Contents

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Arithmetic Operations on Short Finite Sequences

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Summary. In contrast to other proving systems Mizar Mathematical Library, considered as one of the largest formal mathematical libraries [4], is maintained as a single base of theorems, which allows the users to benefit from earlier formalized items [3], [2]. This eventually leads to a development of certain branches of articles using common notation and ideas. Such formalism for finite sequences has been developed since 1989 [1] and further developed despite of the controversy over indexing which excludes zero [6], also for some advanced and new mathematics [5].

The article aims to add some new machinery for dealing with finite sequences, especially those of short length.

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1. Preliminaries

One can verify that every binary relation which is empty is also positive yielding and every binary relation which is empty is also negative yielding and every binary relation which is natural-valued is also $\mathbb{N}$-valued.

Let f be a complex-valued function and k be an object. Note that $(0 \cdot f)(k)$ reduces to 0.

Let us observe that $1 \cdot f$ reduces to f and $(-1) \cdot (-f)$ reduces to f. One can verify that $0 \cdot f$ is empty yielding and $f - f$ is empty yielding.

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Let D be a set. Observe that there exists a D-valued finite sequence which is empty yielding and every finite sequence which is empty yielding is also $\mathbb{N}$-valued and there exists an empty yielding finite sequence which is non empty.

Let n be a natural number. One can verify that there exists an empty yielding, $\mathbb{N}$-valued finite sequence which is n-element and $\min(n, 0)$ is zero.

One can verify that $\max(n, 0)$ reduces to n.

Let a be a non zero natural number. One can verify that $\min(a, 1)$ reduces to 1 and $\max(a, 1)$ reduces to a.

Let a be a non trivial natural number. One can verify that $\min(a, 2)$ reduces to 2 and $\max(a, 2)$ reduces to a.

Let a be a positive real number and b be a positive natural number. One can verify that $b \mapsto a$ is positive and every binary relation which is empty yielding is also function-like and every function which is empty yielding is also natural-valued and every real-valued function which is empty yielding is also non-positive yielding.

Every real-valued function which is empty yielding is also non-negative yielding and every non empty, real-valued function which is empty yielding is also non positive yielding and every non empty, real-valued function which is empty yielding is also non negative yielding and every non empty, real-valued function which is positive yielding is also non non-positive yielding and every non empty, real-valued function which is negative yielding is also non non-negative yielding.

Let f be an empty yielding function and c be a complex number. Note that $c \cdot f$ is empty yielding.

Let g be a complex-valued function. Note that $f \cdot g$ is empty yielding.

2. The Length of Finite Sequences

Let f be a complex-valued finite sequence and x be a complex number. Note that $f+x$ is $(\operatorname{len} f)$-element and $f-x$ is $(\operatorname{len} f)$-element and $|f|$ is $(\operatorname{len} f)$-element and $-f$ is $(\operatorname{len} f)$-element and f^{-1} is $(\operatorname{len} f)$-element.

Let n, m be natural numbers, f be an n-element, complex-valued finite sequence, and g be an m-element, complex-valued finite sequence. One can verify that $f + g$ is $(\min(n, m))$-element and $f \cdot g$ is $(\min(n, m))$-element and $f - g$ is $(\min(n, m))$-element and f/g is $(\min(n, m))$-element.

Let g be an $(n+m)$-element, empty yielding, complex-valued finite sequence. Observe that $f + g$ reduces to f.

Let n be a natural number and g be an n-element, empty yielding, complex-valued finite sequence. One can verify that $f + g$ reduces to f.

Let X be a non empty set. Observe that there exists an X-defined, empty yielding function which is total.

Let f be a total, X-defined, complex-valued function and g be a total, X-defined, empty yielding function. Let us observe that $f + g$ reduces to f.

Let f be a binary relation. Let us observe that there exists a binary relation which is $(\operatorname{dom} f)$-defined and $f \operatorname{null} f$ is $(\operatorname{dom} f)$-defined and there exists a $(\operatorname{dom} f)$-defined binary relation which is total.

Let f be a complex-valued function. Observe that there exists a $(\operatorname{dom} f)$-defined, empty yielding function which is total and $-f$ is $(\operatorname{dom} f)$-defined and $-f$ is total and f^{-1} is $(\operatorname{dom} f)$-defined and f^{-1} is total and $|f|$ is $(\operatorname{dom} f)$-defined and $|f|$ is total.

Let c be a complex number. Let us note that $c+f$ is $(\operatorname{dom} f)$-defined and $c+f$ is total and $f-c$ is $(\operatorname{dom} f)$-defined and $f-c$ is total and $c \cdot f$ is $(\operatorname{dom} f)$-defined and $c \cdot f$ is total.

Let f be a finite sequence. Let us observe that every finite sequence which is $(\operatorname{len} f)$-element is also $(\operatorname{dom} f)$-defined.

Let n be a natural number. Let us observe that every finite sequence which is n-element is also $(\operatorname{Seg} n)$-defined and every finite sequence which is total and $(\operatorname{Seg} n)$-defined is also n-element.

Now we state the proposition:

(1) Let us consider a complex-valued finite sequence f. Then $0 \cdot f = \operatorname{len} f \mapsto 0$.

Let f be a complex-valued finite sequence. Note that $f + \operatorname{len} f \mapsto 0$ reduces to f.

Let n be a natural number, D be a non empty set, and X be a non empty subset of D. One can verify that there exists an X-valued finite sequence which is n-element and there exists a finite sequence of elements of X which is n-element.

3. On Positive and Negative Yielding Functions

Let f be a real-valued function. Let us note that $f + |f|$ is non-negative yielding and $|f| - f$ is non-negative yielding.

Let f be a non-negative yielding, real-valued function and x be an object. Observe that $f(x)$ is non negative.

Let f be a non-positive yielding, real-valued function. Let us observe that $f(x)$ is non positive.

Let f be a non-negative yielding, real-valued function and r be a non negative real number. One can verify that $r \cdot f$ is non-negative yielding and $(-r) \cdot f$ is non-positive yielding and $-f$ is non-positive yielding.

Let f be a non-positive yielding, real-valued function and r be a non negative real number. Let us observe that $r \cdot f$ is non-positive yielding and $(-r) \cdot f$ is non-negative yielding and $-f$ is non-negative yielding and every $\mathbb{Z}$-valued function which is non-negative yielding is also natural-valued.

Let f be a $\mathbb{Z}$-valued function. Let us observe that $\frac{1}{2} \cdot (f + |f|)$ is natural-valued and $\frac{1}{2} \cdot (|f| - f)$ is natural-valued.

Let us consider a binary relation f. Now we state the propositions:

(2) rng f is natural-membered if and only if f is natural-valued.

PROOF: If rng f is natural-membered, then f is natural-valued. □

(3) f is $\mathbb{N}$-valued if and only if rng f is natural-membered. The theorem is a consequence of (2).

(4) rng f is integer-membered if and only if f is $\mathbb{Z}$-valued.

PROOF: If rng f is integer-membered, then f is $\mathbb{Z}$-valued. □

(5) rng f is rational-membered if and only if f is $\mathbb{Q}$-valued.

PROOF: If rng f is rational-membered, then f is $\mathbb{Q}$-valued. □

(6) rng f is real-membered if and only if f is real-valued.

PROOF: If rng f is real-membered, then f is real-valued. □

(7) f is $\mathbb{R}$-valued if and only if rng f is real-membered. The theorem is a consequence of (6).

(8) rng f is complex-membered if and only if f is complex-valued.

PROOF: If rng f is complex-membered, then f is complex-valued. □

(9) f is $\mathbb{C}$-valued if and only if rng f is complex-membered. The theorem is a consequence of (8).

(10) dom f is natural-membered if and only if f is $\mathbb{N}$-defined.

PROOF: If dom f is natural-membered, then f is $\mathbb{N}$-defined. □

Let f be a $\mathbb{Z}$-defined binary relation. Observe that dom f is integer-membered.

Now we state the proposition:

(11) Let us consider a binary relation f. Then dom f is integer-membered if and only if f is $\mathbb{Z}$-defined.

PROOF: If dom f is integer-membered, then f is $\mathbb{Z}$-defined. □

Let f be a $\mathbb{Q}$-defined binary relation. Let us note that dom f is rational-membered.

Now we state the proposition:

(12) Let us consider a binary relation f. Then dom f is rational-membered if and only if f is $\mathbb{Q}$-defined.

PROOF: If dom f is rational-membered, then f is $\mathbb{Q}$-defined. □

Let f be a $\mathbb{R}$-defined binary relation. Note that dom f is real-membered.

Now we state the proposition:

(13) Let us consider a binary relation f. Then dom f is real-membered if and only if f is $\mathbb{R}$-defined.

Proof: If dom f is real-membered, then f is $\mathbb{R}$-defined. □

Let f be a $\mathbb{C}$-defined binary relation. One can check that dom f is complex-membered.

Now we state the propositions:

(14) Let us consider a binary relation f. Then dom f is complex-membered if and only if f is $\mathbb{C}$-defined.

Proof: If dom f is complex-membered, then f is $\mathbb{C}$-defined. □

(15) Let us consider a set D, and a function f. Then f is D-valued if and only if f is a function from dom f into D.

Proof: If f is D-valued, then f is a function from dom f into D. □

(16) Let us consider a set C. Then every total, C-defined function is a function from C into rng f.

(17) Let us consider sets C, D, and a total, C-defined function f. Then f is a function from C into D if and only if f is D-valued. The theorem is a consequence of (16) and (15).

(18) Every real-valued function is a function from dom f into $\mathbb{R}$.

