f_3 = \text{width}\, M$ and $\text{len}\, f_7 = \text{len}\, M$ and $\text{len}\, f_3 = \text{len}\, M$ and $\text{len}\, f_7 = \text{width}\, M$ and $\text{len}\, f_3 > 0$ and $\text{len}\, f_7 > 0$. □

(35) Let us consider elements P, Q, R of the absolute, and points P_1, P_2, P_3, P_4 of the real projective plane. Suppose P, Q, R are mutually different and $P_1 = P$ and $P_2 = Q$ and $P_3 = R$ and $P_4 \in \text{tangent}(P)$ and $P_4 \in \text{tangent}(Q)$. Then

(i) P_1, P_2 and P_3 are not collinear, and

(ii) P_1, P_2 and P_4 are not collinear, and

(iii) P_1, P_3 and P_4 are not collinear, and

(iv) P_2, P_3 and P_4 are not collinear.

PROOF: $P_4 \notin$ the absolute. Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider q being an element of the real projective plane such that $q = Q$ and $\text{tangent}(Q) = \text{Line}(q, \text{PoleInfty}(Q))$. P_1, P_2 and P_4 are not collinear. P_1, P_3 and P_4 are not collinear. P_2, P_3 and P_4 are not collinear. □

(36) Let us consider elements P, Q of the absolute, an element R of the real projective plane, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $R \in \text{tangent}(P)$ and $R \in \text{tangent}(Q)$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 0$. Then

(i) $P = Q$, or

(ii) $u(1) = -v(1)$ and $u(2) = -v(2)$.

The theorem is a consequence of (34).

(37) Let us consider an element P of the absolute, an element R of the real projective plane, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $R \in \mathrm{tangent}(P)$ and $R =$ the direction of u and $u(3) = 0$. Then $R = \mathrm{PoleInfty}(P)$. The theorem is a consequence of (34).

(38) Let us consider a non zero real number a, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $N = \mathrm{symmetric3}(a, a, -a, 0, 0, 0)$. Then (the homography of $N)^\circ$(the absolute) = the absolute.

PROOF: (The homography of $N)^\circ$(the absolute) $\subseteq$ the absolute by [8, (8)]. The absolute $\subseteq$ (the homography of $N)^\circ$(the absolute) by [11, (4), (3)], [7, (89)]. □

(39) Let us consider a non zero element r_1 of $\mathbb{R}_{\mathrm{F}}$, and invertible square matrices M, O over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $O = \mathrm{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r_1 \cdot O$. Then (the homography of $M)^\circ$(the absolute) = the absolute.

PROOF: $r_1 \neq 0$ by [14, (34)]. □

(40) Let us consider an element P of the absolute. Then $\mathrm{tangent}(P)$ misses the BK-model. The theorem is a consequence of (29), (23), and (30).

(41) Let us consider elements P, P_3, P_4 of the real projective plane, elements P_1, P_2 of the absolute, and an element Q of the real projective plane. Suppose $P_1 \neq P_2$ and $P_3 = P_1$ and $P_4 = P_2$ and $P \in$ the BK-model and P, P_3 and P_4 are collinear and $Q \in \mathrm{tangent}(P_1)$ and $Q \in \mathrm{tangent}(P_2)$. Then there exists an element R of the real projective plane such that

(i) $R \in$ the absolute, and

(ii) P, Q and R are collinear.

The theorem is a consequence of (40), (7), (37), (28), and (26).

(42) Let us consider elements P, R, S of the real projective plane, and an element Q of the absolute. Suppose $P \in$ the BK-model and $R \in \mathrm{tangent}(Q)$ and P, S and R are collinear and $R \neq S$. Then $Q \neq S$. The theorem is a consequence of (29), (23), and (30).

3. Subgroup of K-Isometry

Let h be an element of EnsHomography3. We say that h is K-isometry if and only if

(Def. 6) there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of $N)^\circ$(the absolute) = the absolute.

Now we state the proposition:

(43) Let us consider an element h of EnsHomography3.
Suppose $h =$ the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}}$. Then h is K-isometry.

The set of K-isometries yielding a non empty subset of EnsHomography3 is defined by the term

(Def. 7) $\{h$, where h is an element of EnsHomography3 : h is K-isometry$\}$.

The subgroup of K-isometries yielding a strict subgroup of GroupHomography3 is defined by

(Def. 8) the carrier of *it* = the set of K-isometries.

Now we state the propositions:

(44) Let us consider an element h of the set of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3. Suppose $h =$ the homography of N. Then (the homography of N)$^\circ$(the absolute) = the absolute.

(45) (i) the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}} = \mathbf{1}_{\rm GroupHomography3}$, and

(ii) the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}} = \mathbf{1}_\alpha$,
where α is the subgroup of K-isometries.

(46) Let us consider invertible square matrices N_1, N_2 over $\mathbb{R}_{\rm F}$ of dimension 3, and elements h_1, h_2 of the subgroup of K-isometries. Suppose $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2. Then

(i) $h_1 \cdot h_2$ is an element of the subgroup of K-isometries, and

(ii) $h_1 \cdot h_2 =$ the homography of $N_1 \cdot N_2$.

(47) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, and an element h of the subgroup of K-isometries.
Suppose $h =$ the homography of N. Then

(i) $h^{-1} =$ the homography of $N^\smile$, and

(ii) the homography of $N^\smile$ is an element of the subgroup of K-isometries.

The theorem is a consequence of (45).

(48) Let us consider an element s of the projective space over $\mathcal{E}^3_{\rm T}$, and elements p, q, r of the absolute. Suppose p, q, r are mutually different and $s \in$ tangent$(p) \cap$ tangent(q). Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(Dir101) $= p$, and

(iii) (the homography of N)(Dirm101) $= q$, and

(iv) (the homography of N)(Dir011) $= r$, and

(v) (the homography of N)(Dir010) $= s$.

PROOF: Reconsider $P_1 = p$, $P_2 = q$, $P_3 = r$, $P_4 = s$ as a point of the real projective plane. P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Consider N being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N)(Dir101) $= P_1$ and (the homography of N)(Dirm101) $= P_2$ and (the homography of N)(Dir011) $= P_3$ and (the homography of N)(Dir010) $= P_4$. Consider n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N = \langle\langle n_1, n_2, n_3\rangle, \langle n_4, n_5, n_6\rangle, \langle n_7, n_8, n_9\rangle\rangle$. Reconsider $b = -1$ as an element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $a = 1$ as an element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $a = 1$, $b = -1$ as a non zero element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $N_1 = \langle\langle a, 0, 0\rangle, \langle 0, a, 0\rangle, \langle 0, 0, b\rangle\rangle$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Reconsider $M = N^{\mathrm{T}} \cdot N_1 \cdot N$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Consider v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $M = \langle\langle v_1, v_2, v_3\rangle, \langle v_4, v_5, v_6\rangle, \langle v_7, v_8, v_9\rangle\rangle$. Reconsider $r_1 = v_1$, $r_2 = v_2$, $r_3 = v_3$, $r_4 = v_5$, $r_5 = v_6$, $r_6 = v_9$ as a real number. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dir101 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dirm101 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dir011 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. $r_3 = 0$ and $r_1 = -r_6$ and $r_2 = 0$ and $r_5 = 0$ and $r_1 = r_4$. $r_1 \neq 0$. (The homography of $M)^{\circ}$(the absolute) = the absolute. $\square$

(49) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute, and elements s_1, s_2 of the real projective plane. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different and $s_1 \in$ tangent$(p_1) \cap$ tangent(q_1) and $s_2 \in$ tangent$(p_2) \cap$ tangent(q_2). Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of $N)^{\circ}$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$, and

(v) (the homography of N)$(s_1) = s_2$.

The theorem is a consequence of (48) and (47).

(50) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$.

The theorem is a consequence of (33), (48), and (47).

(51) Let us consider a collinearity space C, and elements p, q, r, s of C. If $\mathrm{Line}(p, q) = \mathrm{Line}(r, s)$, then r, s and p are collinear.

(52) Let us consider a collinearity space C, and elements p, q of C. Then $\mathrm{Line}(p, q) = \mathrm{Line}(q, p)$.
PROOF: $\mathrm{Line}(p, q) \subseteq \mathrm{Line}(q, p)$. $\mathrm{Line}(q, p) \subseteq \mathrm{Line}(p, q)$. □

(53) Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3, and elements p, q, r, s of the projective space over $\mathcal{E}^3_{\rm T}$.
Suppose Line((the homography of N)(p), (the homography of N)(q)) = Line((the homography of N)(r), (the homography of N)(s)). Then

(i) p, q and r are collinear, and

(ii) p, q and s are collinear, and

(iii) r, s and p are collinear, and

(iv) r, s and q are collinear.

The theorem is a consequence of (51) and (52).

Let us consider an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 and elements p, q, r, s, t, u, n_1, n_2, n_3, n_4 of the real projective plane. Now we state the propositions:

(54) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear. Then

(i) $u =$ (the homography of N)(t), or

(ii) $\mathrm{Line}(n_1, n_2) = \mathrm{Line}(n_3, n_4)$.

(55) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of $N)(p)$ and $n_2 =$ (the homography of $N)(q)$ and $n_3 =$ (the homography of $N)(r)$ and $n_4 =$ (the homography of $N)(s)$ and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear and p, q and r are not collinear. Then $u =$ (the homography of $N)(t)$. The theorem is a consequence of (54) and (53).

(56) Let us consider elements p, q of the absolute, and elements a, b of the BK-model. Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of $N)^\circ$(the absolute) = the absolute, and

(ii) (the homography of $N)(a) = b$, and

(iii) (the homography of $N)(p) = q$.

Proof: Consider p' being an element of the absolute such that $p \neq p'$ and p, a and p' are collinear. Consider q' being an element of the absolute such that $q \neq q'$ and q, b and q' are collinear. Consider t being an element of the real projective plane such that tangent$(p) \cap$ tangent$(p') = \{t\}$. Consider u being an element of the real projective plane such that tangent$(q) \cap$ tangent$(q') = \{u\}$. Reconsider $a' = a$ as an element of the real projective plane. Consider R_1 being an element of the real projective plane such that $R_1 \in$ the absolute and a', t and R_1 are collinear. Reconsider $b' = b$ as an element of the real projective plane. Consider R_2 being an element of the real projective plane such that $R_2 \in$ the absolute and b', u and R_2 are collinear. p, p', R_1 are mutually different. Consider N being an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3 such that (the homography of $N)^\circ$(the absolute) = the absolute and (the homography of $N)(p) = q$ and (the homography of $N)(p') = q'$ and (the homography of $N)(R_1) = R_2$ and (the homography of $N)(t) = u$. Reconsider $p_5 = p$, $p_6 = p'$, $p_7 = R_1$, $p_8 = t$, $p_9 = a$, $n_1 = q$, $n_2 = q'$, $n_3 = R_2$, $n_4 = u$, $n_5 = b$ as an element of the real projective plane. $n_5 =$ (the homography of $N)(p_9)$. $\square$

(57) Let us consider elements p, q, r, s of the absolute. Suppose p, q, r are mutually different and q, p, s are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3 such that

(i) (the homography of $N)^\circ$(the absolute) = the absolute, and

(ii) (the homography of $N)(p) = q$, and

(iii) (the homography of $N)(q) = p$, and

(iv) (the homography of $N)(r) = s$, and

(v) for every element t of the real projective plane such that $t \in \mathrm{tangent}(p) \cap \mathrm{tangent}(q)$ holds (the homography of $N)(t) = t$.

The theorem is a consequence of (33), (48), and (47).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

(58) Suppose $P \neq Q$. Then there exist elements P_1, P_2, P_3, P_4 of the absolute and there exists an element P_5 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \mathrm{tangent}(P_1) \cap \mathrm{tangent}(P_2)$. The theorem is a consequence of (20), (32), (41), (30), (42), (29), (40), and (7).

(59) Suppose $P \neq Q$. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of $N)^\circ$(the absolute) = the absolute, and

(ii) (the homography of $N)(P) = Q$, and

(iii) (the homography of $N)(Q) = P$, and

(iv) there exist elements P_1, P_2 of the absolute such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and (the homography of $N)(P_1) = P_2$ and (the homography of $N)(P_2) = P_1$.

Proof: Consider P_1, P_2, P_3, P_4 being elements of the absolute, P_5 being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \mathrm{tangent}(P_1) \cap \mathrm{tangent}(P_2)$. Consider N_1 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of $N_1)^\circ$(the absolute) = the absolute and (the homography of $N_1)(\mathrm{Dir101}) = P_1$ and (the homography of $N_1)(\mathrm{Dirm}$ $101) = P_2$ and (the homography of $N_1)(\mathrm{Dir011}) = P_3$ and (the homography of $N_1)(\mathrm{Dir010}) = P_5$. Consider N_2 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of $N_2)^\circ$(the absolute) = the absolute and (the homography of $N_2)(\mathrm{Dir101}) = P_2$ and (the homography of $N_2)(\mathrm{Dirm101}) = P_1$ and (the homography of $N_2)(\mathrm{Dir011}) = P_4$ and (the homography of $N_2)(\mathrm{Dir010}) = P_5$. Reconsider $N = N_2 \cdot (N_1{}^\smile)$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Reconsider $h_1 =$ the homography of N_1 as an element of EnsHomography3. Reconsider $h_5 = h_1$ as an element of the subgroup of K-isometries. Reconsider $h_2 =$ the homography of N_2 as an element of EnsHomography3. Reconsider

$h_6 = h_2$ as an element of the subgroup of K-isometries. Reconsider $h_3 =$ the homography of $N_1{}^\smile$ as an element of EnsHomography3. $h_5{}^{-1} = h_3$. Reconsider $h_7 = h_3$ as an element of the subgroup of K-isometries. Reconsider $h_4 = h_6 \cdot h_7$ as an element of the subgroup of K-isometries. Consider h being an element of EnsHomography3 such that $h_4 = h$ and h is K-isometry. (the homography of $N)(P) = Q$ and (the homography of $N)(Q) = P$ by [5, (102), (57)], [6, (15)]. □

4. Main Lemmas

Now we state the propositions:

(60) Let us consider elements P, Q of the BK-model. Then there exists an element h of the subgroup of K-isometries and there exists an invertible square matrix N over $\mathbb{R}_F$ of dimension 3 such that $h =$ the homography of N and (the homography of $N)(P) = Q$ and (the homography of $N)(Q) = P$. The theorem is a consequence of (43) and (59).