(19) Let us consider a complex-valued finite sequence f. Then

(i) $f - f = 0 \cdot f$, and

(ii) $f - f = \operatorname{len} f \mapsto 0$.

The theorem is a consequence of (1).

(20) Let us consider a complex number a, a finite sequence f, and a natural number k. If $k \in \operatorname{dom} f$, then $(\operatorname{len} f \mapsto a)(k) = a$.

Let a be a real number, k be a non zero natural number, l be a natural number, and f be a $(k+l)$-element finite sequence. One can verify that $(\operatorname{len} f \mapsto a)(k)$ reduces to a.

Let f be a complex-valued function. The functors: delneg f, delpos f, and delall f yielding complex-valued functions are defined by terms

(Def. 1) $\frac{1}{2} \cdot (f + |f|)$,

(Def. 2) $\frac{1}{2} \cdot (|f| - f)$,

(Def. 3) $0 \cdot f$,

respectively. Now we state the propositions:

(21) Let us consider a complex-valued function f. Then

(i) $\operatorname{dom} f = \operatorname{dom}(\operatorname{delpos} f)$, and

(ii) $\operatorname{dom} f = \operatorname{dom}(\operatorname{delneg} f)$, and

(iii) $\operatorname{dom} f = \operatorname{dom}(\operatorname{delall} f)$.

(22) Let us consider a complex-valued function f, and an object x. Then $f(x) = (\operatorname{delneg} f)(x) - (\operatorname{delpos} f)(x)$. The theorem is a consequence of (21).

(23) Let us consider a complex-valued function f. Then $f = \operatorname{delneg} f - \operatorname{delpos} f$. The theorem is a consequence of (21) and (22).

Let us consider a real-valued function f and an object x. Now we state the propositions:

(24) (i) $f(x) = (\operatorname{delneg} f)(x)$, or

(ii) $f(x) = -(\operatorname{delpos} f)(x)$.

The theorem is a consequence of (21).

(25) (i) $(\operatorname{delneg} f)(x) = 0$, or

(ii) $(\operatorname{delpos} f)(x) = 0$.

The theorem is a consequence of (22) and (24).

Let f be a real-valued function. One can verify that $\operatorname{delneg} f \cdot \operatorname{delpos} f$ is empty yielding.

Now we state the proposition:

(26) Let us consider a real-valued function f. Then $\operatorname{delall} f = \operatorname{delneg} f \cdot \operatorname{delpos} f$. The theorem is a consequence of (21).

Let f be a complex-valued function and f_1 be a total, $(\operatorname{dom} f)$-defined, empty yielding function. Let us observe that $f + f_1$ reduces to f and $f - f_1$ reduces to f.

Let f_1 be a total, $(\operatorname{dom} f)$-defined, complex-valued function and f_2 be a total, $(\operatorname{dom} f)$-defined, empty yielding function. One can verify that $f_1 + f_2$ reduces to f_1 and $f_1 - f_2$ reduces to f_1.

Observe that $f - f$ is $(\operatorname{dom} f)$-defined and $f - f$ is total.

Now we state the proposition:

(27) Let us consider a complex-valued function f. Then $|f| = \operatorname{delneg} f + \operatorname{delpos} f$.

Let f be an empty finite sequence. Let us note that $\prod f$ is natural and $\prod f$ is non zero.

Let f be a positive yielding, real-valued finite sequence. One can check that $\prod f$ is positive.

Let f be a complex-valued finite sequence. Let us note that $\operatorname{delneg} f$ is $(\operatorname{len} f)$-element and $\operatorname{delpos} f$ is $(\operatorname{len} f)$-element.

Now we state the proposition:

(28) Let us consider a complex-valued function f. Then $\operatorname{delneg} f = \operatorname{delpos}(-f)$.

Let f be a non-negative yielding, real-valued function. Note that $|f|$ reduces to f and $\operatorname{delneg} f$ reduces to f. We identify $\operatorname{delall} f$ with $\operatorname{delpos} f$. We identify delpos f with delall f. Let f be a non-positive yielding, real-valued function. Observe that $-$delpos f reduces to f. One can verify that delneg f is empty yielding.

We identify delall f with delneg f. We identify delneg f with delall f. Now we state the proposition:

(29) Let us consider a finite sequence f of elements of $\mathbb{Z}$. Then there exist finite sequences f_1, f_2 of elements of $\mathbb{N}$ such that $f = f_1 - f_2$. The theorem is a consequence of (23).

Let a be an integer and n be a natural number. Note that $n \mapsto a$ is $\mathbb{Z}$-valued.

Let f be a non empty, empty yielding finite sequence. Observe that $\prod f$ is zero.

Now we state the propositions:

(30) Let us consider finite sequences f_1, f_2 of elements of $\mathbb{R}$. Suppose len $f_1 =$ len f_2 and for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f_1$ holds $f_1(k) \geqslant f_2(k) > 0$. Then $\prod f_1 \geqslant \prod f_2$.

PROOF: For every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f_2$ holds $f_1(k) \geqslant f_2(k) > 0$. □

(31) Let us consider a real number a, and a finite sequence f of elements of $\mathbb{R}$. Suppose for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f$ holds $0 < f(k) \leqslant a$. Then $\prod f \leqslant \prod(\operatorname{len} f \mapsto a)$. The theorem is a consequence of (20).

(32) Let us consider a non negative real number a, and a finite sequence f of elements of $\mathbb{R}$. Suppose for every natural number k such that $k \in \operatorname{dom} f$ holds $f(k) \geqslant a$. Then $\prod f \geqslant a^{\operatorname{len} f}$. The theorem is a consequence of (20).

(33) Let us consider non-negative yielding finite sequences f_1, f_2 of elements of $\mathbb{R}$. Suppose len $f_1 =$ len f_2 and for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f_2$ holds $f_1(k) \geqslant f_2(k)$. Then $\prod f_1 \geqslant \prod f_2$.

(34) Let us consider finite sequences f_1, f_2 of elements of $\mathbb{R}$. Suppose len $f_1 =$ len f_2 and for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f_2$ holds $f_1(k) \geqslant f_2(k) \geqslant 0$. Then $\prod f_1 \geqslant \prod f_2$.

PROOF: For every real number r such that $r \in \operatorname{rng} f_2$ holds $r \geqslant 0$. For every real number r such that $r \in \operatorname{rng} f_1$ holds $r \geqslant 0$. □

(35) Let us consider a positive real number a, and a non-negative yielding finite sequence f of elements of $\mathbb{R}$. Suppose for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} f$ holds $f(k) \leqslant a$. Then $\prod f \leqslant a^{\operatorname{len} f}$. The theorem is a consequence of (20) and (33).

4. Basic Operations on Short Finsequences

Let a be a complex number. Let us note that $(-\langle -a\rangle)(1)$ reduces to a and $(\langle a^{-1}\rangle^{-1})(1)$ reduces to a.

Let us consider complex numbers a, b. Now we state the propositions:

(36) $\langle a\rangle + \langle b\rangle = \langle a+b\rangle$.

(37) $\langle a\rangle - \langle b\rangle = \langle a-b\rangle$. The theorem is a consequence of (36).

(38) $\langle a\rangle \cdot \langle b\rangle = \langle a\cdot b\rangle$.

(39) $\langle a\rangle / \langle b\rangle = \langle a\cdot (b^{-1})\rangle$. The theorem is a consequence of (38).

Let n be a natural number, f be an n-element finite sequence, and a be a complex number. One can verify that $(f \frown \langle a\rangle)(n+1)$ reduces to a and $(f \frown \langle a\rangle)\restriction n$ reduces to f.

Let a, b, c, d be complex numbers. Let us observe that $\langle a, b, c, d\rangle$ is complex-valued.

Let a, b be complex numbers. Let us observe that $(-\langle -a, b\rangle)(1)$ reduces to a and $(-\langle a, -b\rangle)(2)$ reduces to b and $(\langle a^{-1}, b\rangle^{-1})(1)$ reduces to a and $(\langle a, b^{-1}\rangle^{-1})(2)$ reduces to b.

Let a, b, c be complex numbers. Note that $\langle a, b, c\rangle(1)$ reduces to a and $\langle a, b, c\rangle(2)$ reduces to b and $(-\langle -a, b, c\rangle)(1)$ reduces to a and $(-\langle a, -b, c\rangle)(2)$ reduces to b and $(-\langle a, b, -c\rangle)(3)$ reduces to c and $(\langle a^{-1}, b, c\rangle^{-1})(1)$ reduces to a and $(\langle a, b^{-1}, c\rangle^{-1})(2)$ reduces to b and $(\langle a, b, c^{-1}\rangle^{-1})(3)$ reduces to c.

Now we state the propositions:

(40) Let us consider complex numbers a, b, a natural number n, and n-element, complex-valued finite sequences f, g. Then $f \frown \langle a\rangle + g \frown \langle b\rangle = (f+g) \frown \langle a+b\rangle$.

PROOF: Reconsider $f_3 = f \frown \langle a\rangle$ as an $(n+1)$-element finite sequence of elements of $\mathbb{C}$. Reconsider $g_1 = g \frown \langle b\rangle$ as an $(n+1)$-element finite sequence of elements of $\mathbb{C}$. For every object k such that $k \in \operatorname{dom}(f_3+g_1)$ holds $(f_3+g_1)(k) = ((f+g) \frown \langle a+b\rangle)(k)$. □

(41) Let us consider complex numbers a, b, x, y. Then $\langle a, b\rangle + \langle x, y\rangle = \langle a+x, b+y\rangle$. The theorem is a consequence of (40) and (36).