(61) Let us consider elements P, Q, R, S, T, U of the BK-model. Suppose there exist elements h_1, h_2 of the subgroup of K-isometries and there exist invertible square matrices N_1, N_2 over $\mathbb{R}_F$ of dimension 3 such that $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 and (the homography of $N_1)(P) = R$ and (the homography of $N_1)(Q) = S$ and (the homography of $N_2)(R) = T$ and (the homography of $N_2)(S) = U$. Then there exists an element h_3 of the subgroup of K-isometries and there exists an invertible square matrix N_3 over $\mathbb{R}_F$ of dimension 3 such that $h_3 =$ the homography of N_3 and (the homography of $N_3)(P) = T$ and (the homography of $N_3)(Q) = U$. The theorem is a consequence of (46).

(62) Let us consider elements P, Q, R of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_F$ of dimension 3. Suppose $h =$ the homography of N and (the homography of $N)(P) = R$ and (the homography of $N)(Q) = R$. Then $P = Q$.

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Fubini's Theorem for Non-Negative or Non-Positive Functions

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Summary. The goal of this article is to show Fubini's theorem for non-negative or non-positive measurable functions [10], [2], [3], using the Mizar system [1], [9]. We formalized Fubini's theorem in our previous article [5], but in that case we showed the Fubini's theorem for measurable sets and it was not enough as the integral does not appear explicitly.

On the other hand, the theorems obtained in this paper are more general and it can be easily extended to a general integrable function. Furthermore, it also can be easy to extend to functional space L^p [12]. It should be mentioned also that Hölzl and Heller [11] have introduced the Lebesgue integration theory in Isabelle/HOL and have proved Fubini's theorem there.

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1. Extended Real-Valued Characteristic Function

Let A, X be sets and e be an extended real. The functor $\chi_{e,A,X}$ yielding a function from X into $\overline{\mathbb{R}}$ is defined by

(Def. 1) for every object x such that $x \in X$ holds if $x \in A$, then $it(x) = e$ and if $x \notin A$, then $it(x) = 0$.

Now we state the propositions:

(1) Let us consider a non empty set X, a set A, and a real number r. Then $r \cdot \chi_{A,X} = \chi_{r,A,X}$.

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(2) Let us consider a non empty set X, and a set A. Then

(i) $\chi_{+\infty,A,X} = \overline{\chi}_{A,X}$, and

(ii) $\chi_{-\infty,A,X} = -\overline{\chi}_{A,X}$.

(3) Let us consider sets X, A. Then $\chi_{A,X}$ is without $+\infty$ and without $-\infty$.

(4) Let us consider a non empty set X, a set A, and a real number r. Then

(i) $\operatorname{rng}\chi_{r,A,X} \subseteq \{0, r\}$, and

(ii) $\chi_{r,A,X}$ is without $+\infty$ and without $-\infty$.

The theorem is a consequence of (3) and (1).

(5) Let us consider a non empty set X, a σ-field S of subsets of X, a non empty partial function f from X to $\overline{\mathbb{R}}$, and a σ-measure M on S. Suppose f is simple function in S. Then there exists a non empty finite sequence E of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ and there exists a finite sequence r of elements of $\mathbb{R}$ such that E and a are representation of f and for every natural number n, $a(n) = r(n)$ and $f{\restriction}E(n) = \chi_{r(n),E(n),X}{\restriction}E(n)$ and if $E(n) = \emptyset$, then $r(n) = 0$.

PROOF: Consider E being a finite sequence of separated subsets of S, b being a finite sequence of elements of $\overline{\mathbb{R}}$ such that E and b are representation of f. For every natural number n such that $E(n) \neq \emptyset$ holds $b(n) \in \mathbb{R}$ by [8, (32)]. Define $\mathcal{Q}[\text{natural number}, \text{object}] \equiv$ if $E(\$_1) \neq \emptyset$, then $\$_2 = b(\$_1)$ and if $E(\$_1) = \emptyset$, then $\$_2 = 0$. For every natural number n such that $n \in \operatorname{Seg}\operatorname{len} E$ there exists an element a of $\overline{\mathbb{R}}$ such that $\mathcal{Q}[n, a]$. Consider a being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a = \operatorname{Seg}\operatorname{len} E$ and for every natural number n such that $n \in \operatorname{Seg}\operatorname{len} E$ holds $\mathcal{Q}[n, a(n)]$. Define $\mathcal{R}[\text{natural number}, \text{object}] \equiv \$_2 = a(\$_1)$. For every natural number n such that $n \in \operatorname{Seg}\operatorname{len} E$ there exists an element r of $\mathbb{R}$ such that $\mathcal{R}[n, r]$. Consider r being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg}\operatorname{len} E$ and for every natural number n such that $n \in \operatorname{Seg}\operatorname{len} E$ holds $\mathcal{R}[n, r(n)]$. For every natural number n such that $n \in \operatorname{dom} E$ for every object x such that $x \in E(n)$ holds $f(x) = a(n)$. For every natural number n, $a(n) = r(n)$ and $f{\restriction}E(n) = \chi_{r(n),E(n),X}{\restriction}E(n)$ and if $E(n) = \emptyset$, then $r(n) = 0$. □

Let F be a finite sequence-like function. Let us observe that F is disjoint valued if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $m \neq n$ holds $F(m)$ misses $F(n)$.

Now we state the propositions:

(6) Let us consider a non empty set X, a σ-field S of subsets of X, and elements E_1, E_2 of S. Suppose E_1 misses E_2. Then $\langle E_1, E_2\rangle$ is a finite sequence of separated subsets of S.

(7) Let us consider a non empty set X, subsets A_1, A_2 of X, and real numbers r_1, r_2. Then $\langle \chi_{r_1,A_1,X}, \chi_{r_2,A_2,X}\rangle$ is a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. The theorem is a consequence of (4).

(8) Let us consider a non empty set X, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose $\operatorname{len} F \geqslant 2$. Then $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/2} = F_{/1}+F_{/2}$.

(9) Let us consider a non empty set X, and a function f from X into $\overline{\mathbb{R}}$. Then $f + (X \longmapsto 0_{\overline{\mathbb{R}}}) = f$.

(10) Let us consider a non empty set X, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then

(i) $\operatorname{dom} F = \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) for every natural number n such that $n \in \operatorname{dom} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n} = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$, and

(iii) for every natural number n and for every element x of X such that $1 \leqslant n < \operatorname{len} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n}(x)+ F_{/n+1}(x)$.

Proof: For every natural number n and for every element x of X such that $1 \leqslant n < \operatorname{len} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n}(x)+ F_{/n+1}(x)$. $\square$

(11) Let us consider a non empty set X, a σ-field S of subsets of X, a function f from X into $\overline{\mathbb{R}}$, a finite sequence E of separated subsets of S, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose $\operatorname{dom} E = \operatorname{dom} F$ and $\operatorname{dom} f = \bigcup \operatorname{rng} E$ and for every natural number n such that $n \in \operatorname{dom} F$ there exists a real number r such that $F_{/n} = r \cdot \chi_{E(n),X}$ and $f = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/\operatorname{len} F}$. Then

(i) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $m \neq n$ holds $F_{/n}(x) = 0$, and

(ii) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $n < m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n}(x) = 0$, and

(iii) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $n \geqslant m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_{/n}(x) = f(x)$, and

(iv) for every element x of X and for every natural number m such that $m \in \operatorname{dom} F$ and $x \in E(m)$ holds $F_{/m}(x) = f(x)$, and

(v) f is simple function in S.

Proof: For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $m \neq n$ holds $F_{/n}(x) = 0$. For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $n < m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = 0$. For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $n \geqslant m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = f(x)$. For every element x of X and for every natural number m such that $m \in \operatorname{dom} F$ and $x \in E(m)$ holds $F_{/m}(x) = f(x)$. For every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$ by [7, (41)]. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} E$ and x, $y \in E(n)$ holds $f(x) = f(y)$. $\square$

(12) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and an element E of S. Then $\chi_{E,X}$ is simple function in S.

Proof: Reconsider $E_2 = X \setminus E$ as an element of S. Reconsider $E_3 = \langle E, E_2 \rangle$ as a finite sequence of separated subsets of S. $1 \cdot \chi_{E,X} = \chi_{1,E,X}$ and $0 \cdot \chi_{E_2,X} = \chi_{0,E_2,X}$. Reconsider $F = \langle 1 \cdot \chi_{E,X}, 0 \cdot \chi_{E_2,X} \rangle$ as a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. For every natural number n such that $n \in \operatorname{dom} F$ there exists a real number r such that $F_{/n} = r \cdot \chi_{E_3(n),X}$. $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/\operatorname{len} F} = F_{/1} + F_{/2}$. $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/\operatorname{len} F} = \chi_{E,X}$. $\square$

(13) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements A, B of S, and an extended real e. Then $\chi_{e,A,X}$ is measurable on B. The theorem is a consequence of (2) and (1).

(14) Let us consider a set X, subsets A_1, A_2 of X, and an extended real e. Then $\chi_{e,A_1,X} \restriction A_2 = \chi_{e,A_1 \cap A_2,X} \restriction A_2$.

(15) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements A, B, C of S, and an extended real e. If $C \subseteq B$, then $\chi_{e,A,X} \restriction B$ is measurable on C. The theorem is a consequence of (13).

(16) Let us consider a set X, subsets A_1, A_2 of X, an extended real e, and an object x. If A_1 misses A_2, then $(\chi_{e,A_1,X} \restriction A_2)(x) = 0$.

(17) Let us consider a set X, a subset A of X, and an extended real e. Then

(i) if $e \geqslant 0$, then $\chi_{e,A,X}$ is non-negative, and

(ii) if $e \leqslant 0$, then $\chi_{e,A,X}$ is non-positive.

(18) Let us consider sets A, X, and a subset B of X. Then $\operatorname{dom}(\chi_{A,X} \restriction B) = B$.

2. Some Properties of Integration

Now we state the propositions:

(19) Let us consider a non empty set X, a σ-field S of subsets of X, and a partial function f from X to $\overline{\mathbb{R}}$. If f is empty, then f is simple function in S.

Proof: Reconsider $E_4 = \emptyset$ as an element of S. Reconsider $F = \langle E_4 \rangle$ as a finite sequence of separated subsets of S. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$. $\square$

(20) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and elements E_1, E_2 of S. Then $\int \chi_{E_1,X} \restriction E_2 \, \mathrm{d}M = M(E_1 \cap E_2)$.

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements E_1, E_2 of S, and partial functions f, g from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(21) Suppose $E_1 = \operatorname{dom} f$ and f is non-negative and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-negative and g is measurable on E_2. Then $\int f + g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f+g) \, \mathrm{d}M + \int g \restriction \operatorname{dom}(f+g) \, \mathrm{d}M$.

(22) Suppose $E_1 = \operatorname{dom} f$ and f is non-positive and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-positive and g is measurable on E_2. Then $\int f + g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f+g) \, \mathrm{d}M + \int g \restriction \operatorname{dom}(f+g) \, \mathrm{d}M$. The theorem is a consequence of (21).

(23) Suppose $E_1 = \operatorname{dom} f$ and f is non-negative and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-positive and g is measurable on E_2. Then

(i) $\int f - g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f-g) \, \mathrm{d}M - \int g \restriction \operatorname{dom}(f-g) \, \mathrm{d}M$, and

(ii) $\int g - f \, \mathrm{d}M = \int g \restriction \operatorname{dom}(g-f) \, \mathrm{d}M - \int f \restriction \operatorname{dom}(g-f) \, \mathrm{d}M$.

The theorem is a consequence of (21).

(24) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, a partial function f from X to $\overline{\mathbb{R}}$, and a real number r. Suppose $E = \operatorname{dom} f$ and f is non-positive or non-negative and f is measurable on E. Then $\int r \cdot f \, \mathrm{d}M = r \cdot \int f \, \mathrm{d}M$.

3. Sections of Partial Function

Now we state the proposition:

(25) Let us consider non empty sets X, Y, an element A of $2^{X \times Y}$, and sets x, y. Suppose $x \in X$ and $y \in Y$. Then

(i) $\langle x, y\rangle \in A$ iff $x \in \mathrm{Ysection}(A, y)$, and

(ii) $\langle x, y\rangle \in A$ iff $y \in \mathrm{Xsection}(A, x)$.

Let X_1, X_2 be non empty sets, Y be a set, f be a partial function from $X_1 \times X_2$ to Y, and x be an element of X_1. The functor $\mathrm{ProjPMap1}(f, x)$ yielding a partial function from X_2 to Y is defined by

(Def. 3) $\mathrm{dom}\, it = \mathrm{Xsection}(\mathrm{dom}\, f, x)$ and for every element y of X_2 such that $\langle x, y\rangle \in \mathrm{dom}\, f$ holds $it(y) = f(x, y)$.