(42) Let us consider complex numbers a, b, c, x, y, z. Then $\langle a, b, c\rangle + \langle x, y, z\rangle = \langle a+x, b+y, c+z\rangle$. The theorem is a consequence of (40) and (41).

(43) Let us consider complex numbers a, b, c, d, x, y, z, v. Then $\langle a, b, c, d\rangle + \langle x, y, z, v\rangle = \langle a+x, b+y, c+z, d+v\rangle$. The theorem is a consequence of (40) and (42).

(44) Let us consider complex numbers a, b, a natural number n, and n-element, complex-valued finite sequences f, g. Then $(f \frown \langle a\rangle) \cdot (g \frown \langle b\rangle) = (f\cdot g) \frown \langle a\cdot b\rangle$.

PROOF: Reconsider $f_3 = f \frown \langle a \rangle$ as an $(n+1)$-element finite sequence of elements of $\mathbb{C}$. Reconsider $g_1 = g \frown \langle b \rangle$ as an $(n+1)$-element finite sequence of elements of $\mathbb{C}$. For every object k such that $k \in \operatorname{dom}(f_3 \cdot g_1)$ holds $(f_3 \cdot g_1)(k) = ((f \cdot g) \frown \langle a \cdot b \rangle)(k)$. □

(45) Let us consider complex numbers a, b, x, y. Then $\langle a, b \rangle \cdot \langle x, y \rangle = \langle a \cdot x, b \cdot y \rangle$. The theorem is a consequence of (44) and (38).

(46) Let us consider complex numbers a, b, c, x, y, z. Then $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a \cdot x, b \cdot y, c \cdot z \rangle$. The theorem is a consequence of (44) and (45).

(47) Let us consider complex numbers a, b, c, d, x, y, z, v. Then $\langle a, b, c, d \rangle \cdot \langle x, y, z, v \rangle = \langle a \cdot x, b \cdot y, c \cdot z, d \cdot v \rangle$. The theorem is a consequence of (44) and (46).

(48) Let us consider a complex number a, a non zero natural number n, and an n-element, complex-valued finite sequence f. Then $\langle a \rangle + f = \langle a + f(1) \rangle$.

(49) Let us consider complex numbers a, b, a non trivial natural number n, and an n-element, complex-valued finite sequence f. Then $\langle a, b \rangle + f = \langle a + f(1), b + f(2) \rangle$.

(50) Let us consider a complex number a, a non zero natural number n, and an n-element, complex-valued finite sequence f. Then $\langle a \rangle \cdot f = \langle a \cdot f(1) \rangle$.

(51) Let us consider complex numbers a, b, a non trivial natural number n, and an n-element, complex-valued finite sequence f. Then $\langle a, b \rangle \cdot f = \langle a \cdot f(1), b \cdot f(2) \rangle$.

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Some Remarks about Product Spaces

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Summary. This article covers some technical aspects about the product topology which are usually not given much of a thought in mathematics and standard literature like [7] and [6], not even by Bourbaki in [4].

Let $\{\mathcal{T}_i\}_{i\in I}$ be a family of topological spaces. The prebasis of the product space $\mathcal{T} = \prod_{i\in I} \mathcal{T}_i$ is defined in [5] as the set of all $\pi_i^{-1}(V)$ with $i \in I$ and V open in $\mathcal{T}_i$. Here it is shown that the basis generated by this prebasis consists exactly of the sets $\prod_{i\in I} V_i$ with V_i open in $\mathcal{T}_i$ and for all but finitely many $i \in I$ holds $V_i = \mathcal{T}_i$. Given $I = \{a\}$ we have $\mathcal{T} \cong \mathcal{T}_a$, given $I = \{a, b\}$ with $a \neq b$ we have $\mathcal{T} \cong \mathcal{T}_a \times \mathcal{T}_b$. Given another family of topological spaces $\{\mathcal{S}_i\}_{i\in I}$ such that $\mathcal{S}_i \cong \mathcal{T}_i$ for all $i \in I$, we have $\mathcal{S} = \prod_{i\in I} \mathcal{S}_i \cong \mathcal{T}$. If instead S_i is a subspace of T_i for each $i \in I$, then $\mathcal{S}$ is a subspace of $\mathcal{T}$.

These results are obvious for mathematicians, but formally proven here by means of the Mizar system [3], [2].

MSC: 54B10 68T99 03B35

Keywords: topology; product spaces

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a one-to-one function f, and an object y. Suppose $\operatorname{rng} f = \{y\}$. Then $\operatorname{dom} f = \{(f^{-1})(y)\}$.
PROOF: Consider x_0 being an object such that $x_0 \in \operatorname{dom} f$ and $f(x_0) = y$. For every object x, $x \in \operatorname{dom} f$ iff $x = (f^{-1})(y)$. □

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(2) Let us consider a one-to-one function f, and objects y_1, y_2. Suppose $\operatorname{rng} f = \{y_1, y_2\}$. Then $\operatorname{dom} f = \{(f^{-1})(y_1), (f^{-1})(y_2)\}$.
PROOF: Consider x_1 being an object such that $x_1 \in \operatorname{dom} f$ and $f(x_1) = y_1$. Consider x_2 being an object such that $x_2 \in \operatorname{dom} f$ and $f(x_2) = y_2$. For every object x, $x \in \operatorname{dom} f$ iff $x = (f^{-1})(y_1)$ or $x = (f^{-1})(y_2)$. □

Let X, Y be sets. Note that there exists a function which is empty, X-defined, Y-valued, and one-to-one.

Let T, S be sets, f be a function from T into S, and G be a finite family of subsets of T. Let us note that $f^{\circ}G$ is finite.

Now we state the propositions:

(3) Let us consider a set A, a family F of subsets of A, and a binary relation R. Then $R^{\circ}(\bigcap F) \subseteq \bigcap\{R^{\circ}X$, where X is a subset of $A : X \in F\}$.

(4) Let us consider a set A, a family F of subsets of A, and a one-to-one function f. Then $f^{\circ}(\bigcap F) = \bigcap\{f^{\circ}X$, where X is a subset of $A : X \in F\}$.
PROOF: Set $S = \{f^{\circ}X$, where X is a subset of $A : X \in F\}$. $\bigcap S \subseteq f^{\circ}(\bigcap F)$. $f^{\circ}(\bigcap F) \subseteq \bigcap S$. □

(5) Let us consider a set X, a non empty set Y, and a function f from X into Y. Then $\{f^{-1}(\{y\})$, where y is an element of $Y : y \in \operatorname{rng} f\}$ is a partition of X.
PROOF: Set $P = \{f^{-1}(\{y\})$, where y is an element of $Y : y \in \operatorname{rng} f\}$. For every object x, $x \in X$ iff there exists a set A such that $x \in A$ and $A \in P$. For every subset A of X such that $A \in P$ holds $A \neq \emptyset$ and for every subset B of X such that $B \in P$ holds $A = B$ or A misses B. $P \subseteq 2^X$. □

(6) Let us consider a non empty set X, and objects x, y. If $X \longmapsto x = X \longmapsto y$, then $x = y$.

(7) Let us consider an object i, and a many sorted set J indexed by $\{i\}$. Then $J = \{i\} \longmapsto J(i)$.
PROOF: For every object x such that $x \in \operatorname{dom} J$ holds $J(x) = (\{i\} \longmapsto J(i))(x)$. □

(8) Let us consider a 2-element set I, and elements i, j of I. If $i \neq j$, then $I = \{i, j\}$.
PROOF: For every object x, $x = i$ or $x = j$ iff $x \in I$. □

(9) Let us consider a 2-element set I, a many sorted set f indexed by I, and elements i, j of I. If $i \neq j$, then $f = [i \longmapsto f(i), j \longmapsto f(j)]$. The theorem is a consequence of (8).

(10) Let us consider objects a, b, c, d. If $a \neq b$, then $[a \longmapsto c, b \longmapsto d] = [b \longmapsto d, a \longmapsto c]$.
PROOF: For every object x such that $x \in \operatorname{dom}[a \longmapsto c, b \longmapsto d]$ holds $[a \longmapsto c, b \longmapsto d](x) = [b \longmapsto d, a \longmapsto c](x)$. □

(11) Let us consider a function f, and objects i, j. If i, $j \in \operatorname{dom} f$, then $f = f{+\cdot}[i \longmapsto f(i), j \longmapsto f(j)]$.

(12) Let us consider objects x, y, z. Then $x \dot{\longmapsto} y {+\cdot} (x \dot{\longmapsto} z) = x \dot{\longmapsto} z$.

Let us observe that there exists a function which is non non-empty.

Now we state the propositions:

(13) Let us consider non empty sets X, Y, and an element y of Y. Then $X \longmapsto y \in \prod(X \longmapsto Y)$.

PROOF: Set $f = X \longmapsto y$. For every object x such that $x \in \operatorname{dom}(X \longmapsto Y)$ holds $f(x) \in (X \longmapsto Y)(x)$. □

(14) Let us consider a non empty set X, a set Y, and a subset Z of Y. Then $\prod(X \longmapsto Z) \subseteq \prod(X \longmapsto Y)$.

(15) Let us consider a non empty set X, and an object i. Then $\prod(\{i\} \longmapsto X) = \{\{i\} \longmapsto x$, where x is an element of $X\}$.

PROOF: Set $S = \{\{i\} \longmapsto x$, where x is an element of $X\}$. For every object z, $z \in \prod(\{i\} \longmapsto X)$ iff $z \in S$. □

(16) Let us consider a non empty set X, and objects i, f. Then $f \in \prod(\{i\} \longmapsto X)$ if and only if there exists an element x of X such that $f = \{i\} \longmapsto x$. The theorem is a consequence of (15).

(17) Let us consider a non empty set X, an object i, and an element x of X. Then $(\operatorname{proj}(\{i\} \longmapsto X, i))(\{i\} \longmapsto x) = x$. The theorem is a consequence of (13).

(18) Let us consider sets X, Y. Then $X \neq \emptyset$ and $Y = \emptyset$ if and only if $\prod(X \longmapsto Y) = \emptyset$.

Let f be an empty function and x be an object. Let us note that $\operatorname{proj}(f, x)$ is trivial.

Now we state the proposition:

(19) Let us consider a trivial function f, and an object x. If $x \in \operatorname{dom} f$, then $\operatorname{proj}(f, x)$ is one-to-one.

PROOF: Consider t being an object such that $\operatorname{dom} f = \{t\}$. Set $F = \operatorname{proj}(f, x)$. For every objects y, z such that y, $z \in \operatorname{dom} F$ and $F(y) = F(z)$ holds $y = z$. □

Let x, y be objects. Note that $\operatorname{proj}(x \dot{\longmapsto} y, x)$ is one-to-one.

Let I be a 1-element set, J be a many sorted set indexed by I, and i be an element of I. One can verify that $\operatorname{proj}(J, i)$ is one-to-one.

Now we state the propositions:

(20) Let us consider a non empty set X, a subset Y of X, and an object i. Then $(\operatorname{proj}(\{i\} \longmapsto X, i))^{\circ}(\prod(\{i\} \longmapsto Y)) = Y$. The theorem is a consequence of (16), (13), and (14).

(21) Let us consider non-empty functions f, g, and objects i, x. Suppose $x \in \prod f \cap \prod(f{+\cdot}g)$. Then $(\mathrm{proj}(f,i))(x) = (\mathrm{proj}(f{+\cdot}g,i))(x)$.

(22) Let us consider non-empty functions f, g, an object i, and a set A. Suppose $A \subseteq \prod f \cap \prod(f{+\cdot}g)$. Then $(\mathrm{proj}(f,i))^{\circ}A = (\mathrm{proj}(f{+\cdot}g,i))^{\circ}A$. The theorem is a consequence of (21).

(23) Let us consider non-empty functions f, g. Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then $\prod(f{+\cdot}g) \subseteq \prod f$.

Let us consider non-empty functions f, g and an object i. Now we state the propositions:

(24) Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, f \setminus \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod(f{+\cdot}g)) = f(i)$. The theorem is a consequence of (23) and (22).

(25) Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod(f{+\cdot}g)) = g(i)$. The theorem is a consequence of (23) and (22).

(26) Suppose $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod g) = g(i)$. The theorem is a consequence of (25).

(27) Let us consider a function f, sets X, Y, and an object i. Suppose $X \subseteq Y$. Then $\prod(f{+\cdot}(i \overset{\cdot}{\longmapsto} X)) \subseteq \prod(f{+\cdot}(i \overset{\cdot}{\longmapsto} Y))$.

(28) Let us consider objects i, j, and sets A, B, C, D. Suppose $A \subseteq C$ and $B \subseteq D$. Then $\prod[i \longmapsto A, j \longmapsto B] \subseteq \prod[i \longmapsto C, j \longmapsto D]$. The theorem is a consequence of (14).

(29) Let us consider sets X, Y, and objects f, i, j. Suppose $i \neq j$. Then $f \in \prod[i \longmapsto X, j \longmapsto Y]$ if and only if there exist objects x, y such that $x \in X$ and $y \in Y$ and $f = [i \longmapsto x, j \longmapsto y]$.

PROOF: If $f \in \prod[i \longmapsto X, j \longmapsto Y]$, then there exist objects x, y such that $x \in X$ and $y \in Y$ and $f = [i \longmapsto x, j \longmapsto y]$. Reconsider $g = f$ as a function. For every object z such that $z \in \mathrm{dom}[i \longmapsto X, j \longmapsto Y]$ holds $g(z) \in [i \longmapsto X, j \longmapsto Y](z)$. □

(30) Let us consider a non-empty function f, sets X, Y, objects i, j, x, y, and a function g. Suppose $x \in X$ and $y \in Y$ and $i \neq j$ and $g \in \prod f$. Then $g{+\cdot}[i \longmapsto x, j \longmapsto y] \in \prod(f{+\cdot}[i \longmapsto X, j \longmapsto Y])$.

PROOF: For every object z such that $z \in \mathrm{dom}(f{+\cdot}[i \longmapsto X, j \longmapsto Y])$ holds $(g{+\cdot}[i \longmapsto x, j \longmapsto y])(z) \in (f{+\cdot}[i \longmapsto X, j \longmapsto Y])(z)$. □

(31) Let us consider a function f, sets A, B, C, D, and objects i, j. Suppose $A \subseteq C$ and $B \subseteq D$. Then $\prod(f{+\cdot}[i \longmapsto A, j \longmapsto B]) \subseteq \prod(f{+\cdot}[i \longmapsto$

$C, j \longmapsto D]$). The theorem is a consequence of (27).

(32) Let us consider a function f, sets A, B, and objects i, j. Suppose i, $j \in \operatorname{dom} f$ and $A \subseteq f(i)$ and $B \subseteq f(j)$. Then $\prod(f {+\cdot} [i \longmapsto A, j \longmapsto B]) \subseteq \prod f$. The theorem is a consequence of (11) and (31).

(33) Let us consider a set I, and many sorted sets f, g indexed by I. Then $\prod f \cap \prod g = \prod(f \cap g)$.
Proof: For every object x, $x \in \prod f \cap \prod g$ iff there exists a function h such that $h = x$ and $\operatorname{dom} h = \operatorname{dom}(f \cap g)$ and for every object y such that $y \in \operatorname{dom}(f \cap g)$ holds $h(y) \in (f \cap g)(y)$. □

(34) Let us consider a 2-element set I, a many sorted set f indexed by I, elements i, j of I, and an object x. Suppose $i \neq j$. Then

(i) $f {+\cdot} (i, x) = [i \longmapsto x, j \longmapsto f(j)]$, and

(ii) $f {+\cdot} (j, x) = [i \longmapsto f(i), j \longmapsto x]$.

The theorem is a consequence of (10).

Let us consider a non-empty function f, a set X, and an object i. Now we state the propositions:

(35) If $i \in \operatorname{dom} f$, then $f {+\cdot} (i, X)$ is non-empty iff X is not empty.
Proof: For every object x such that $x \in \operatorname{dom}(f {+\cdot} (i, X))$ holds $(f {+\cdot} (i, X))(x)$ is not empty. □

(36) If $i \in \operatorname{dom} f$, then $\prod(f {+\cdot} (i, X)) = \emptyset$ iff X is empty. The theorem is a consequence of (35).

(37) Let us consider a non-empty function f, a set X, objects i, x, and a function g. Suppose $i \in \operatorname{dom} f$ and $x \in X$ and $g \in \prod f$. Then $g {+\cdot} (i, x) \in \prod(f {+\cdot} (i, X))$.
Proof: For every object y such that $y \in \operatorname{dom}(f {+\cdot} (i, X))$ holds $(g {+\cdot} (i, x))(y) \in (f {+\cdot} (i, X))(y)$. □

(38) Let us consider a function f, sets X, Y, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq Y$. Then $\prod(f {+\cdot} (i, X)) \subseteq \prod(f {+\cdot} (i, Y))$. The theorem is a consequence of (27).

(39) Let us consider a function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq f(i)$. Then $\prod(f {+\cdot} (i, X)) \subseteq \prod f$. The theorem is a consequence of (38).

(40) Let us consider a non-empty function f, non empty sets X, Y, and objects i, j. Suppose $i, j \in \operatorname{dom} f$ and ($X \not\subseteq f(i)$ or $f(j) \not\subseteq Y$) and $\prod(f {+\cdot} (i, X)) \subseteq \prod(f {+\cdot} (j, Y))$. Then

(i) $i = j$, and

(ii) $X \subseteq Y$.

Proof: $f +\cdot (i, X)$ is non-empty and $f +\cdot (j, Y)$ is non-empty. $i = j$. Set $g =$ the element of $\prod f$. $g +\cdot (i, x) \in \prod(f +\cdot (i, X))$. $\square$

(41) Let us consider a non-empty function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $\prod(f +\cdot (i, X)) \subseteq \prod f$. Then $X \subseteq f(i)$. The theorem is a consequence of (37).

(42) Let us consider a non-empty function f, non empty sets X, Y, and objects i, j. Suppose $i, j \in \operatorname{dom} f$ and ($X \neq f(i)$ or $Y \neq f(j)$) and $\prod(f +\cdot (i, X)) = \prod(f +\cdot (j, Y))$. Then

(i) $i = j$, and

(ii) $X = Y$.

Proof: $f +\cdot (i, X)$ is non-empty and $f +\cdot (j, Y)$ is non-empty. $i = j$. $\square$

(43) Let us consider a non-empty function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq f(i)$. Then $(\operatorname{proj}(f, i))^{\circ}(\prod(f +\cdot (i, X))) = X$. The theorem is a consequence of (25).

(44) Let us consider objects x, y, z. Then $x \dot{\longmapsto} y +\cdot (x, z) = x \dot{\longmapsto} z$. The theorem is a consequence of (12).

Let I be a non empty set and J be a 1-sorted yielding, nonempty many sorted set indexed by I. Let us observe that the support of J is non-empty.