Let y be an element of X_2. The functor $\mathrm{ProjPMap2}(f, y)$ yielding a partial function from X_1 to Y is defined by

(Def. 4) $\mathrm{dom}\, it = \mathrm{Ysection}(\mathrm{dom}\, f, y)$ and for every element x of X_1 such that $\langle x, y\rangle \in \mathrm{dom}\, f$ holds $it(x) = f(x, y)$.

Now we state the propositions:

(26) Let us consider non empty sets X_1, X_2, a set Y, a partial function f from $X_1 \times X_2$ to Y, an element x of X_1, and an element y of X_2. Then

(i) if $x \in \mathrm{dom}\, \mathrm{ProjPMap2}(f, y)$, then $(\mathrm{ProjPMap2}(f, y))(x) = f(x, y)$, and

(ii) if $y \in \mathrm{dom}\, \mathrm{ProjPMap1}(f, x)$, then $(\mathrm{ProjPMap1}(f, x))(y) = f(x, y)$.

(27) Let us consider non empty sets X_1, X_2, Y, a function f from $X_1 \times X_2$ into Y, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(f, x) = \mathrm{curry}(f, x)$, and

(ii) $\mathrm{ProjPMap2}(f, y) = \mathrm{curry}'(f, y)$.

The theorem is a consequence of (26).

(28) Let us consider non empty sets X, Y, Z, a partial function f from $X \times Y$ to Z, an element x of X, an element y of Y, and a set A. Then

(i) $\mathrm{Xsection}(f^{-1}(A), x) = (\mathrm{ProjPMap1}(f, x))^{-1}(A)$, and

(ii) $\mathrm{Ysection}(f^{-1}(A), y) = (\mathrm{ProjPMap2}(f, y))^{-1}(A)$.

(29) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a real number r, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\mathrm{ProjPMap1}(r \cdot f, x) = r \cdot \mathrm{ProjPMap1}(f, x)$, and

(ii) $\mathrm{ProjPMap2}(r \cdot f, y) = r \cdot \mathrm{ProjPMap2}(f, y)$.

(30) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, and an element y of X_2. Suppose for every element z of $X_1 \times X_2$ such that $z \in \operatorname{dom} f$ holds $f(z) = 0$. Then

(i) $(\mathrm{ProjPMap2}(f, y))(x) = 0$, and

(ii) $(\mathrm{ProjPMap1}(f, x))(y) = 0$.

(31) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element x of X_1, an element y of X_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose f is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Then

(i) $\mathrm{ProjPMap1}(f, x)$ is simple function in S_2, and

(ii) $\mathrm{ProjPMap2}(f, y)$ is simple function in S_1.

Proof: Consider F being a finite sequence of separated subsets of σ(Meas–Rect(S_1, S_2)) such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for every elements z_1, z_2 of $X_1 \times X_2$ such that $n \in \operatorname{dom} F$ and z_1, $z_2 \in F(n)$ holds $f(z_1) = f(z_2)$. Define $\mathcal{H}(\text{natural number}) = \mathrm{MeasurableXsection}(F(\$_1), x)$. Consider H being a finite sequence of elements of S_2 such that $\operatorname{len} H = \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{dom} H$ holds $H(n) = \mathcal{H}(n)$. Reconsider $F_1 = F$ as a finite sequence of elements of $2^{X_1 \times X_2}$. Reconsider $F_2 = H$ as a finite sequence of elements of 2^{X_2}. For every natural number n such that $n \in \operatorname{dom} F_2$ holds $F_2(n) = \mathrm{Xsection}(F_1(n), x)$. For every natural number n and for every elements y_1, y_2 of X_2 such that $n \in \operatorname{dom} H$ and y_1, $y_2 \in H(n)$ holds $(\mathrm{ProjPMap1}(f, x))(y_1) = (\mathrm{ProjPMap1}(f, x))(y_2)$. Define $\mathcal{G}(\text{natural number}) = \mathrm{MeasurableYsection}(F(\$_1), y)$. Consider G being a finite sequence of elements of S_1 such that $\operatorname{len} G = \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = \mathcal{G}(n)$. Reconsider $G_1 = G$ as a finite sequence of elements of 2^{X_1}. For every natural number n such that $n \in \operatorname{dom} G_1$ holds $G_1(n) = \mathrm{Ysection}(F_1(n), y)$. For every natural number n and for every elements x_1, x_2 of X_1 such that $n \in \operatorname{dom} G$ and x_1, $x_2 \in G(n)$ holds $(\mathrm{ProjPMap2}(f, y))(x_1) = (\mathrm{ProjPMap2}(f, y))(x_2)$. □

Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(32) If f is non-negative, then $\mathrm{ProjPMap1}(f, x)$ is non-negative and $\mathrm{ProjPMap2}(f, y)$ is non-negative.

Proof: For every object q such that $q \in \operatorname{dom} \mathrm{ProjPMap1}(f, x)$ holds $0 \leqslant (\mathrm{ProjPMap1}(f, x))(q)$. For every object p such that $p \in \operatorname{dom} \mathrm{ProjPMap2}(f, y)$ holds $0 \leqslant (\mathrm{ProjPMap2}(f, y))(p)$. □

(33) If f is non-positive, then ProjPMap1(f, x) is non-positive and ProjPMap2(f, y) is non-positive.
PROOF: For every set q such that $q \in \operatorname{dom} \text{ProjPMap1}(f, x)$ holds $0 \geqslant (\text{ProjPMap1}(f, x))(q)$. For every set p such that $p \in \operatorname{dom} \text{ProjPMap2}(f, y)$ holds $0 \geqslant (\text{ProjPMap2}(f, y))(p)$ by [6, (8)]. □

(34) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a subset A of $X_1 \times X_2$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\text{ProjPMap1}(f \restriction A, x) = \text{ProjPMap1}(f, x) \restriction \text{Xsection}(A, x)$, and

(ii) $\text{ProjPMap2}(f \restriction A, y) = \text{ProjPMap2}(f, y) \restriction \text{Ysection}(A, y)$.

The theorem is a consequence of (25).

(35) Let us consider non empty sets X_1, X_2, a subset A of $X_1 \times X_2$, an element x of X_1, and an element y of X_2. Then

(i) $\text{ProjPMap1}(\overline{\chi}_{A, X_1 \times X_2}, x) = \overline{\chi}_{\text{Xsection}(A,x), X_2}$, and

(ii) $\text{ProjPMap2}(\overline{\chi}_{A, X_1 \times X_2}, y) = \overline{\chi}_{\text{Ysection}(A,y), X_1}$.

The theorem is a consequence of (27) and (25).

(36) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $f \restriction E = g \restriction E$ and $E \subseteq \operatorname{dom} f$ and $E \subseteq \operatorname{dom} g$ and f is measurable on E. Then g is measurable on E.

(37) Let us consider non empty sets X_1, X_2, a subset A of $X_1 \times X_2$, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, an element y of X_2, and a sequence F of partial functions from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. Suppose $A \subseteq \operatorname{dom} f$ and for every natural number n, $\operatorname{dom}(F(n)) = A$ and for every element z of $X_1 \times X_2$ such that $z \in A$ holds $F\#z$ is convergent and $\lim(F\#z) = f(z)$. Then

(i) there exists a sequence F_1 of partial functions from X_1 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_1(n) = \text{ProjPMap2}(F(n), y)$ and for every element x of X_1 such that $x \in \text{Ysection}(A, y)$ holds $F_1\#x$ is convergent and $(\text{ProjPMap2}(f, y))(x) = \lim(F_1\#x)$, and

(ii) there exists a sequence F_2 of partial functions from X_2 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_2(n) = \text{ProjPMap1}(F(n), x)$ and for every element y of X_2 such that $y \in \text{Xsection}(A, x)$ holds $F_2\#y$ is convergent and $(\text{ProjPMap1}(f, x))(y) = \lim(F_2\#y)$.

PROOF: Define $\mathcal{R}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \text{ProjPMap2}(F(\$_1), y)$. For every element n of $\mathbb{N}$, there exists an element f of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that

$\mathcal{R}[n, f]$. There exists a sequence F_1 of partial functions from X_1 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_1(n) = \text{ProjPMap2}(F(n), y)$ and for every element x of X_1 such that $x \in \text{Ysection}(A, y)$ holds $F_1\#x$ is convergent and $(\text{ProjPMap2}(f, y))(x) = \lim(F_1\#x)$. Define $\mathcal{Q}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \text{ProjPMap1}(F(\$_1), x)$. For every element n of $\mathbb{N}$, there exists an element f of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{Q}[n, f]$. Consider F_2 being a sequence of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{Q}[n, F_2(n)]$. For every natural number n, $\text{dom}(F_2(n)) = \text{Xsection}(A, x)$. For every natural numbers m, n, $\text{dom}(F_2(m)) = \text{dom}(F_2(n))$. For every natural number n, $F_2(n) = \text{ProjPMap1}(F(n), x)$. For every element y of X_2 such that $y \in \text{Xsection}(A, x)$ holds $F_2\#y$ is convergent and $(\text{ProjPMap1}(f, x))(y) = \lim(F_2\#y)$. $\square$

(38) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, a σ-measure M_2 on S_2, an element A of S_1, an element B of S_2, and an element x of X_1. Then $M_2(B \cap \text{MeasurableXsection}(E, x)) \cdot (\chi_{A,X_1}(x)) = \int \text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2} \restriction E, x)\, \mathrm{d}M_2$.

PROOF: Set $C_1 = \chi_{A \times B, X_1 \times X_2} \restriction E$. $\text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) = \text{curry}(\chi_{A \times B, X_1 \times X_2}, x)$. $\text{ProjPMap1}(C_1, x) = \text{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) \restriction \text{Xsection}(E, x)$. For every element y of X_2, $(\text{ProjPMap1}(C_1, x))(y) = (\chi_{A,X_1} \restriction \text{MeasurableYsection}(E, y))(x) \cdot (\chi_{B,X_2}(y))$. $\square$

(39) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, a σ-measure M_1 on S_1, an element A of S_1, an element B of S_2, and an element y of X_2. Then $M_1(A \cap \text{MeasurableYsection}(E, y)) \cdot (\chi_{B,X_2}(y)) = \int \text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2} \restriction E, y)\, \mathrm{d}M_1$.

PROOF: Set $C_1 = \chi_{A \times B, X_1 \times X_2} \restriction E$. $\text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) = \text{curry}'(\chi_{A \times B, X_1 \times X_2}, y)$. $\text{ProjPMap2}(C_1, y) = \text{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) \restriction \text{Ysection}(E, y)$. For every element x of X_1, $(\text{ProjPMap2}(C_1, y))(x) = (\chi_{B,X_2} \restriction \text{MeasurableXsection}(E, x))(y) \cdot (\chi_{A,X_1}(x))$ by [4, (2)]. $\square$

(40) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an extended real e. Then

(i) $\langle x, y \rangle \in \text{dom} f$ and $f(x, y) = e$ iff $y \in \text{dom}\,\text{ProjPMap1}(f, x)$ and $(\text{ProjPMap1}(f, x))(y) = e$, and

(ii) $\langle x, y \rangle \in \text{dom} f$ and $f(x, y) = e$ iff $x \in \text{dom}\,\text{ProjPMap2}(f, y)$ and $(\text{ProjPMap2}(f, y))(x) = e$.

The theorem is a consequence of (25) and (26).

(41) Let us consider non empty sets X_1, X_2, sets A, Z, a partial function f from $X_1 \times X_2$ to Z, and an element x of X_1. Then $\mathrm{Xsection}(f^{-1}(A), x) = (\mathrm{ProjPMap1}(f, x))^{-1}(A)$.

(42) Let us consider non empty sets X_1, X_2, sets A, Z, a partial function f from $X_1 \times X_2$ to Z, and an element y of X_2. Then $\mathrm{Ysection}(f^{-1}(A), y) = (\mathrm{ProjPMap2}(f, y))^{-1}(A)$.

(43) Let us consider non empty sets X_1, X_2, subsets A, B of $X_1 \times X_2$, and a set p. Then

(i) $\mathrm{Xsection}(A \setminus B, p) = \mathrm{Xsection}(A, p) \setminus \mathrm{Xsection}(B, p)$, and

(ii) $\mathrm{Ysection}(A \setminus B, p) = \mathrm{Ysection}(A, p) \setminus \mathrm{Ysection}(B, p)$.

(44) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, and partial functions f_1, f_2 from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\mathrm{ProjPMap1}(f_1 + f_2, x) = \mathrm{ProjPMap1}(f_1, x) + \mathrm{ProjPMap1}(f_2, x)$, and

(ii) $\mathrm{ProjPMap1}(f_1 - f_2, x) = \mathrm{ProjPMap1}(f_1, x) - \mathrm{ProjPMap1}(f_2, x)$, and

(iii) $\mathrm{ProjPMap2}(f_1 + f_2, y) = \mathrm{ProjPMap2}(f_1, y) + \mathrm{ProjPMap2}(f_2, y)$, and

(iv) $\mathrm{ProjPMap2}(f_1 - f_2, y) = \mathrm{ProjPMap2}(f_1, y) - \mathrm{ProjPMap2}(f_2, y)$.

The theorem is a consequence of (42), (41), (43), (26), and (40).

(45) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element x of X_1. Then

(i) $\mathrm{ProjPMap1}(\max_+(f), x) = \max_+(\mathrm{ProjPMap1}(f, x))$, and

(ii) $\mathrm{ProjPMap1}(\max_-(f), x) = \max_-(\mathrm{ProjPMap1}(f, x))$.

(46) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element y of X_2. Then

(i) $\mathrm{ProjPMap2}(\max_+(f), y) = \max_+(\mathrm{ProjPMap2}(f, y))$, and

(ii) $\mathrm{ProjPMap2}(\max_-(f), y) = \max_-(\mathrm{ProjPMap2}(f, y))$.

(47) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose $E \subseteq \mathrm{dom}\, f$ and f is measurable on E. Then

(i) $\mathrm{ProjPMap1}(f, x)$ is measurable on $\mathrm{MeasurableXsection}(E, x)$, and

(ii) $\mathrm{ProjPMap2}(f, y)$ is measurable on $\mathrm{MeasurableYsection}(E, y)$.

The theorem is a consequence of (45) and (46).

Let X_1, X_2, Y be non empty sets, F be a sequence of partial functions from $X_1 \times X_2$ into Y, and x be an element of X_1. The functor ProjPMap1(F, x) yielding a sequence of partial functions from X_2 into Y is defined by

(Def. 5) for every natural number n, $it(n) = \mathrm{ProjPMap1}(F(n), x)$.

Let y be an element of X_2. The functor ProjPMap2(F, y) yielding a sequence of partial functions from X_1 into Y is defined by

(Def. 6) for every natural number n, $it(n) = \mathrm{ProjPMap2}(F(n), y)$.

(48) Let us consider non empty sets X_1, X_2, a subset E of $X_1 \times X_2$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{\mathrm{Xsection}(E,x), X_2}$, and

(ii) $\mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{\mathrm{Ysection}(E,y), X_1}$.

The theorem is a consequence of (25) and (27).

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and an extended real e. Now we state the propositions:

(49) $\int \chi_{e,E,X} \, \mathrm{d}M = e \cdot M(E)$. The theorem is a consequence of (2), (12), and (1).

(50) $\int \chi_{e,E,X} \restriction E \, \mathrm{d}M = e \cdot M(E)$. The theorem is a consequence of (15), (2), (13), (49), (16), (1), and (12).

(51) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements E_1, E_2 of S, and an extended real e. Then $\int \chi_{e,E_1,X} \restriction E_2 \, \mathrm{d}M = e \cdot M(E_1 \cap E_2)$. The theorem is a consequence of (14), (17), (13), (16), (15), and (50).

(52) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then

(i) $(\mathrm{Yvol}(E, M_2))(x) = \int \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$, and

(ii) $(\mathrm{Yvol}(E, M_2))(x) = \int^{+} \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$, and

(iii) $(\mathrm{Yvol}(E, M_2))(x) = \int' \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$.

The theorem is a consequence of (48), (12), and (27).

(53) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then

(i) $(\mathrm{Xvol}(E, M_1))(y) = \int \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$, and

(ii) $(\mathrm{Xvol}(E, M_1))(y) = \int^{+} \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$, and

(iii) $(\mathrm{Xvol}(E, M_1))(y) = \int' \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$.

The theorem is a consequence of (48), (12), and (27).

(54) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a real number r. Then $\int r \cdot \chi_{E,X} \, \mathrm{d}M = r \cdot \int \chi_{E,X} \, \mathrm{d}M$. The theorem is a consequence of (12).

(55) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a real number r. Suppose M_1 is σ-finite. Then

(i) $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \, \mathrm{d}M_1$, and

(ii) if $r \geqslant 0$, then $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int^+ \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \, \mathrm{d}M_1$.

PROOF: Set $p_2 = \mathrm{ProjPMap2}(\chi_{E,X_1 \times X_2}, y)$. $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$. $\mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) = r \cdot p_2$. p_2 is non-negative. $\chi_{E,X_1 \times X_2}$ is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. $\int \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \, \mathrm{d}M_1 = r \cdot (\int' p_2 \, \mathrm{d}M_1)$. If $r \geqslant 0$, then $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int^+ \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \, \mathrm{d}M_1$. $\square$

(56) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a real number r. Suppose M_2 is σ-finite. Then

(i) $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \, \mathrm{d}M_2$, and

(ii) if $r \geqslant 0$, then $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int^+ \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \, \mathrm{d}M_2$.

PROOF: Set $p_2 = \mathrm{ProjPMap1}(\chi_{E,X_1 \times X_2}, x)$. $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$. $\mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) = r \cdot p_2$. p_2 is non-negative. $\chi_{E,X_1 \times X_2}$ is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. $\int \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \, \mathrm{d}M_2 = r \cdot (\int' p_2 \, \mathrm{d}M_2)$. If $r \geqslant 0$, then $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int^+ \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \, \mathrm{d}M_2$. $\square$

(57) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $\mathrm{dom} f \in S$ and for every element x of X such that $x \in \mathrm{dom} f$ holds $0 = f(x)$. Then

(i) for every element E of S such that $E \subseteq \mathrm{dom} f$ holds f is measurable on E, and

(ii) $\int f \, \mathrm{d}M = 0$.

The theorem is a consequence of (15) and (50).

(58) Let us consider non empty sets X_1, X_2, Y, a sequence F of partial functions from $X_1 \times X_2$ into Y, an element x of X_1, and an element y of X_2. Suppose F has the same dom. Then

(i) $\mathrm{ProjPMap1}(F, x)$ has the same dom, and

(ii) $\mathrm{ProjPMap2}(F, y)$ has the same dom.

4. Fubini's Theorem for Non-negative or Non-positive Functions

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, M_1 be a σ-measure on S_1, and f be a partial function from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. The functor Integral1(M_1, f) yielding a function from X_2 into $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every element y of X_2, $it(y) = \int \mathrm{ProjPMap2}(f, y)\,\mathrm{d}M_1$.

Let S_2 be a σ-field of subsets of X_2 and M_2 be a σ-measure on S_2. The functor Integral2(M_2, f) yielding a function from X_1 into $\overline{\mathbb{R}}$ is defined by

(Def. 8) for every element x of X_1, $it(x) = \int \mathrm{ProjPMap1}(f, x)\,\mathrm{d}M_2$.

Now we state the propositions:

(59) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element V of S_2. Suppose M_1 is σ-finite and f is non-negative or non-positive and $E = \mathrm{dom}\, f$ and f is measurable on E. Then Integral1(M_1, f) is measurable on V.

(60) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element U of S_1. Suppose M_2 is σ-finite and f is non-negative or non-positive and $E = \mathrm{dom}\, f$ and f is measurable on E. Then Integral2(M_2, f) is measurable on U.

(61) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then $(\mathrm{Xvol}(E, M_1))(y) = \int \chi_{\mathrm{MeasurableYsection}(E,y), X_1}\,\mathrm{d}M_1$.

(62) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then $(\mathrm{Yvol}(E, M_2))(x) = \int \chi_{\mathrm{MeasurableXsection}(E,x), X_2}\,\mathrm{d}M_2$.

(63) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{\mathrm{MeasurableXsection}(E,x), X_2}$, and

(ii) $\mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{\mathrm{MeasurableYsection}(E,y), X_1}$.

(64) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element

E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then $\text{Xvol}(E, M_1) = \text{Integral1}(M_1, \chi_{E, X_1 \times X_2})$. The theorem is a consequence of (61) and (63).

(65) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then $\text{Yvol}(E, M_2) = \text{Integral2}(M_2, \chi_{E, X_1 \times X_2})$. The theorem is a consequence of (62) and (63).

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, M_1 be a σ-measure on S_1, and M_2 be a σ-measure on S_2. The functor $\text{ProdMeas}(M_1, M_2)$ yielding a σ-measure on $\sigma(\text{MeasRect}(S_1, S_2))$ is defined by the term

(Def. 9) $\text{Prod}\,\sigma\text{-Meas}(M_1, M_2)$.

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$. Now we state the propositions:

(66) Suppose $E_1 = \text{dom}\, f$ and f is non-negative and f is measurable on E_1. Then

(i) $\text{Integral1}(M_1, f)$ is non-negative, and

(ii) $\text{Integral1}(M_1, f \restriction E_2)$ is non-negative, and

(iii) $\text{Integral2}(M_2, f)$ is non-negative, and

(iv) $\text{Integral2}(M_2, f \restriction E_2)$ is non-negative.

The theorem is a consequence of (47) and (32).

(67) Suppose $E_1 = \text{dom}\, f$ and f is non-positive and f is measurable on E_1. Then

(i) $\text{Integral1}(M_1, f)$ is non-positive, and

(ii) $\text{Integral1}(M_1, f \restriction E_2)$ is non-positive, and

(iii) $\text{Integral2}(M_2, f)$ is non-positive, and

(iv) $\text{Integral2}(M_2, f \restriction E_2)$ is non-positive.

The theorem is a consequence of (47) and (33).

(68) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element V of S_2. Suppose M_1 is σ-finite and f is non-negative or non-positive and $E_1 = \text{dom}\, f$ and f is measurable on E_1. Then $\text{Integral1}(M_1, f \restriction E_2)$ is measurable on V. The theorem is a consequence of (59).

(69) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, elements E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element U of S_1. Suppose M_2 is σ-finite and f is non-negative or non-positive and $E_1 = \text{dom}\, f$ and f is measurable on E_1. Then $\text{Integral2}(M_2, f \restriction E_2)$ is measurable on U. The theorem is a consequence of (60).

(70) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2. Suppose $E = \text{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and for every element x of X_1 such that $x \in \text{dom}\,\text{ProjPMap2}(f, y)$ holds $(\text{ProjPMap2}(f, y))(x) = 0$. Then $(\text{Integral1}(M_1, f))(y) = 0$. The theorem is a consequence of (47), (32), and (33).

(71) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element x of X_1. Suppose $E = \text{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and for every element y of X_2 such that $y \in \text{dom}\,\text{ProjPMap1}(f, x)$ holds $(\text{ProjPMap1}(f, x))(y) = 0$. Then $(\text{Integral2}(M_2, f))(x) = 0$. The theorem is a consequence of (47), (32), and (33).

(72) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, elements E, E_1, E_2 of $\sigma(\text{MeasRect}(S_1, S_2))$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose $E = \text{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and E_1 misses E_2. Then

- (i) $\text{Integral1}(M_1, f \restriction (E_1 \cup E_2)) =$
  $\text{Integral1}(M_1, f \restriction E_1) + \text{Integral1}(M_1, f \restriction E_2)$, and
- (ii) $\text{Integral2}(M_2, f \restriction (E_1 \cup E_2)) =$
  $\text{Integral2}(M_2, f \restriction E_1) + \text{Integral2}(M_2, f \restriction E_2)$.

(73) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E = \text{dom}\, f$ and f is measurable on E. Then

- (i) $\text{Integral1}(M_1, -f) = -\text{Integral1}(M_1, f)$, and
- (ii) $\text{Integral2}(M_2, -f) = -\text{Integral2}(M_2, f)$.

The theorem is a consequence of (29) and (47).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, partial functions f, g from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(74) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-negative and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-negative and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f + g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f + g)) + \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f + g))$, and

(ii) $\mathrm{Integral2}(M_2, f + g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f + g)) + \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f + g))$.

PROOF: Set $f_1 = f \restriction (A \cap B)$. Set $g_1 = g \restriction (A \cap B)$. $\mathrm{Integral1}(M_1, f_1)$ is non-negative and $\mathrm{Integral1}(M_1, g_1)$ is non-negative and $\mathrm{Integral2}(M_2, f_1)$ is non-negative and $\mathrm{Integral2}(M_2, g_1)$ is non-negative. For every element y of X_2, $(\mathrm{Integral1}(M_1, f_1) + \mathrm{Integral1}(M_1, g_1))(y) = (\mathrm{Integral1}(M_1, f + g))(y)$. For every element x of X_1, $(\mathrm{Integral2}(M_2, f_1) + \mathrm{Integral2}(M_2, g_1))(x) = (\mathrm{Integral2}(M_2, f + g))(x)$. □

(75) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-positive and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-positive and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f + g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f + g)) + \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f + g))$, and

(ii) $\mathrm{Integral2}(M_2, f + g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f + g)) + \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f + g))$.

The theorem is a consequence of (73) and (74).

(76) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-negative and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-positive and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f - g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f - g)) - \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f - g))$, and

(ii) $\mathrm{Integral1}(M_1, g - f) =$
$\mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(g - f)) - \mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(g - f))$, and

(iii) $\mathrm{Integral2}(M_2, f - g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f - g)) - \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f - g))$, and

(iv) $\mathrm{Integral2}(M_2, g - f) =$
$\mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(g - f)) - \mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(g - f))$.

The theorem is a consequence of (74) and (73).

(77) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \mathrm{Yvol}(E, M_2)\,\mathrm{d}M_1 = \int \chi_{E,X_1 \times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$, and

(ii) $\int \mathrm{Xvol}(E, M_1)\,\mathrm{d}M_2 = \int \chi_{E,X_1 \times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$.

(78) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and a real number r. Suppose $E = \mathrm{dom}\, f$ and f is non-negative or non-positive and f is measurable on E. Then

(i) $\mathrm{Integral1}(M_1, r \cdot f) = r \cdot \mathrm{Integral1}(M_1, f)$, and

(ii) $\mathrm{Integral2}(M_2, r \cdot f) = r \cdot \mathrm{Integral2}(M_2, f)$.

The theorem is a consequence of (32), (33), (29), and (47).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(79) (i) $\mathrm{Integral1}(M_1, \chi_{E,X_1 \times X_2} \restriction E) = \mathrm{Integral1}(M_1, \chi_{E,X_1 \times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \chi_{E,X_1 \times X_2} \restriction E) = \mathrm{Integral2}(M_2, \chi_{E,X_1 \times X_2})$.