2. Remarks about Product Spaces

Now we state the propositions:

(45) Let us consider topological spaces T, S, and a function f from T into S. Then f is open if and only if there exists a basis B of T such that for every subset V of T such that $V \in B$ holds $f^{\circ}V$ is open.

(46) Let us consider non empty topological spaces T_1, T_2, S_1, S_2, a function f from T_1 into S_1, and a function g from T_2 into S_2. If f is open and g is open, then $f \times g$ is open.

Proof: There exists a basis B of $T_1 \times T_2$ such that for every subset P of $T_1 \times T_2$ such that $P \in B$ holds $(f \times g)^{\circ}P$ is open. $\square$

Let us consider non empty topological spaces S, T and a function f from S into T. Now we state the propositions:

(47) If f is bijective and there exists a basis K of S and there exists a basis L of T such that $f^{\circ}K = L$, then f is a homeomorphism.

Proof: For every subset W of T such that $W \in L$ holds $f^{-1}(W)$ is open. For every subset V of S such that $V \in K$ holds $f^{\circ}V$ is open. f is open. $\square$

(48) If f is bijective and there exists a prebasis K of S and there exists a prebasis L of T such that $f^{\circ}K = L$, then f is a homeomorphism.
PROOF: Reconsider $K_0 = \text{FinMeetCl}(K)$ as a basis of S. Reconsider $L_0 = \text{FinMeetCl}(L)$ as a basis of T. For every subset W of T, $W \in L_0$ iff there exists a subset V of S such that $V \in K_0$ and $f^{\circ}V = W$. □

Let us consider topological spaces S, T. Now we state the propositions:

(49) If there exists a basis K of S and there exists a basis L of T such that $K = L \Cap \{\Omega_S\}$, then S is a subspace of T.
PROOF: For every subset A of S, $A \in$ the topology of S iff there exists a subset B of T such that $B \in$ the topology of T and $A = B \cap \Omega_S$. Consider B being a subset of T such that $B \in$ the topology of T and the carrier of $S = B \cap \Omega_S$. □

(50) Suppose $\Omega_S \subseteq \Omega_T$ and there exists a prebasis K of S and there exists a prebasis L of T such that $K = L \Cap \{\Omega_S\}$. Then S is a subspace of T.
PROOF: Reconsider $K_0 = \text{FinMeetCl}(K)$ as a basis of S. Reconsider $L_0 = \text{FinMeetCl}(L)$ as a basis of T. For every object x, $x \in K_0$ iff $x \in L_0 \Cap \{\Omega_S\}$. □

(51) If there exists a prebasis K of S and there exists a prebasis L of T such that $\Omega_S \in K$ and $K = L \Cap \{\Omega_S\}$, then S is a subspace of T. The theorem is a consequence of (50).

(52) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and an element i of I. Then $\text{rng}\,\text{proj}(J, i) =$ the carrier of $J(i)$.

Let X be a set and T be a topological structure. Observe that $X \longmapsto T$ is topological structure yielding.

Let F be a binary relation. We say that F is topological space yielding if and only if

(Def. 1) for every object x such that $x \in \text{rng}\,F$ holds x is a topological space.

Note that every binary relation which is topological space yielding is also topological structure yielding and every function which is topological space yielding is also 1-sorted yielding.

Let X be a set and T be a topological space. One can verify that $X \longmapsto T$ is topological space yielding.

Let I be a set. One can verify that there exists a many sorted set indexed by I which is topological space yielding and nonempty.

Let I be a non empty set, J be a topological space yielding, nonempty many sorted set indexed by I, and i be an element of I. Let us note that the functor $J(i)$ yields a non empty topological space. Let f be a function. The functor $\text{ProjMap}\,f$ yielding a many sorted function indexed by $\text{dom}\,f$ is defined by

(Def. 2) for every object x such that $x \in \operatorname{dom} f$ holds $it(x) = \operatorname{proj}(f, x)$.

Let f be an empty function. One can verify that ProjMap f is empty.

Let f be a non-empty function. Note that ProjMap f is non-empty.

Let f be a non non-empty function. Let us note that ProjMap f is empty yielding.

Let I be a non empty set and J be a topological structure yielding, nonempty many sorted set indexed by I. The functor ProjMap J yielding a many sorted set indexed by I is defined by the term

(Def. 3) ProjMap(the support of J).

Observe that ProjMap J is function yielding, non empty, and non-empty.

Now we state the proposition:

(53) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and an element i of I. Then $(\operatorname{ProjMap} J)(i) = \operatorname{proj}(J, i)$.

Let I be a non empty set, J be a topological structure yielding, nonempty many sorted set indexed by I, and f be a one-to-one, I-valued function. The functor ProdBasSel(J, f) yielding a many sorted set indexed by rng f is defined by the term

(Def. 4) $(\operatorname{ProjMap} J) \circ (I\text{-indexing}\, f^{-1}) \restriction \operatorname{rng} f$.

Let f be an empty, one-to-one, I-valued function. Note that ProdBasSel(J, f) is empty.

Now we state the propositions:

(54) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, a one-to-one, I-valued function f, and an element i of I. Suppose $i \in \operatorname{rng} f$. Then $(\operatorname{ProdBasSel}(J, f))(i) = (\operatorname{proj}(J, i))^{\circ}(f^{-1})(i)$. The theorem is a consequence of (53).

(55) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and a one-to-one, I-valued function f. Suppose f^{-1} is non-empty and $\operatorname{dom} f \subseteq 2^{\prod \alpha}$. Then ProdBasSel$(J, f)$ is non-empty, where α is the support of J. The theorem is a consequence of (54).

(56) Let us consider a non empty set I, and a topological space yielding, nonempty many sorted set J indexed by I. Then $\emptyset \in$ the product prebasis for J. The theorem is a consequence of (36).

(57) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and a non empty subset P of $\prod$(the support of J). Suppose $P \in$ the product prebasis for J. Then there exists an element i of I such that

(i) $(\mathrm{proj}(J,i))^{\circ}P$ is open, and

(ii) for every element j of I such that $j \neq i$ holds $(\mathrm{proj}(J,j))^{\circ}P = \Omega_{J(j)}$.

Proof: Consider i being a set, T being a topological structure, V being a subset of T such that $i \in I$ and V is open and $T = J(i)$ and $P = \prod((\text{the support of } J) +\cdot (i, V))$. $\mathrm{rng}\,\mathrm{proj}(J,i) =$ the carrier of $J(i)$. For every object x, $x \in (\mathrm{proj}(J,j))^{\circ}P$ iff $x \in \Omega_{J(j)}$ by [1, (30), (32)], [9, (8)], [8, (7)]. □

(58) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a non empty subset P of $\prod(\text{the support of } J)$. Suppose $P \in$ the product prebasis for J. Then

(i) for every element j of I, $(\mathrm{proj}(J,j))^{\circ}P$ is open, and

(ii) there exists an element i of I such that for every element j of I such that $j \neq i$ holds $(\mathrm{proj}(J,j))^{\circ}P = \Omega_{J(j)}$.

The theorem is a consequence of (57).

(59) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, a one-to-one, I-valued function f, and a family X of subsets of $\prod(\text{the support of } J)$. Suppose $X \subseteq$ the product prebasis for J and $\mathrm{dom}\, f = X$ and f^{-1} is non-empty and for every subset A of $\prod(\text{the support of } J)$ such that $A \in X$ holds $(\mathrm{proj}(J, f_{/A}))^{\circ}A$ is open. Let us consider an element i of I. Then

(i) if $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f))) = \Omega_{J(i)}$, and

(ii) if $i \in \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f)))$ is open.

Proof: Set $g = \mathrm{ProdBasSel}(J,f)$. Set $P = \prod((\text{the support of } J) +\cdot g)$. g is non-empty. If $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}P = \Omega_{J(i)}$. □

(60) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, a one-to-one, I-valued function f, and a family X of subsets of $\prod(\text{the support of } J)$. Suppose $X \subseteq$ the product prebasis for J and $\mathrm{dom}\, f = X$ and f^{-1} is non-empty and for every subset A of $\prod(\text{the support of } J)$ such that $A \in X$ holds $(\mathrm{proj}(J, f_{/A}))^{\circ}A$ is open. Let us consider an element i of I. Then

(i) $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f)))$ is open, and

(ii) if $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f))) = \Omega_{J(i)}$.

The theorem is a consequence of (59).

(61) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a subset P of $\prod$(the support of J). Then $P \in$ FinMeetCl(the product prebasis for J) if and only if there exists a family X of subsets of $\prod$(the support of J) and there exists a one-to-one, I-valued function f such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and $P = \prod$((the support of J)$+\cdot$ ProdBasSel(J, f)).

Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a non empty subset P of $\prod$(the support of J). Now we state the propositions:

(62) Suppose $P \in$ FinMeetCl(the product prebasis for J). Then there exists a family X of subsets of $\prod$(the support of J) and there exists a one-to-one, I-valued function f such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and for every element i of I, $(\text{proj}(J, i))^{\circ}P$ is open and if $i \notin \text{rng}\, f$, then $(\text{proj}(J, i))^{\circ}P = \Omega_{J(i)}$.
Proof: Consider X being a family of subsets of $\prod$(the support of J), f being a one-to-one, I-valued function such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and $P = \prod$((the support of J)$+\cdot$ ProdBasSel(J, f)). f^{-1} is non-empty. □

(63) Suppose $P \in$ FinMeetCl(the product prebasis for J). Then there exists a finite subset I_0 of I such that for every element i of I, $(\text{proj}(J, i))^{\circ}P$ is open and if $i \notin I_0$, then $(\text{proj}(J, i))^{\circ}P = \Omega_{J(i)}$. The theorem is a consequence of (62).