The theorem is a consequence of (34) and (48).

(80) (i) $\mathrm{Integral1}(M_1, \overline{\chi}_{E,X_1 \times X_2} \restriction E) = \mathrm{Integral1}(M_1, \overline{\chi}_{E,X_1 \times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \overline{\chi}_{E,X_1 \times X_2} \restriction E) = \mathrm{Integral2}(M_2, \overline{\chi}_{E,X_1 \times X_2})$.

The theorem is a consequence of (34), (35), (2), (50), and (49).

(81) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an extended real e. Then

(i) $\mathrm{Integral1}(M_1, \chi_{e,E,X_1 \times X_2} \restriction E) = \mathrm{Integral1}(M_1, \chi_{e,E,X_1 \times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \chi_{e,E,X_1 \times X_2} \restriction E) = \mathrm{Integral2}(M_2, \chi_{e,E,X_1 \times X_2})$.

The theorem is a consequence of (1), (78), (79), (2), and (80).

(82) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \chi_{E,X_1 \times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, \chi_{E,X_1 \times X_2})\,\mathrm{d}M_2$, and

(ii) $\int \chi_{E,X_1 \times X_2} \restriction E \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, \chi_{E,X_1 \times X_2} \restriction E) \, \mathrm{d}M_2$, and

(iii) $\int \chi_{E,X_1 \times X_2} \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, \chi_{E,X_1 \times X_2}) \, \mathrm{d}M_1$, and

(iv) $\int \chi_{E,X_1 \times X_2} \restriction E \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, \chi_{E,X_1 \times X_2} \restriction E) \, \mathrm{d}M_1$.

The theorem is a consequence of (64), (77), (79), and (65).

(83) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a real number r. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \chi_{r,E,X_1 \times X_2} \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, \chi_{r,E,X_1 \times X_2}) \, \mathrm{d}M_2$, and

(ii) $\int \chi_{r,E,X_1 \times X_2} \restriction E \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, \chi_{r,E,X_1 \times X_2} \restriction E) \, \mathrm{d}M_2$, and

(iii) $\int \chi_{r,E,X_1 \times X_2} \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, \chi_{r,E,X_1 \times X_2}) \, \mathrm{d}M_1$, and

(iv) $\int \chi_{r,E,X_1 \times X_2} \restriction E \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, \chi_{r,E,X_1 \times X_2} \restriction E) \, \mathrm{d}M_1$.

The theorem is a consequence of (1), (12), (64), (82), (78), (81), and (65).

(84) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element A of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is non-negative or non-positive and $A = \mathrm{dom}\, f$ and f is measurable on A. Then

(i) $\int f \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, f) \, \mathrm{d}M_2$, and

(ii) $\int f \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, f) \, \mathrm{d}M_1$.

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Sequences of Prime Reciprocals. Preliminaries

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Summary. In the article we formalize some properties needed to prove that sequences of prime reciprocals are divergent. The aim is to show that the series exhibits log-log growth. We introduce some auxiliary notions as harmonic numbers, telescoping series, and prove some standard properties of logarithms and exponents absent in the Mizar Mathematical Library. At the end we proceed with square-free and square-containing parts of a natural number and reciprocals of corresponding products.

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0. Introduction

The aim of this article is to provide preliminaries needed to prove that sequences of reciprocals of prime numbers are divergent. One of the proofs (which we follow) relies on the unique decomposition of natural numbers into the square and its square-free part (similarly to Euler's 1737 original proof [4]). Essentially, it is the proof that the series exhibits log-log growth.

We start with preliminary lemmas, mainly on integrals. Section 2 introduces the notion of n-th harmonic number,

$$H_n = \sum_{i=0}^{n} \frac{1}{i},$$

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as well as its basic properties.

We proved main steps of the proof that prime harmonic series diverges:

- the lower estimate $1+x < \exp(x)$ for the exponential function (13), which holds for all $x > 0$,
- $\int_n^{n+1} \frac{1}{x} dx < \frac{1}{n}$ (21),
- $\ln(n+1) < H_n$ (22),
- the formula for telescoping sum (24).

Although the solution of the Basel problem states [11] that

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6},$$

we proved rough upper bound, using a telescoping sum, namely, for $n \geqslant 2$,

$$\sum_{k=1}^{n} \frac{1}{k^2} < \frac{5}{3}.$$

Evidently however, as in the Mizar Mathematical Library there are at least four strategies for counting the (finite) sum (and the product, similarly): to treat counted objects as

- elements of certain finite subsets;
- values of finite sequences (which is probably most frequent in MML [1], but needs recounting after concantenation or deletion of elements);
- values of the type `Real_Sequence` which are functions from $\mathbb{N}$ into $\mathbb{R}$ (this could be generalized into partial functions and is obviously more general than the previous one);
- bags – objects quite well developed during formalization of polynomials, in terms of bags the fundamental theorem of arithmetic is expressed in Mizar;

there is a need to propose unified approach which will be suggested to use, even if the differences between all of them are purely technical. This probably needs some revisions [6] resulting in removing existing duplications [5] of the Mizar repository. Section 6 discusses some of the details, and also introduces a Mizar functor generating bag from a given finite subset of primes. Additionally, we define two sequences of reciprocals needed for proper summing (and multiplying) later on.

The formula we still definitely need to show is

$$\sum_{i=1}^{n} \frac{1}{i} \leqslant \prod_{p \leqslant n} (1 + \frac{1}{p}) \sum_{k=1}^{n} \frac{1}{k^2}.$$

The product in the formula above corresponds to the square-free part of i and the sum corresponds to the square part of i.

In order to simplify the operations on $\mathbb{P}_n$ (introduced in Section 4 as the set of all primes less or equal n for arbitrary $n \in \mathbb{N}$), we conclude our article with selected properties of reciprocals of products of sets of prime numbers.

1. PRELIMINARIES

From now on n, i, k, m denote natural numbers and p denotes a prime number.

One can check that there exists a natural number which is non zero, square, and non trivial.

Let Z be a subset of $\mathbb{R}$, f be a partial function from Z to $\mathbb{R}$, and A be a subset of $\mathbb{R}$. Let us note that $f \restriction A$ is A-defined as a partial function from Z to $\mathbb{R}$.

Let us consider a subset Z of $\mathbb{R}$. Now we state the propositions:

(1) If $0 \in Z$, then $(\mathrm{id}_Z)^{-1}(\{0\}) = \{0\}$.

(2) If $0 \notin Z$, then $(\mathrm{id}_Z)^{-1}(\{0\}) = \emptyset$.

PROOF: $(\mathrm{id}_Z)^{-1}(\{0\}) \subseteq \emptyset$. □

Let us consider an open subset Z of $\mathbb{R}$ and a non empty, closed interval subset A of $\mathbb{R}$. Now we state the propositions:

(3) If $0 \notin Z$ and $A \subseteq Z$, then $\frac{1}{\mathrm{id}_Z} \restriction A$ is continuous.

(4) Suppose $Z =]0, +\infty[$ and $A = [1, n+1]$. Then $\int\limits_A \frac{1}{\mathrm{id}_Z}(x)dx =$ (the function ln)$(n+1)$. The theorem is a consequence of (2) and (3).

(5) Suppose $Z =]0, +\infty[$ and $0 < n$ and $A = [n, n+1]$. Then $\int\limits_A \frac{1}{\mathrm{id}_Z}(x)dx =$ (the function ln)$(\frac{n+1}{n})$. The theorem is a consequence of (2) and (3).

(6) Let us consider real numbers x, r. Suppose $x > 0$ and $r > 0$. Then Maclaurin(the function exp, $]-r, r[$, x) is positive yielding.

PROOF: Set $f =$ Maclaurin(the function exp, $]-r, r[$, x) by [10, (8)]. For every real number r such that $r \in \mathrm{rng}\, f$ holds $0 < r$. □

(7) Let us consider a summable sequence f of real numbers, and a natural number n. If f is positive yielding, then $\sum(f \uparrow (n+1)) > 0$.

PROOF: Set $L = f \uparrow (n+1)$. For every natural number i, $0 \leqslant L(i)$. There exists a natural number i such that $i \in \mathrm{dom}\, L$ and $0 < L(i)$. Consider k being a natural number such that $k \in \mathrm{dom}\, L$ and $L(k) > 0$. □

2. Harmonic Numbers

Let n be a natural number. The functor H_n yielding a real number is defined by the term

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(\mathrm{inv}_{\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Now we state the propositions:

(8) $H_0 = 0$.

(9) $H_{n+1} = H_n + \frac{1}{n+1}$.

(10) $H_1 = 1$. The theorem is a consequence of (9) and (8).

(11) $H_2 = \frac{3}{2}$. The theorem is a consequence of (9) and (10).

3. On Exponents and Logarithms

Now we state the proposition:

(12) (The function ln)(1) $= 0$.

Let us consider a real number x. Now we state the propositions:

(13) If $x > 0$, then (the function exp)$(x) > x + 1$. The theorem is a consequence of (6) and (7).

(14) If $x > 0$, then (the function ln)$(x + 1) < x$.

(15) Let us consider a natural number n. If $n > 0$, then (the function ln)$(\frac{n+1}{n}) < \frac{1}{n}$. The theorem is a consequence of (14).

(16) Let us consider a real number x. Then (the function ln)((the function exp)$(x)) = x$.

(17) Let us consider real numbers x, y. Suppose $0 < x < y$. Then (the function ln)$(x) <$ (the function ln)(y).

(18) Let us consider a non zero natural number n. Then (the function ln)$(n+1) > 0$. The theorem is a consequence of (12) and (17).

(19) Let us consider real numbers x, y. Suppose $0 < x$ and $0 < y$. Then (the function ln)$(x \cdot y) =$ (the function ln)$(x) +$ (the function ln)(y).

(20) Let us consider a real number x. Then there exists a non zero natural number y such that $x <$ (the function ln)((the function ln)$(y + 1)$). The theorem is a consequence of (17) and (16).

(21) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, an open subset Z of $\mathbb{R}$, and a non zero natural number n. Suppose $Z =]0, +\infty[$ and $A = [n, n + 1]$. Then $\int\limits_A \frac{1}{\mathrm{id}_Z}(x)dx < \frac{1}{n}$. The theorem is a consequence of (2), (3), and (15).

(22) Let us consider a non zero natural number n. Then (the function ln)$(n+1) < H_n$.

Proof: Set $A = [1, n+1]$. Reconsider $Z =]0, +\infty[$ as an open subset of $\mathbb{R}$. $A \subseteq Z$. $\frac{1}{\mathrm{id}_Z} \restriction A$ is continuous. Set $g = \frac{1}{\mathrm{id}_Z}$. Define $\mathcal{P}[\text{natural number}] \equiv \int_{1}^{\$_1+1} g(x)dx < H_{\$_1}$. $\int_{A_1} g(x)dx < \frac{1}{1}$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (11), (10)], [3, (17)]. For every non zero natural number n, $\mathcal{P}[n]$. □

(23) Let us consider natural numbers n_1, n_2. If $n_1{}^{\mathbf{2}} = n_2{}^{\mathbf{2}}$, then $n_1 = n_2$.

Let n be a non trivial natural number. Let us note that $n^{\mathbf{2}}$ is non trivial.

(24) Telescoping series:

Let us consider sequences a, b, s of real numbers. Suppose for every natural number n, $s(n) = a(n) + b(n)$ and for every natural number k, $b(k) = -a(k+1)$. Let us consider a natural number n. Then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = a(0) + b(n)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1) = a(0) + b(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. □

(25) Let us consider sequences f_1, f_2 of real numbers, and a non trivial natural number n. Suppose for every non trivial natural number k such that $k \leqslant n$ holds $f_1(k) < f_2(k)$. Then $\sum_{\kappa=1+1}^{n} f_1(\kappa) < \sum_{\kappa=1+1}^{n} f_2(\kappa)$.

Proof: Define $\mathcal{X}[\text{natural number}] \equiv$ if for every non trivial natural number k such that $k \leqslant \$_1$ holds $f_1(k) < f_2(k)$, then $(\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}(\$_1) - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}(1) < (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}(\$_1) - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}(1)$. For every non trivial natural number n such that $\mathcal{X}[n]$ holds $\mathcal{X}[n+1]$. For every non trivial natural number n, $\mathcal{X}[n]$. □

4. Some Special Sequences

The functor Reci-seq1 yielding a sequence of real numbers is defined by

(Def. 2) for every natural number n, $it(n) = \frac{1}{n^{\mathbf{2}} - \frac{1}{4}}$.

Now we state the propositions:

(26) Let us consider a natural number n. Then $(\text{Reci-seq1})(n) = \frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}}$.

(27) Reci-seq1 $= \text{rseq}(0, 1, 1, -\frac{1}{2}) + -\text{rseq}(0, 1, 1, \frac{1}{2})$.

Let us consider a natural number n.

(28) $(\sum_{\alpha=0}^{\kappa}(\text{Reci-seq1})(\alpha))_{\kappa \in \mathbb{N}}(n) < -2$. The theorem is a consequence of (24).

(29) $\sum_{\kappa=1+1}^{n} \text{Reci-seq1}(\kappa) < \frac{2}{3}$. The theorem is a consequence of (24).

Note that Basel-seq is summable.

(30) Let us consider a natural number n. Then $(\sum_{\alpha=0}^{\kappa}(\text{Reci-seq1})(\alpha))_{\kappa\in\mathbb{N}}(n) = -2 + -\frac{1}{n+\frac{1}{2}}$. The theorem is a consequence of (24).