(64) Let us consider a 1-element set I, a topological structure yielding, nonempty many sorted set J indexed by I, an element i of I, and a subset P of $\prod$(the support of J). Then $P \in$ the product prebasis for J if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod(\{i\} \longmapsto V)$. The theorem is a consequence of (7) and (44).

(65) Let us consider a 1-element set I, and a topological space yielding, nonempty many sorted set J indexed by I. Then the topology of $\prod J =$ the product prebasis for J.

(66) Let us consider a 1-element set I, a topological space yielding, nonempty many sorted set J indexed by I, an element i of I, and a subset P of $\prod J$. Then P is open if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod(\{i\} \longmapsto V)$. The theorem is a consequence of (65) and (64).

Let I be a non empty set, J be a topological structure yielding, nonempty many sorted set indexed by I, and i be an element of I. Note that $\text{proj}(J, i)$ is continuous and onto.

Let J be a topological space yielding, nonempty many sorted set indexed by I. Note that $\mathrm{proj}(J,i)$ is open.

Let us consider a 1-element set I, a topological space yielding, nonempty many sorted set J indexed by I, and an element i of I. Now we state the propositions:

(67) $\mathrm{proj}(J,i)$ is a homeomorphism. The theorem is a consequence of (7).

(68) $\prod J$ and $J(i)$ are homeomorphic. The theorem is a consequence of (67).

Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a subset P of $\prod$(the support of J). Now we state the propositions:

(69) Suppose $i \neq j$. Then $P \in$ the product prebasis for J if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod[i \longmapsto V, j \longmapsto \Omega_{J(j)}]$ or there exists a subset W of $J(j)$ such that W is open and $P = \prod[i \longmapsto \Omega_{J(i)}, j \longmapsto W]$. The theorem is a consequence of (34).

(70) Suppose $i \neq j$. Then $P \in$ FinMeetCl(the product prebasis for J) if and only if there exists a subset V of $J(i)$ and there exists a subset W of $J(j)$ such that V is open and W is open and $P = \prod[i \longmapsto V, j \longmapsto W]$.

Proof: There exists a family Y of subsets of $\prod$(the support of J) such that $Y \subseteq$ the product prebasis for J and Y is finite and $P = \mathrm{Intersect}(Y)$. □

(71) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and elements i, j of I. Then $\langle \mathrm{proj}(J,i), \mathrm{proj}(J,j) \rangle$ is a function from $\prod J$ into $J(i) \times J(j)$.

(72) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, a subset P of $\prod$(the support of J), and elements i, j of I. Suppose $i \neq j$ and there exists a many sorted set F indexed by I such that $P = \prod F$ and for every element k of I, $F(k) \subseteq$ (the support of J)(k). Then $\langle \mathrm{proj}(J,i), \mathrm{proj}(J,j) \rangle^{\circ} P = (\mathrm{proj}(J,i))^{\circ} P \times (\mathrm{proj}(J,j))^{\circ} P$. The theorem is a consequence of (26), (30), and (11).

(73) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a function f from $\prod J$ into $J(i) \times J(j)$. Suppose $i \neq j$ and $f = \langle \mathrm{proj}(J,i), \mathrm{proj}(J,j) \rangle$. Then f is onto and open.

Proof: For every element k of I, $(\mathrm{proj}(J,k))^{\circ}(\Omega_{\prod \alpha}) =$ the carrier of $J(k)$, where α is the support of J. There exists a basis B of $\prod J$ such that for every subset P of $\prod J$ such that $P \in B$ holds $f^{\circ} P$ is open. □

(74) Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a function f from

$\prod J$ into $J(i) \times J(j)$. Suppose $i \neq j$ and $f = \langle \mathrm{proj}(J, i), \mathrm{proj}(J, j) \rangle$. Then f is a homeomorphism.

PROOF: f is onto and open. For every objects x_1, x_2 such that x_1, $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. □

(75) Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, and elements i, j of I. If $i \neq j$, then $\prod J$ and $J(i) \times J(j)$ are homeomorphic. The theorem is a consequence of (74).

Let I_1, I_2 be non empty sets, J be a topological space yielding, nonempty many sorted set indexed by I_2, and f be a function from I_1 into I_2. One can check that $J \cdot f$ is topological space yielding and nonempty.

Let J_1 be a topological space yielding, nonempty many sorted set indexed by I_1, J_2 be a topological space yielding, nonempty many sorted set indexed by I_2, and p be a function from I_1 into I_2. Assume p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic.

A product homeomorphism of J_1, J_2 and p is a function from $\prod J_1$ into $\prod J_2$ defined by

(Def. 5) there exists a many sorted function F indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(it(g))(p(i)) = F(i)(g(i))$.

Now we state the proposition:

(76) Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, a function p from I_1 into I_2, a product homeomorphism H of J_1, J_2 and p, and a many sorted function F indexed by I_1. Suppose p is bijective and for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. Let us consider an element i of I_1, and a subset U of $J_1(i)$. Then $H^\circ(\prod((\text{the support of } J_1) +\cdot (i, U))) = \prod((\text{the support of } J_2) +\cdot (p(i), F(i)^\circ U))$.

PROOF: Reconsider $j = p(i)$ as an element of I_2. Consider f being a function from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism. For every object y, $y \in H^\circ(\prod((\text{the support of } J_1) +\cdot (i, U)))$ iff $y \in \prod((\text{the support of } J_2) +\cdot (j, F(i)^\circ U))$. □

Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, a function p from I_1 into I_2, and a product homeomorphism H of J_1, J_2 and p. Now we state the propositions:

(77) If p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic, then H is bijective.
PROOF: Consider F being a many sorted function indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} H$ and $H(x_1) = H(x_2)$ holds $x_1 = x_2$. Set $i_0 =$ the element of I_1. Consider f_0 being a function from $J_1(i_0)$ into $(J_2 \cdot p)(i_0)$ such that $F(i_0) = f_0$ and f_0 is a homeomorphism. $\square$

(78) If p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic, then H is a homeomorphism.
PROOF: Consider F being a many sorted function indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. H is bijective. There exists a prebasis K of $\prod J_1$ and there exists a prebasis L of $\prod J_2$ such that $H^\circ K = L$. $\square$

(79) Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, and a function p from I_1 into I_2. Suppose p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic. Then $\prod J_1$ and $\prod J_2$ are homeomorphic. The theorem is a consequence of (78).

(80) Let us consider a non empty set I, topological space yielding, nonempty many sorted sets J_1, J_2 indexed by I, and a permutation p of I. Suppose for every element i of I, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic. Then $\prod J_1$ and $\prod J_2$ are homeomorphic.

(81) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a permutation p of I. Then $\prod J$ and $\prod J \cdot p$ are homeomorphic. The theorem is a consequence of (79).

(82) Let us consider a non empty set I, and topological space yielding, nonempty many sorted sets J_1, J_2 indexed by I. Suppose for every element i of I, $J_1(i)$ is a subspace of $J_2(i)$. Then $\prod J_1$ is a subspace of $\prod J_2$.
PROOF: There exists a prebasis K_1 of $\prod J_1$ and there exists a prebasis K_2 of $\prod J_2$ such that $\Omega_{\prod J_1} \in K_1$ and $K_1 = K_2 \Cap \{\Omega_{\prod J_1}\}$. $\square$

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Binary Representation of Natural Numbers

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Summary. Binary representation of integers [5], [3] and arithmetic operations on them have already been introduced in Mizar Mathematical Library [8, 7, 6, 4]. However, these articles formalize the notion of integers as mapped into a certain length tuple of boolean values.

In this article we formalize, by means of Mizar system [2], [1], the binary representation of natural numbers which maps $\mathbb{N}$ into bitstreams.

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1. Preliminaries

Let us consider a natural number x. Now we state the propositions:

(1) There exists a natural number m such that $x < 2^m$.

(2) If $x \neq 0$, then there exists a natural number n such that $2^n \leqslant x < 2^{n+1}$.
PROOF: Define $\mathcal{Q}$[natural number] $\equiv x < 2^{\$_1}$. There exists a natural number m such that $\mathcal{Q}[m]$. Consider k being a natural number such that $\mathcal{Q}[k]$ and for every natural number n such that $\mathcal{Q}[n]$ holds $k \leqslant n$. Reconsider $k_1 = k - 1$ as a natural number. $2^{k_1} \leqslant x$. □

(3) Let us consider a natural number x, and natural numbers n_1, n_2. If $2^{n_1} \leqslant x < 2^{n_1+1}$ and $2^{n_2} \leqslant x < 2^{n_2+1}$, then $n_1 = n_2$.

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(4) $\langle 0 \rangle = \langle \underbrace{0, \ldots, 0}_{1} \rangle$.

(5) Let us consider natural numbers n_1, n_2. Then $\langle \underbrace{0, \ldots, 0}_{n_1} \rangle \frown \langle \underbrace{0, \ldots, 0}_{n_2} \rangle = \langle \underbrace{0, \ldots, 0}_{n_1+n_2} \rangle$.

2. Homomorphism from the Natural Numbers to the Bitstreams

Let x be a natural number. The functor LenBinSeq(x) yielding a non zero natural number is defined by

(Def. 1) (i) $it = 1$, **if** $x = 0$,

(ii) there exists a natural number n such that $2^n \leqslant x < 2^{n+1}$ and $it = n + 1$, **otherwise**.

Let us consider a natural number x. Now we state the propositions:

(6) $x < 2^{\mathrm{LenBinSeq}(x)}$.

(7) $x = \mathrm{AbsVal}(\mathrm{LenBinSeq}(x)\text{-BinarySequence}(x))$. The theorem is a consequence of (6).

(8) Let us consider a natural number n, and an $(n + 1)$-tuple x of *Boolean*. If $x(n + 1) = 1$, then $2^n \leqslant \mathrm{AbsVal}(x) < 2^{n+1}$.

(9) There exists a function F from *Boolean** into $\mathbb{N}$ such that for every element x of *Boolean**, there exists a $(\mathrm{len}\, x)$-tuple x_0 of *Boolean* such that $x = x_0$ and $F(x) = \mathrm{AbsVal}(x_0)$.

PROOF: Define $\mathcal{P}$[element of *Boolean**, object] $\equiv$ there exists a $(\mathrm{len}\,\$_1)$-tuple x_0 of *Boolean* such that $\$_1 = x_0$ and $\$_2 = \mathrm{AbsVal}(x_0)$. For every element x of *Boolean**, there exists an element y of $\mathbb{N}$ such that $\mathcal{P}[x, y]$. Consider f being a function from *Boolean** into $\mathbb{N}$ such that for every element x of *Boolean**, $\mathcal{P}[x, f(x)]$. □

The functor Nat2BinLen yielding a function from $\mathbb{N}$ into *Boolean** is defined by

(Def. 2) for every element x of $\mathbb{N}$, $it(x) = \mathrm{LenBinSeq}(x)\text{-BinarySequence}(x)$.

Now we state the propositions:

(10) Let us consider an element x of $\mathbb{N}$, and a $(\mathrm{LenBinSeq}(x))$-tuple y of *Boolean*. If $(\mathrm{Nat2BinLen})(x) = y$, then $\mathrm{AbsVal}(y) = x$. The theorem is a consequence of (7).

(11) $\mathrm{rng}\,\mathrm{Nat2BinLen} = \{x$, where x is an element of *Boolean*$^* : x(\mathrm{len}\, x) = 1\} \cup \{\langle 0 \rangle\}$.

Proof: For every object z, $z \in \operatorname{rng}\mathrm{Nat2BinLen}$ iff $z \in \{x$, where x is an element of $\mathit{Boolean}^* : x(\operatorname{len} x) = 1\} \cup \{\langle 0 \rangle\}$. □

(12) Nat2BinLen is one-to-one.

Let x, y be elements of $\mathit{Boolean}^*$. Assume $\operatorname{len} x \neq 0$ and $\operatorname{len} y \neq 0$. The functor $\mathrm{MaxLen}(x, y)$ yielding a non zero natural number is defined by the term

(Def. 3) $\max(\operatorname{len} x, \operatorname{len} y)$.

Let K be a natural number and x be an element of $\mathit{Boolean}^*$. The functor $\mathrm{ExtBit}(x, K)$ yielding a K-tuple of $\mathit{Boolean}$ is defined by the term

(Def. 4) $$\begin{cases} x \frown \underbrace{\langle 0, \ldots, 0 \rangle}_{K -' \operatorname{len} x}, & \textbf{if } \operatorname{len} x \leqslant K, \\ x \restriction K, & \textbf{otherwise}. \end{cases}$$

Now we state the propositions:

(13) Let us consider a natural number K, and an element x of $\mathit{Boolean}^*$. Suppose $\operatorname{len} x \leqslant K$. Then $\mathrm{ExtBit}(x, K+1) = \mathrm{ExtBit}(x, K) \frown \langle 0 \rangle$.

(14) Let us consider a non zero natural number K, and an element x of $\mathit{Boolean}^*$. If $\operatorname{len} x = K$, then $\mathrm{ExtBit}(x, K) = x$.

(15) Let us consider a non zero natural number K, K-tuples x, y of $\mathit{Boolean}$, and $(K+1)$-tuples x_1, y_1 of $\mathit{Boolean}$. Suppose $x_1 = x \frown \langle 0 \rangle$ and $y_1 = y \frown \langle 0 \rangle$. Then x_1 and y_1 are summable.

(16) Let us consider a non zero natural number K, and a K-tuple y of $\mathit{Boolean}$. Suppose $y = \underbrace{\langle 0, \ldots, 0 \rangle}_{K}$. Let us consider a non zero natural number n. If $n \leqslant K$, then $y_{/n} = 0$.

(17) Let us consider a non zero natural number K, and K-tuples x, y of $\mathit{Boolean}$. Then $\mathrm{carry}(x, y) = \mathrm{carry}(y, x)$.

(18) Let us consider a non zero natural number K, and K-tuples x, y of $\mathit{Boolean}$. Suppose $y = \underbrace{\langle 0, \ldots, 0 \rangle}_{K}$. Let us consider a non zero natural number n. Suppose $n \leqslant K$. Then

(i) $(\mathrm{carry}(x, y))_{/n} = 0$, and

(ii) $(\mathrm{carry}(y, x))_{/n} = 0$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant K$, then $(\mathrm{carry}(x, y))_{/\$_1} = 0$. For every non zero natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every non zero natural number k, $\mathcal{P}[k]$. □

Let us consider a non zero natural number K and K-tuples x, y of $\mathit{Boolean}$. Now we state the propositions:

(19) $x + y = y + x$. The theorem is a consequence of (17).

(20) If $y = \langle \underbrace{0, \ldots, 0}_{K} \rangle$, then $x + y = x$ and $y + x = x$.

PROOF: For every natural number i such that $i \in \operatorname{Seg} K$ holds $(x+y)(i) = x(i)$. □

(21) Let us consider a non zero natural number K, and K-tuples x, y of *Boolean*. If $x(\operatorname{len} x) = 1$ and $y(\operatorname{len} y) = 1$, then x and y are not summable.

Let us consider a non zero natural number K and K-tuples x, y of *Boolean*. Now we state the propositions:

(22) If x and y are summable, then y and x are summable. The theorem is a consequence of (17).

(23) If x and y are summable and ($x(\operatorname{len} x) = 1$ or $y(\operatorname{len} y) = 1$), then $(x + y)(\operatorname{len}(x + y)) = 1$. The theorem is a consequence of (19) and (22).

(24) Let us consider a non zero natural number K, K-tuples x, y of *Boolean*, and $(K + 1)$-tuples x_1, y_1 of *Boolean*. Suppose x and y are not summable and $x_1 = x \frown \langle 0 \rangle$ and $y_1 = y \frown \langle 0 \rangle$. Then $(x_1 + y_1)(\operatorname{len}(x_1 + y_1)) = 1$.

PROOF: Set $K_1 = K + 1$. Reconsider $S = \operatorname{carry}(x, y) \frown \langle 1 \rangle$ as a K_1-tuple of *Boolean*. $S_{/1} = \mathit{false}$. For every natural number i such that $1 \leqslant i < K_1$ holds $S_{/i+1} = (x_{1/i} \wedge y_{1/i} \vee x_{1/i} \wedge S_{/i}) \vee y_{1/i} \wedge S_{/i}$. □

Let x, y be elements of *Boolean**. The functor $x + y$ yielding an element of *Boolean** is defined by the term

$$\text{(Def. 5)} \quad \begin{cases} y, \textbf{if } \operatorname{len} x = 0, \\ x, \textbf{if } \operatorname{len} y = 0, \\ \operatorname{ExtBit}(x, \operatorname{MaxLen}(x, y)) + \operatorname{ExtBit}(y, \operatorname{MaxLen}(x, y)), \\ \quad \textbf{if } \operatorname{ExtBit}(x, \operatorname{MaxLen}(x, y)) \text{ and } \operatorname{ExtBit}(y, \operatorname{MaxLen}(x, y)) \\ \quad \text{are summable and } \operatorname{len} x \neq 0 \text{ and } \operatorname{len} y \neq 0, \\ \operatorname{ExtBit}(x, \operatorname{MaxLen}(x, y) + 1) + \operatorname{ExtBit}(y, \operatorname{MaxLen}(x, y) + 1), \\ \quad \textbf{otherwise}. \end{cases}$$

Let F be a function from $\mathbb{N}$ into *Boolean** and x be an element of $\mathbb{N}$. Let us note that the functor $F(x)$ yields an element of *Boolean**. Now we state the propositions:

(25) Let us consider an element x of *Boolean**. If $x \in \operatorname{rng} \operatorname{Nat2BinLen}$, then $1 \leqslant \operatorname{len} x$.

(26) Let us consider elements x, y of *Boolean**. Suppose $x, y \in \operatorname{rng} \operatorname{Nat2BinLen}$. Then $x+y \in \operatorname{rng} \operatorname{Nat2BinLen}$. The theorem is a consequence of (11), (25), (4), (18), (16), (20), (14), (21), (23), (13), and (24).

(27) Let us consider a non zero natural number n, an n-tuple x of *Boolean*, natural numbers m, l, and an l-tuple y of *Boolean*. Suppose $y = x \frown \langle \underbrace{0, \ldots, 0}_{m} \rangle$. Then $\operatorname{AbsVal}(y) = \operatorname{AbsVal}(x)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number l for every l-tuple y of *Boolean* such that $y = x \frown \langle \underbrace{0, \ldots, 0}_{\$_1} \rangle$ holds $\mathrm{AbsVal}(y) = \mathrm{AbsVal}(x)$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. $\mathcal{P}[0]$. For every natural number m, $\mathcal{P}[m]$. □

(28) Let us consider a natural number n, an element x of $\mathbb{N}$, and an n-tuple y of *Boolean*. Suppose $y = (\mathrm{Nat2BinLen})(x)$. Then

(i) $n = \mathrm{LenBinSeq}(x)$, and

(ii) $\mathrm{AbsVal}(y) = x$, and

(iii) $(\mathrm{Nat2BinLen})(\mathrm{AbsVal}(y)) = y$.