Let us consider a non trivial natural number n.

(31) $\sum_{\kappa=1+1}^{n} \text{Basel-seq}(\kappa) < \sum_{\kappa=1+1}^{n} \text{Reci-seq1}(\kappa)$.
PROOF: For every non trivial natural number k such that $k \leqslant n$ holds $(\text{Basel-seq})(k) < (\text{Reci-seq1})(k)$ by [9, (29)]. □

(32) $\sum_{\kappa=0}^{n} \text{Basel-seq}(\kappa) < \frac{5}{3}$. The theorem is a consequence of (31) and (29).

(33) $(\sum_{\alpha=0}^{\kappa}(\text{Basel-seq})(\alpha))_{\kappa\in\mathbb{N}}(n) < \frac{5}{3}$. The theorem is a consequence of (32).

The functor Reci-seq2 yielding a sequence of real numbers is defined by

(Def. 3) for every natural number n, $it(n) = 1 + \frac{1}{\text{pr}(n)}$.

Now we state the proposition:

(34) $\sum \text{Sgm}\{1\} = 1$.

Let n be a natural number. The functor $\mathbb{P}_n$ yielding a subset of $\mathbb{N}$ is defined by the term

(Def. 4) $\mathbb{P} \cap \text{Seg}\, n$.

One can verify that $\mathbb{P}_n$ is finite.

Now we state the propositions:

(35) Let us consider natural numbers m, n. If $m \leqslant n$, then $\mathbb{P}_m \subseteq \mathbb{P}_n$.

(36) If $n+1$ is not a prime number, then $\mathbb{P}_{n+1} = \mathbb{P}_n$.

(37) (i) $\mathbb{P}_0 = \emptyset$, and

(ii) $\mathbb{P}_1 = \emptyset$.

The theorem is a consequence of (36).

(38) If $n+1$ is a prime number, then $\mathbb{P}_{n+1} = \mathbb{P}_n \cup \{n+1\}$.

(39) Let us consider a prime number p. If $p > 2$, then $p+1$ is not a prime number.

(40) $\mathbb{P}_2 = \{2\}$.

(41) $n+1 \notin \mathbb{P}_n$.

Let n be a natural number. The functor indexp(n) yielding a natural number is defined by the term

(Def. 5) $\overline{\overline{\mathbb{P}_n}}$.

Now we state the proposition:

(42) Let us consider a natural number n. Then $\text{indexp}(n) \leqslant n$.

5. Square-free and Square-containing Parts of a Natural Number

Let us consider a non zero natural number n. Now we state the propositions:

(43) $n = (\mathrm{TSqF}\, n) \cdot (n \operatorname{div} \mathrm{TSqF}\, n)$.

(44) $(\mathrm{SqF}\, n)^2 \mid n$.

PROOF: Define $\mathcal{F}(\text{non zero natural number}) = (\prod \mathrm{SqFactors}\,\$_1)^2$. Define $\mathcal{G}(\text{non zero natural number}) = \mathrm{SqFactors}\,\$_1$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every non zero natural number n such that support $\mathcal{G}(n) \subseteq \mathrm{Seg}\,\$_1$ holds $\mathcal{F}(n) \mid n$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$ by [8, (20)]. For every natural number k, $\mathcal{P}[k]$. □

(45) Let us consider a finite-support, natural-valued many sorted set m indexed by $\mathbb{P}$, and a prime number p. If support $m = \{p\}$, then $\prod m = m(p)$.

PROOF: Consider f being a finite sequence of elements of $\mathbb{C}$ such that $\prod m = \prod f$ and $f = m \cdot \mathrm{CFS}(\text{support}\, m)$. $m \cdot \langle p \rangle = \langle m(p) \rangle$. □

(46) Let us consider a non zero natural number n. Then $(\mathrm{SqF}\, n)^2 = \mathrm{TSqF}\, n$.

PROOF: Define $\mathcal{F}(\text{non zero natural number}) = (\prod \mathrm{SqFactors}\,\$_1)^2$. Define $\mathcal{G}(\text{non zero natural number}) = \mathrm{SqFactors}\,\$_1$. Define $\mathcal{H}(\text{non zero natural number}) = \prod \mathrm{TSqFactors}\,\$_1$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every non zero natural number n such that support $\mathcal{G}(n) \subseteq \mathrm{Seg}\,\$_1$ holds $\mathcal{F}(n) = \mathcal{H}(n)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. □

Let n be a non zero natural number. Note that $n \operatorname{div} (\mathrm{SqF}\, n)^2$ is square-free as a natural number.

The functor SquarefreePart(n) yielding a non zero natural number is defined by the term

(Def. 6) $n \operatorname{div} \mathrm{TSqF}\, n$.

Let us observe that SquarefreePart(n) is square-free.

Let us consider a non zero natural number n. Now we state the propositions:

(47) $n = \mathrm{SquarefreePart}(n) \cdot (\mathrm{SqF}\, n)^{\mathbf{2}}$. The theorem is a consequence of (44) and (46).

(48) support $\mathrm{PFExp}(n) \subseteq \mathrm{Seg}\, n$.

(49) support $\mathrm{PPF}(n) \subseteq \mathrm{Seg}\, n$.

(50) $\mathrm{Seg}\, \mathrm{SquarefreePart}(n) \subseteq \mathrm{Seg}\, n$. The theorem is a consequence of (47).

(51) Let us consider non zero natural numbers k, n. Then $k^{\mathbf{2}} \mid \mathrm{SquarefreePart}(n)$ if and only if $k = 1$.

(52) Let us consider non zero natural numbers m, n. Suppose $\mathrm{SquarefreePart}(n) = \mathrm{SquarefreePart}(m)$ and $\mathrm{TSqF}\, m = \mathrm{TSqF}\, n$. Then $m = n$. The theorem is a consequence of (47) and (46).

6. Generating Bags from Subsets of Prime Numbers

Let A be a finite subset of $\mathbb{P}$. The functor A-bag yielding a bag of $\mathbb{P}$ is defined by the term

(Def. 7) $\mathrm{EmptyBag}\,\mathbb{P} {+\cdot} \mathrm{id}_A$.

Let us consider a finite subset A of $\mathbb{P}$. Now we state the propositions:

(53) support A-bag $= A$.
Proof: Set $f = A$-bag. support $f \subseteq A$. $A \subseteq$ support f. □

(54) If $A = \emptyset$, then A-bag $=$ EmptyBag $\mathbb{P}$. The theorem is a consequence of (53).

(55) Let us consider a finite subset A of $\mathbb{P}$, and an object i. If $i \in$ support A-bag, then $(A\text{-bag})(i) = i$. The theorem is a consequence of (53).

(56) Let us consider finite subsets A, B of $\mathbb{P}$. If A-bag $= B$-bag, then $A = B$. The theorem is a consequence of (53).

Let A be a finite subset of $\mathbb{P}$. Let us observe that A-bag is prime-factorization-like and $\prod A$-bag is square-free as a natural number.

Let us consider a non zero natural number n and an object x. Now we state the propositions:

(57) If $x \in 2^{\mathbb{P}_n}$, then x is a finite subset of $\mathbb{P}$.

(58) If $x \in 2^{\mathbb{P}_n} \times \mathrm{Seg}\, n$, then $(x)_{\mathbf{1}}$ is a finite subset of $\mathbb{P}$.

(59) $\mathrm{rseq}(0, 1, 1, 0) = \mathrm{inv}_{\mathbb{N}}$.

(60) $\mathrm{indexp}(0) = 0$.

Let us consider a natural number n.

(61) (The partial product of Reci-seq2)$(n) > 0$.

(62) (The function ln)((the partial product of Reci-seq2)$(n)) \leqslant (\sum_{\alpha=0}^{\kappa} (\mathrm{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(n)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ (the function ln)((the partial product of Reci-seq2)$(\$_1)) \leqslant (\sum_{\alpha=0}^{\kappa} (\mathrm{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. □

(63) (The function ln)((the partial product of Reci-seq2)(indexp$(n))) \leqslant (\sum_{\alpha=0}^{\kappa} (\mathrm{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(n)$. The theorem is a consequence of (62) and (42).

The functors: Reci-Sqf and Reci-TSq yielding sequences of real numbers are defined by conditions

(Def. 8) Reci-Sqf$(0) = 0$ and for every non zero natural number i, Reci-Sqf$(i) = \frac{1}{\mathrm{SquarefreePart}(i)}$, (Def. 9) Reci-TSq(0) = 0 and for every non zero natural number i, Reci-TSq(i) = $\frac{1}{\mathrm{TSqF}\, i}$,

respectively. Now we state the proposition:

(64) rseq(0, 1, 1, 0) = Reci-Sqf · Reci-TSq.

From now on s, s_1, s_2 denote sequences of real numbers.

Let us consider a natural number n.

(65) (Reci-Sqf)(n) $\geqslant 0$.

(66) (Reci-TSq)(n) $\geqslant 0$.

(67) (Basel-seq)(n) $\geqslant 0$.

(68) $(\sum_{\alpha=0}^{\kappa}(\mathrm{rseq}(0,1,1,0))(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant (\sum_{\alpha=0}^{\kappa}(\text{Reci-Sqf})(\alpha))_{\kappa\in\mathbb{N}}(n) \cdot (\sum_{\alpha=0}^{\kappa}(\text{Reci-TSq})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Let n be a non zero natural number. The functor Compose(n) yielding a function from $2^{\mathbb{P}_n} \times \operatorname{Seg} n$ into $\mathbb{N}$ is defined by

(Def. 10) for every element x of $2^{\mathbb{P}_n} \times \operatorname{Seg} n$ and for every finite subset A of $\mathbb{P}$ and for every natural number k such that $x = \langle A, k\rangle$ holds $it(x) = \prod(A, 1)\text{-bag} \cdot k^{\mathbf{2}}$.

Now we state the proposition:

(69) $(\sum_{\alpha=0}^{\kappa}(\text{Basel-seq})(\alpha))_{\kappa\in\mathbb{N}}(n) \geqslant 0$.

Proof: For every natural number n, (Basel-seq)(n) $\geqslant 0$ by [7, (31)]. □

7. On Reciprocals of Products of Prime Numbers

Let n be a natural number. The functor ReciProducts(n) yielding a subset of $\mathbb{R}$ is defined by the term

(Def. 11) the set of all $\frac{1}{\prod \operatorname{Sgm} X}$ where X is a subset of $\mathbb{P}_n$.

Let us note that ReciProducts(n) is finite.

Now we state the propositions:

(70) ReciProducts(0) = {1}.

Proof: the set of all $\frac{1}{\prod \operatorname{Sgm} X}$ where X is a subset of $\mathbb{P}_0 = \{1\}$. □

(71) Let us consider a prime number p. If $p > 2$, then ReciProducts($p+1$) = ReciProducts(p). The theorem is a consequence of (39) and (36).

(72) Let us consider a natural number p. Suppose $p+1$ is not a prime number. Then ReciProducts($p+1$) = ReciProducts(p). The theorem is a consequence of (36).

(73) ReciProducts(1) = {1}.

Proof: the set of all $\frac{1}{\prod \operatorname{Sgm} X}$ where X is a subset of $\mathbb{P}_1 = \{1\}$. □

(74) $\text{ReciProducts}(2) = \{\frac{1}{2}, 1\}$.

PROOF: $\{2\} \subseteq \mathbb{P}_2$. $\text{ReciProducts}(2) \subseteq \{\frac{1}{2}, 1\}$. $\square$

Let us consider a natural number n.

(75) $\text{ReciProducts}(n) \subseteq \text{ReciProducts}(n+1)$.

(76) Suppose $n+1$ is a prime number. Then $\text{ReciProducts}(n+1) = \text{ReciProducts}(n) \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\}$.

PROOF: $\text{ReciProducts}(n+1) \subseteq \text{ReciProducts}(n) \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\}$. $\text{ReciProducts}(n) \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\} \subseteq \text{ReciProducts}(n+1)$. $\square$

(77) Suppose $n+1$ is a prime number. Then $\text{ReciProducts}(n+1) = \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_n : n+1 \notin X\} \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\}$.

PROOF: $\text{ReciProducts}(n+1) \subseteq \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_n : n+1 \notin X\} \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\}$. $\{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_n : n+1 \notin X\} \cup \{\frac{1}{\prod \text{Sgm}\, X}$, where X is a subset of $\mathbb{P}_{n+1} : n+1 \in X\} \subseteq \text{ReciProducts}(n+1)$. $\square$

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Diophantine sets. Preliminaries[1]

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Summary. In this article, we define Diophantine sets using the Mizar formalism. We focus on selected properties of multivariate polynomials, i.e., functions of several variables to show finally that the class of Diophantine sets is closed with respect to the operations of union and intersection.

This article is the next in a series [1], [5] aiming to formalize the proof of Matiyasevich's negative solution of Hilbert's tenth problem.

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0. Introduction

Multivariate polynomials are often interpreted in informal mathematical practice as a finite sum of terms with each term being a product of a nonzero coefficient $c \in \mathfrak{F} \setminus \{\mathbf{0}\}$ and a monomial $x_1^{e_1} \cdot x_2^{e_2} \cdot \ldots \cdot x_n^{e_n}$ determined by an exponent vector $\langle e_1, e_2, \ldots, e_n \rangle \in \mathbb{N}^n$ where n is a given natural number.

Formal interpretation of multivariate polynomials developed in the Mizar Mathematical Library [4] can be considered as a generalization of the informal approach, where the natural number n is replaced by an ordinal number λ. Additionally, to avoid problems that occur when multiplying an infinite number of nonzero factors, each exponent vector $e : \lambda \mapsto \mathbb{N}$ has only a finite number of nonzero coordinates. Such exponent vectors are called *bags* of λ and the set of all

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bags for a given λ is denoted by *Bags* λ. It is important to note that for a finite λ, each bag b corresponds to a finite sequence (with 0-based numbering). Using bags, multivariate polynomials have been defined in [6] as functions that assign a coefficient (an element of $\mathfrak{F}$) to each bag of λ and are zero almost everywhere. Moreover, the evaluation for a multivariate polynomial p and vector $x : \lambda \mapsto \mathfrak{F}$ has been defined as

$$\mathrm{eval}(p, x) = \sum_{b \in Bags\,\lambda} p(b) \ \cdot \ \prod_{i \in \lambda} x(i)^{b(i)}. \tag{0.1}$$

Based on this approach we define Diophantine sets in Def. 6 as follows. Let us consider a natural number n that plays the role of the dimension and a subset D of all finite sequences of length n numbered from 0 (see Def. 5). We call D *Diophantine* if there exist a natural number k and a $n+k$–variable polynomial p such that each coefficient is an integer number and

$$\forall_{x:n \mapsto \mathbb{N}}\, x \in D \iff \exists_{y:k \mapsto \mathbb{N}}\, \mathrm{eval}(p, x^\frown y) = 0. \tag{0.2}$$

The main aim of our article is to show that the union and intersection of two n–dimension Diophantine sets D_1, D_2 is also Diophantine. The informal proof of these facts as presented by Z. Adamowicz and P. Zbierski in [2] or C. Smorynski [7] is quite obvious. Suppose that p_i is $n + k_i$–variable polynomial which determines D_i for $i = 1, 2$. Then $p'_1 \cdot p'_2$ determines $D_1 \cap D_2$ and $(p'_1)^2 + (p'_2)^2$ determines $D_1 \cup D_2$ where p'_i is the $n + k_1 + k_2$–variable polynomial obtained from p_i by modifying the order of variables and adding insignificant variables. The property of p'_1, p'_2, used in [2] can be formally formulated as follows:

$$\mathrm{eval}(p'_1, x^\frown y_1{}^\frown y_2) = \mathrm{eval}(p_1, x^\frown y_1) \ \wedge \ \mathrm{eval}(p'_2, x^\frown y_1{}^\frown y_2) = \mathrm{eval}(p_2, x^\frown y_2) \tag{0.3}$$

for arbitrary $x : n \mapsto \mathbb{N}$, $y_1 : k_1 \mapsto \mathbb{N}$, $y_2 : k_2 \mapsto \mathbb{N}$. The existence of such polynomials have been showed in Th. 27, 28. The construction of these polynomials is useful for the further development of multivariate polynomials in the Mizar Mathematical Library. Therefore we define and provide basic properties of two transformations that

- add an additional variable to the polynomial, preserving its value, i.e.

$$\forall_{x:n \mapsto \mathbb{N}, a \in \mathbb{N}}\, \mathrm{eval}(p \text{ extended by } 0, x^\frown \langle a \rangle) = \mathrm{eval}(p, x), \tag{0.4}$$

- permute the order of variables, preserving its value, i.e.

$$\forall_{x:n \mapsto \mathbb{N}}\, \mathrm{eval}(p \text{ permuted by } \sigma, x) = \mathrm{eval}(p, x \cdot \sigma^{-1}). \tag{0.5}$$

1. Preliminaries

From now on i, j, k, n, m denote natural numbers and b, b_1, b_2 denote bags of n.

Let X be a non empty set and n be a natural number. Note that there exists a finite 0-sequence of X which is n-element and there exists a finite 0-sequence which is n-element and real-valued.

Let n, m be natural numbers, p be an n-element finite 0-sequence, and q be an m-element finite 0-sequence. One can check that $p \frown q$ is $(n+m)$-element.

Let p be a real-valued finite 0-sequence and q be a real-valued finite 0-sequence. Let us observe that $p \frown q$ is real-valued.

Let n be a natural number and p be an n-element, real-valued finite 0-sequence. The functor ${}^{@}p$ yielding a function from n into $\mathbb{R}_{\rm F}$ is defined by the term

(Def. 1) p.

Let X be a non empty set and p be a function from n into X. The functor ${}^{@}p$ yielding an n-element finite 0-sequence of X is defined by the term

(Def. 2) p.

Let X be a set, p be a permutation of X, and M be a many sorted set indexed by X. Observe that $M \cdot p$ is total.

Let F be a finite-support function and f be a one-to-one function. Let us observe that $F \cdot f$ is finite-support.

Now we state the propositions:

(1) Let us consider finite 0-sequences F, G. Suppose $F \frown G$ is one-to-one. Then $\operatorname{rng} F$ misses $\operatorname{rng} G$.

(2) Let us consider a set X, an X-defined function f, and a permutation σ of X. Then $\overline{\overline{\operatorname{support} f \cdot \sigma}} = \overline{\overline{\operatorname{support} f}}$.

Proof: Set $P = \sigma^{-1}$. $P^{\circ}(\operatorname{support} f) \subseteq \operatorname{support} f \cdot \sigma$. $\operatorname{support} f \cdot \sigma \subseteq P^{\circ}(\operatorname{support} f)$. □

Let X be a set. Observe that $0_X(\mathbb{R}_{\rm F})$ is natural-valued and $1\_(X, \mathbb{R}_{\rm F})$ is natural-valued.

Let x be an element of X. Note that $1\_1(x, \mathbb{R}_{\rm F})$ is natural-valued and there exists a series of X, $\mathbb{R}_{\rm F}$ which is $\mathbb{Z}$-valued.

Let O be an ordinal number. Let us note that there exists a polynomial of O,$\mathbb{R}_{\rm F}$ which is $\mathbb{Z}$-valued.

Let X be a set and p be a $\mathbb{Z}$-valued series of X, $\mathbb{R}_{\rm F}$. One can check that $-p$ is $\mathbb{Z}$-valued.

Let q be a $\mathbb{Z}$-valued series of X, $\mathbb{R}_{\rm F}$. Observe that $p+q$ is $\mathbb{Z}$-valued and $p-q$ is $\mathbb{Z}$-valued.

Let X be an ordinal number and p, q be $\mathbb{Z}$-valued series of X, $\mathbb{R}_{\rm F}$. One can verify that $p * q$ is $\mathbb{Z}$-valued.

Let X be a set. Let us note that there exists a function from X into $\mathbb{R}_{\rm F}$ which is natural-valued.

Let O be an ordinal number. One can check that there exists a function from O into $\mathbb{R}_{\rm F}$ which is $\mathbb{Z}$-valued.

Let b be a bag of O and x be a $\mathbb{Z}$-valued function from O into $\mathbb{R}_{\rm F}$. Note that $\mathrm{eval}(b, x)$ is integer.

Let p be a $\mathbb{Z}$-valued polynomial of O,$\mathbb{R}_{\rm F}$. One can check that $\mathrm{eval}(p, x)$ is integer.

2. Polynomial Extended by 0

Now we state the propositions:

(3) Let us consider a many sorted set b indexed by n. If $k \leqslant n$, then $\langle b(0), \ldots, b(k) \rangle = b \restriction k$.

PROOF: For every object x such that $x \in k$ holds $\langle b(0), \ldots, b(k) \rangle(x) = (b \restriction k)(x)$. □

(4) Let us consider a bag b of $n+1$. Then $b = \langle b(0), \ldots, b(n) \rangle$ extended by $b(n)$.

PROOF: Set $C = \langle b(0), \ldots, b(n) \rangle$. Set $B = C$ extended by $b(n)$. $C = b \restriction n$. For every object x such that $x \in n + 1$ holds $B(x) = b(x)$ by [8, (2)]. □

(5) $\langle b$ extended by $k(0), \ldots, b$ extended by $k(n) \rangle = b$. The theorem is a consequence of (3).

Let us consider n. Let L be a non empty zero structure and p be a series of n, L. The p extended by 0 yielding a series of $n + 1$, L is defined by

(Def. 3) for every bag b of $n + 1$, if $b(n) \neq 0$, then $it(b) = 0_L$ and if $b(n) = 0$, then $it(b) = p(\langle b(0), \ldots, b(n) \rangle)$.

Now we state the propositions:

(6) Let us consider a non empty zero structure L, and a series p of n, L. Then (the p extended by 0)(b extended by 0) $= p(b)$. The theorem is a consequence of (5).

(7) Let us consider a non empty zero structure L, a series p of n, L, and a bag b of $n+1$. Suppose $b \in$ Support(the p extended by 0). Then $b(n) = 0$.

(8) Let us consider a non empty zero structure L, and a series p of n, L. Then b extended by $0 \in$ Support(the p extended by 0) if and only if $b \in$ Support p. The theorem is a consequence of (5).

(9) Let us consider a non empty zero structure L, a series p of n, L, and a bag b of $n+1$. Suppose $b(n) = 0$. Then $b \in$ Support(the p extended by 0)

if and only if $\langle b(0), \ldots, b(n)\rangle \in \operatorname{Support} p$. The theorem is a consequence of (4) and (8).

Let us consider n. Let L be a non empty zero structure and p be a polynomial of n,L. Let us observe that the p extended by 0 is finite-Support.

Now we state the propositions:

(10) Let us consider a non empty zero structure L, and a series p of n, L. Then $\{0_L\} \cup \operatorname{rng} p = \operatorname{rng}(\text{the } p \text{ extended by } 0)$. The theorem is a consequence of (6).

(11) $\operatorname{support} b = \operatorname{support}(b \text{ extended by } 0)$.
PROOF: Set $E = b$ extended by 0. $\operatorname{support} b \subseteq \operatorname{support} E$. □

(12) $\operatorname{SgmX}(\subseteq_n, \operatorname{support} b) = \operatorname{SgmX}(\subseteq_{n+1}, \operatorname{support}(b \text{ extended by } 0))$. The theorem is a consequence of (11).

(13) Let us consider a well unital, non trivial double loop structure L, a function x from n into L, and a function y from $n+1$ into L. Suppose $y \restriction n = x$. Then $\operatorname{eval}(b, x) = \operatorname{eval}(b \text{ extended by } 0, y)$.
PROOF: Set $S = \operatorname{SgmX}(\subseteq_n, \operatorname{support} b)$. Set $B = b$ extended by 0. Set $S_1 = \operatorname{SgmX}(\subseteq_{n+1}, \operatorname{support} B)$. Consider P being a finite sequence of elements of L such that $\operatorname{len} P = \operatorname{len} S$ and $\operatorname{eval}(b, x) = \prod P$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} P$ holds $P_{/i} = \operatorname{power}_L(x \cdot S_{/i}, b \cdot S_{/i})$. Consider P_1 being a finite sequence of elements of L such that $\operatorname{len} P_1 = \operatorname{len} S_1$ and $\operatorname{eval}(B, y) = \prod P_1$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} P_1$ holds $P_{1/i} = \operatorname{power}_L(y \cdot S_{1/i}, B \cdot S_{1/i})$. $S = S_1$. $b = \langle B(0), \ldots, B(n)\rangle$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} P$ holds $P(i) = P_1(i)$. □

(14) $b_1 < b_2$ if and only if b_1 extended by $k < b_2$ extended by k.
PROOF: Set $B_1 = b_1$ extended by k. Set $B_2 = b_2$ extended by k. If $b_1 < b_2$, then b_1 extended by $k < b_2$ extended by k. Consider o being an ordinal number such that $B_1(o) < B_2(o)$ and for every ordinal number l such that $l \in o$ holds $B_1(l) = B_2(l)$. For every ordinal number l such that $l \in o$ holds $b_1(l) = b_2(l)$. □

(15) Let us consider a non empty set X, a finite subset A of X, and an order R in X. Suppose R linearly orders A. Suppose $1 \leqslant i \leqslant k \leqslant \overline{\overline{A}}$. Then $(\operatorname{SgmX}(R, \operatorname{rng}(\operatorname{SgmX}(R, A) \restriction k)))_{/i} = (\operatorname{SgmX}(R, A))_{/i}$.
PROOF: Set $S_1 = \operatorname{SgmX}(R, A)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every i such that $1 \leqslant i \leqslant \$_1 \leqslant \overline{\overline{A}}$ holds $(\operatorname{SgmX}(R, \operatorname{rng}(S_1 \restriction \$_1)))_{/i} = S_{1/i}$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (83)]. For every k, $\mathcal{P}[k]$. □

(16) Let us consider an ordinal number O, and a finite subset A of $\operatorname{Bags} O$. Suppose n, $m \in \operatorname{dom} \operatorname{SgmX}(\operatorname{BagOrder} O, A)$ and $n < m$.
Then $(\operatorname{SgmX}(\operatorname{BagOrder} O, A))_{/n} < (\operatorname{SgmX}(\operatorname{BagOrder} O, A))_{/m}$.

(17) Let us consider a right zeroed, add-associative, right complementable, well unital, distributive, non trivial double loop structure L, and a polynomial p of n,L. Then

(i) $\operatorname{len}\operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p) =$ $\operatorname{len}\operatorname{SgmX}(\operatorname{BagOrder}(n+1), \operatorname{Support}(\text{the } p \text{ extended by } 0))$, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len}\operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p)$ holds $(\operatorname{SgmX}(\operatorname{BagOrder}(n+1), \operatorname{Support}(\text{the } p \text{ extended by } 0)))_{/i} =$ $(\operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p))_{/i}$ extended by 0.

Proof: Set $B = \operatorname{BagOrder} n$. Set $B_1 = \operatorname{BagOrder}(n+1)$. Set $P =$ the p extended by 0. Define $\mathcal{F}(\text{bag of } n) = \$_1$ extended by 0. Consider f being a function from $\operatorname{Bags} n$ into $\operatorname{Bags}(n+1)$ such that for every element x of $\operatorname{Bags} n$, $f(x) = \mathcal{F}(x)$. Set $F = f \restriction \operatorname{Support} p$. Set $S_1 = \operatorname{SgmX}(B, \operatorname{Support} p)$. Set $S_2 = \operatorname{SgmX}(B_1, \operatorname{Support} P)$. $\operatorname{rng} F \subseteq \operatorname{Support} P$. $\operatorname{Support} P \subseteq \operatorname{rng} F$. F is one-to-one. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} S_1$, then for every i such that $1 \leqslant i \leqslant \$_1$ holds $S_{2/i} = S_{1/i}$ extended by0. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. $\square$

(18) Let us consider a right zeroed, add-associative, right complementable, well unital, distributive, non trivial double loop structure L, a polynomial p of n,L, a function x from n into L, and a function y from $n+1$ into L. Suppose $y \restriction n = x$. Then $\operatorname{eval}(p, x) = \operatorname{eval}(\text{the } p \text{ extended by } 0, y)$.

Proof: Set $n_1 = n + 1$. Set $S = \operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p)$. Set $P =$ the p extended by 0. Set $S_1 = \operatorname{SgmX}(\operatorname{BagOrder} n_1, \operatorname{Support} P)$. Consider T being a finite sequence of elements of L such that $\operatorname{len} T = \operatorname{len} S$ and $\operatorname{eval}(p, x) = \sum T$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} T$ holds $T_{/i} = p \cdot S_{/i} \cdot \operatorname{eval}(S_{/i}, x)$. Consider T_1 being a finite sequence of elements of L such that $\operatorname{len} T_1 = \operatorname{len} S_1$ and $\operatorname{eval}(P, y) = \sum T_1$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} T_1$ holds $T_{1/i} = P \cdot S_{1/i} \cdot \operatorname{eval}(S_{1/i}, y)$. $\operatorname{len} S = \operatorname{len} S_1$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} S$ holds $S_{1/i} = S_{/i}$ extended by 0. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} S$ holds $T(i) = T_1(i)$. $\square$

3. Polynomial Permuted by Permutation

Now we state the propositions:

(19) Let us consider an ordinal number O, a well unital, commutative, associative, non trivial double loop structure L, a function x from O into L, a bag b of O, and a one-to-one finite sequence S of elements of O. Suppose

rng S = support b. Then there exists a finite sequence y of elements of L such that

(i) $\operatorname{len} y = \overline{\overline{\text{support}\, b}}$, and

(ii) $\operatorname{eval}(b, x) = \prod y$, and

(iii) for every i such that $1 \leqslant i \leqslant \operatorname{len} y$ holds $y_{/i} = \operatorname{power}_L(x \cdot S_{/i}, b \cdot S_{/i})$.

(20) Let us consider an ordinal number O, a well unital, commutative, associative, non trivial double loop structure L, a function x from O into L, a bag b of O, and a permutation σ of O. Then $\operatorname{eval}(b, x) = \operatorname{eval}(b \cdot \sigma, x \cdot \sigma)$.
Proof: Set $S_1 = \operatorname{SgmX}(\subseteq_n, \text{support}\, b)$. Consider y being a finite sequence of elements of L such that $\operatorname{len} y = \operatorname{len} S_1$ and $\operatorname{eval}(b, x) = \prod y$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} y$ holds $y_{/i} = \operatorname{power}_L(x \cdot S_{1/i}, b \cdot S_{1/i})$. Set $P = \sigma^{-1}$. rng $P \cdot S_1 \subseteq$ support $b \cdot \sigma$. support $b \cdot \sigma \subseteq$ rng $P \cdot S_1$. Reconsider $S = P \cdot S_1$ as a one-to-one finite sequence of elements of n. Consider Y being a finite sequence of elements of L such that $\operatorname{len} Y = \overline{\overline{\text{support}\, b \cdot \sigma}}$ and $\operatorname{eval}(b \cdot \sigma, x \cdot \sigma) = \prod Y$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} Y$ holds $Y_{/i} = \operatorname{power}_L(x \cdot \sigma \cdot S_{/i}, b \cdot \sigma \cdot S_{/i})$. $\operatorname{len} Y = \operatorname{len} y$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} Y$ holds $Y(i) = y(i)$. □

Let O be an ordinal number, L be a non empty zero structure, s be a series of O, L, and σ be a permutation of O. The s permuted by σ yielding a series of O, L is defined by

(Def. 4) for every bag b of O, $it(b) = s(b \cdot \sigma)$.

Let us consider an ordinal number O, a non empty zero structure L, a permutation σ of O, a series s of O, L, and a bag b of O. Now we state the propositions:

(21) $b \in$ Support(the s permuted by σ) if and only if $b \cdot \sigma \in$ Support s.

(22) $b \cdot \sigma^{-1} \in$ Support(the s permuted by σ) if and only if $b \in$ Support s.

(23) Let us consider an ordinal number O, a non empty zero structure L, a permutation σ of O, and a series s of O, L. Then $\overline{\overline{\text{Support}\, s}} = \overline{\overline{\text{Support}\, \alpha}}$, where α is the s permuted by σ.
Proof: Set P = the s permuted by σ. Define $\mathcal{R}[\text{bag of } O, \text{bag of } O] \equiv \$_2 = \$_1 \cdot \sigma$. For every element x of Bags O, there exists an element y of Bags O such that $\mathcal{R}[x, y]$. Consider f being a function from Bags O into Bags O such that for every element x of Bags O, $\mathcal{R}[x, f(x)]$. f is one-to-one. f°(Support P) $\subseteq$ Support s. Support $s \subseteq f^\circ$(Support P). □

(24) Let us consider an ordinal number O, an Abelian, right zeroed, add-associative, right complementable, well unital, distributive, non trivial double loop structure L, a polynomial p of O,L, a function x from O into L, and a one-to-one finite sequence S of elements of Bags O. Suppose

rng S = Support p. Then there exists a finite sequence y of elements of L such that

(i) $\operatorname{len} y = \overline{\overline{\text{Support}\, p}}$, and

(ii) $\text{eval}(p, x) = \sum y$, and

(iii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} y$ holds $y_{/i} = p \cdot S_{/i} \cdot \text{eval}(S_{/i}, x)$.

Let O be an ordinal number, L be a non empty zero structure, σ be a permutation of O, and p be a polynomial of O,L. One can check that the p permuted by σ is finite-Support.

(25) Let us consider an ordinal number O, an Abelian, right zeroed, add-associative, right complementable, well unital, distributive, commutative, associative, non trivial double loop structure L, a polynomial p of O,L, a function x from O into L, and a permutation σ of O. Then $\text{eval}(p, x) = \text{eval}(\text{the } p \text{ permuted by } \sigma, x \cdot (\sigma^{-1}))$.

Proof: Set S_2 = SgmX(BagOrder O, Support p). Consider y being a finite sequence of elements of L such that $\operatorname{len} y = \operatorname{len} S_2$ and $\text{eval}(p, x) = \sum y$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} y$ holds $y_{/i} = p \cdot S_{2/i} \cdot \text{eval}(S_{2/i}, x)$. Set P = the p permuted by σ. Define $\mathcal{R}[\text{bag of } O, \text{bag of } O] \equiv \$_2 = \$_1 \cdot \sigma^{-1}$. For every element x of Bags O, there exists an element y of Bags O such that $\mathcal{R}[x, y]$. Consider f being a function from Bags O into Bags O such that for every element x of Bags O, $\mathcal{R}[x, f(x)]$. f is one-to-one. Reconsider $f_1 = f \cdot S_2$ as a one-to-one finite sequence of elements of Bags O. rng $f_1 \subseteq$ Support P. Support $P \subseteq$ rng f_1. Consider z being a finite sequence of elements of L such that $\operatorname{len} z = \overline{\overline{\text{Support}\, P}}$ and $\text{eval}(P, x \cdot \sigma^{-1}) = \sum z$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} z$ holds $z_{/i} = P \cdot f_{1/i} \cdot \text{eval}(f_{1/i}, x \cdot \sigma^{-1})$. $\operatorname{len} y = \operatorname{len} z$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} y$ holds $y(i) = z(i)$. □

(26) Let us consider an ordinal number O, a non empty zero structure L, a series s of O, L, and a permutation σ of O. Then rng(the s permuted by σ) = rng s.

4. Main Lemmas

Now we state the propositions:

(27) Let us consider a right zeroed, add-associative, right complementable, well unital, distributive, non trivial double loop structure L, and a polynomial p of n,L. Then there exists a polynomial q of $n+m$,L such that

(i) $\operatorname{rng} q \subseteq \operatorname{rng} p \cup \{0_L\}$, and

(ii) for every function x from n into L and for every function y from $n + m$ into L such that $y \restriction n = x$ holds $\mathrm{eval}(p, x) = \mathrm{eval}(q, y)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ there exists a polynomial q of $n+\$_1$,$L$ such that $\mathrm{rng}\, q \subseteq \mathrm{rng}\, p \cup \{0_L\}$ and for every function x from n into L and for every function y from $n + \$_1$ into L such that $y \restriction n = x$ holds $\mathrm{eval}(p, x) = \mathrm{eval}(q, y)$. $\mathcal{P}[0]$. If $\mathcal{P}[k]$, then $\mathcal{P}[k + 1]$. $\mathcal{P}[k]$. □

(28) Let us consider an Abelian, right zeroed, add-associative, right complementable, well unital, distributive, commutative, associative, non trivial double loop structure L, and a polynomial p of $n+m$,L. Then there exists a polynomial q of $n + k + m$,L such that

(i) $\mathrm{rng}\, q \subseteq \mathrm{rng}\, p \cup \{0_L\}$, and

(ii) for every function X_1 from $n + m$ into L and for every function X_2 from $n + k + m$ into L such that $X_1 \restriction n = X_2 \restriction n$ and $({}^{@}X_1)_{\lfloor n} = ({}^{@}X_2)_{\lfloor n+k}$ holds $\mathrm{eval}(p, X_1) = \mathrm{eval}(q, X_2)$.

Proof: Consider P being a polynomial of $n + m + k$,L such that $\mathrm{rng}\, P \subseteq \mathrm{rng}\, p \cup \{0_L\}$ and for every function x from $n + m$ into L and for every function y from $n+m+k$ into L such that $y \restriction (n+m) = x$ holds $\mathrm{eval}(p, x) = \mathrm{eval}(P, y)$. Reconsider $P_1 = P$ as a polynomial of $n + k + m$,L. Set $I = \mathrm{id}_{n+k+m}$. Set $n_1 = n + m$. Set $I_2 = I \restriction n_1$. $\mathrm{rng}\, I_2$ misses $\mathrm{rng}\, I_{\lfloor n_1}$. $\mathrm{rng}(I_2 \restriction n)$ misses $\mathrm{rng}\, I_{2\lfloor n}$. Reconsider $I_1 = ((I_2 \restriction n) \frown I_{\lfloor n_1}) \frown I_{2\lfloor n}$ as a function from $n+k+m$ into $n+k+m$. Reconsider $R = {}^{@}X_1 \frown ({}^{@}X_2 \restriction (n+k))_{\lfloor n}$ as a function from $n+m+k$ into L. Reconsider $r = R$ as a function from $n+k+m$ into L. $\mathrm{eval}(P_1, r) = \mathrm{eval}(T, r \cdot I_1)$. For every k such that $k \in \mathrm{dom}\, {}^{@}X_2$ holds $({}^{@}r \cdot I_1)(k) = ({}^{@}X_2)(k)$. □

5. Diophantine Sets

From now on x, s denote objects.

Let D be a non empty set and n be a natural number. The n-xtuples of D yielding a subset of D^{ω} is defined by

(Def. 5) $x \in it$ iff x is an n-element finite 0-sequence of D.

Observe that the n-xtuples of D is non empty and every element of the n-xtuples of D is n-element and D-valued.

Let A be a subset of the n-xtuples of $\mathbb{N}$. We say that A is diophantine if and only if

(Def. 6) there exists a natural number m and there exists a $\mathbb{Z}$-valued polynomial p of $n + m$,$\mathbb{R}_{\mathrm{F}}$ such that for every s, $s \in A$ iff there exists an n-element finite 0-sequence x of $\mathbb{N}$ and there exists an m-element finite 0-sequence y of $\mathbb{N}$ such that $s = x$ and $\mathrm{eval}(p, {}^{@}(x \frown y)) = 0$.

One can verify that every subset of the n-xtuples of $\mathbb{N}$ which is empty is also diophantine and $\Omega_{\text{the } n\text{-xtuples of } \mathbb{N}}$ is diophantine.

Let n be a zero natural number. One can verify that every subset of the n-xtuples of $\mathbb{N}$ is diophantine.

Let n be a natural number. Let us observe that there exists a subset of the n-xtuples of $\mathbb{N}$ which is non empty and diophantine and there exists a subset of the n-xtuples of $\mathbb{N}$ which is empty and diophantine.

Let A, B be diophantine subsets of the n-xtuples of $\mathbb{N}$. One can check that $A \cap B$ is diophantine as a subset of the n-xtuples of $\mathbb{N}$ and $A \cup B$ is diophantine as a subset of the n-xtuples of $\mathbb{N}$.

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