The theorem is a consequence of (6).

(29) Let us consider elements x, y of $\mathbb{N}$. Then $(\mathrm{Nat2BinLen})(x + y) = (\mathrm{Nat2BinLen})(x) + (\mathrm{Nat2BinLen})(y)$. The theorem is a consequence of (7), (27), (26), (28), (13), and (15).

(30) Let us consider elements x, y of $\mathit{Boolean}^*$. If x, $y \in \mathrm{rng}\,\mathrm{Nat2BinLen}$, then $x + y = y + x$. The theorem is a consequence of (29).

(31) Let us consider elements x, y, z of $\mathit{Boolean}^*$. If x, y, $z \in \mathrm{rng}\,\mathrm{Nat2BinLen}$, then $(x + y) + z = x + (y + z)$. The theorem is a consequence of (29).

3. Homomorphism from the Bitstreams to the Natural Numbers

Let x be an element of $\mathit{Boolean}^*$. The functor $\mathrm{ExtAbsVal}(x)$ yielding a natural number is defined by

(Def. 6) there exists a natural number n and there exists an n-tuple y of *Boolean* such that $y = x$ and $\mathit{it} = \mathrm{AbsVal}(y)$.

Now we state the proposition:

(32) There exists a function F from $\mathit{Boolean}^*$ into $\mathbb{N}$ such that for every element x of $\mathit{Boolean}^*$, $F(x) = \mathrm{ExtAbsVal}(x)$.

Proof: Define $\mathcal{P}[\text{element of } \mathit{Boolean}^*, \text{object}] \equiv \$_2 = \mathrm{ExtAbsVal}(\$_1)$. For every element x of $\mathit{Boolean}^*$, there exists an element y of $\mathbb{N}$ such that $\mathcal{P}[x, y]$. Consider f being a function from $\mathit{Boolean}^*$ into $\mathbb{N}$ such that for every element x of $\mathit{Boolean}^*$, $\mathcal{P}[x, f(x)]$. □

The functor BinLen2Nat yielding a function from $\mathit{Boolean}^*$ into $\mathbb{N}$ is defined by

(Def. 7) for every element x of $\mathit{Boolean}^*$, $\mathit{it}(x) = \mathrm{ExtAbsVal}(x)$.

Let F be a function from *Boolean** into $\mathbb{N}$ and x be an element of *Boolean**. Let us observe that the functor $F(x)$ yields an element of $\mathbb{N}$. Observe that BinLen2Nat is onto.

Now we state the propositions:

(33) Let us consider an element x of *Boolean**, and a natural number K. Suppose $\operatorname{len} x \neq 0$ and $\operatorname{len} x \leqslant K$. Then $\mathrm{ExtAbsVal}(x) = \mathrm{AbsVal}(\mathrm{ExtBit}(x, K))$. The theorem is a consequence of (27).

(34) Let us consider elements x, y of *Boolean**. Then $(\mathrm{BinLen2Nat})(x+y) = (\mathrm{BinLen2Nat})(x) + (\mathrm{BinLen2Nat})(y)$. The theorem is a consequence of (33), (13), and (15).

The functor EqBinLen2Nat yielding an equivalence relation of *Boolean** is defined by

(Def. 8) for every objects x, y, $\langle x, y \rangle \in it$ iff $x, y \in$ *Boolean** and $(\mathrm{BinLen2Nat})(x) = (\mathrm{BinLen2Nat})(y)$.

The functor QuBinLen2Nat yielding a function from Classes EqBinLen2Nat into $\mathbb{N}$ is defined by

(Def. 9) for every element A of Classes EqBinLen2Nat, there exists an object x such that $x \in A$ and $it(A) = (\mathrm{BinLen2Nat})(x)$.

Let us observe that QuBinLen2Nat is one-to-one and onto.

Now we state the proposition:

(35) Let us consider an element x of *Boolean**.
Then $(\mathrm{QuBinLen2Nat})([x]_{\mathrm{EqBinLen2Nat}}) = (\mathrm{BinLen2Nat})(x)$.

Let A, B be elements of Classes EqBinLen2Nat. The functor $A+B$ yielding an element of Classes EqBinLen2Nat is defined by

(Def. 10) there exist elements x, y of *Boolean** such that $x \in A$ and $y \in B$ and $it = [x+y]_{\mathrm{EqBinLen2Nat}}$.

Now we state the proposition:

(36) Let us consider elements A, B of Classes EqBinLen2Nat, and elements x, y of *Boolean**. If $x \in A$ and $y \in B$, then $A + B = [x+y]_{\mathrm{EqBinLen2Nat}}$. The theorem is a consequence of (34).

Let us consider elements A, B of Classes EqBinLen2Nat. Now we state the propositions:

(37) $(\mathrm{QuBinLen2Nat})(A+B) = (\mathrm{QuBinLen2Nat})(A) + (\mathrm{QuBinLen2Nat})(B)$. The theorem is a consequence of (36), (35), and (34).

(38) $A+B = B+A$. The theorem is a consequence of (36), (35), and (34).

(39) Let us consider elements A, B, C of Classes EqBinLen2Nat. Then $(A+B)+C = A+(B+C)$. The theorem is a consequence of (36), (35), and (34).

(40) Let us consider a natural number n, and elements z, z_1 of *Boolean**. Suppose $z = \varepsilon_{Boolean}$ and $z_1 = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

Then $[z]_{\mathrm{EqBinLen2Nat}} = [z_1]_{\mathrm{EqBinLen2Nat}}$.

(41) Let us consider elements A, Z of Classes EqBinLen2Nat, a natural number n, and an element z of *Boolean**. Suppose $Z = [z]_{\mathrm{EqBinLen2Nat}}$ and $z = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then

(i) $A + Z = A$, and

(ii) $Z + A = A$.

The theorem is a consequence of (40), (36), and (38).

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Continuity of Bounded Linear Operators on Normed Linear Spaces[1]

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Summary. In this article, using the Mizar system [1], [2], we discuss the continuity of bounded linear operators on normed linear spaces. In the first section, it is discussed that bounded linear operators on normed linear spaces are uniformly continuous and Lipschitz continuous. Especially, a bounded linear operator on the dense subset of a complete normed linear space has a unique natural extension over the whole space. In the next section, several basic currying properties are formalized.

In the last section, we formalized that continuity of bilinear operator is equivalent to both Lipschitz continuity and local continuity. We referred to [4], [13], and [3] in this formalization.

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1. Uniform Continuity and Lipschitz Continuity of Bounded Linear Operators

From now on S, T, W, Y denote real normed spaces, f denotes partial function from S to T, Z denotes a subset of S, and i, n denote natural numbers.

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Now we state the propositions:

(1) Let us consider real normed spaces E, F, a subset E_1 of E, and a partial function f from E to F. Suppose E_1 is dense and F is complete and $\mathrm{dom}\, f = E_1$ and f is uniformly continuous on E_1. Then

(i) there exists a function g from E into F such that $g \restriction E_1 = f$ and g is uniformly continuous on the carrier of E and for every point x of E, there exists a sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ and $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$ and for every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ holds $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$, and

(ii) for every functions g_1, g_2 from E into F such that $g_1 \restriction E_1 = f$ and g_1 is continuous on the carrier of E and $g_2 \restriction E_1 = f$ and g_2 is continuous on the carrier of E holds $g_1 = g_2$.

Proof: For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent for every real number s such that $0 < s$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(f_* s_0)(m) - (f_* s_0)(n)\| < s$. For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent holds $f_* s_0$ is convergent by [12, (5)]. For every point x of E and for every sequences s_1, s_2 of E such that $\mathrm{rng}\, s_1 \subseteq E_1$ and s_1 is convergent and $\lim s_1 = x$ and $\mathrm{rng}\, s_2 \subseteq E_1$ and s_2 is convergent and $\lim s_2 = x$ holds $\lim(f_* s_1) = \lim(f_* s_2)$ by [7, (14)].

Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = \$_1$ and $f_* s_0$ is convergent and $\$_2 = \lim(f_* s_0)$. For every element x of E, there exists an element y of F such that $\mathcal{P}[x, y]$. Consider g being a function from E into F such that for every element x of E, $\mathcal{P}[x, g(x)]$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$. For every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq E_1$ and s_0 is convergent and $\lim s_0 = x$ holds $f_* s_0$ is convergent and $g(x) = \lim(f_* s_0)$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every points x_1, x_2 of E such that x_1, $x_2 \in$ the carrier of E and $\|x_1 - x_2\| < s$ holds $\|g_{/x_1} - g_{/x_2}\| < r$. For every element x of E, $g_1(x) = g_2(x)$ by [5, (14)], [9, (18)]. $\square$

(2) Let us consider real normed spaces E, F, G, a point f of the real norm space of bounded linear operators from E into F, and a point g of the real norm space of bounded linear operators from F into G. Then there exists a point h of the real norm space of bounded linear operators from E into

G such that

(i) $h = g \cdot f$, and

(ii) $\|h\| \leqslant \|g\| \cdot \|f\|$.

PROOF: Reconsider $L_1 = f$ as a Lipschitzian linear operator from E into F. Reconsider $L_2 = g$ as a Lipschitzian linear operator from F into G. Set $L_3 = L_2 \cdot L_1$. For every real number t such that $t \in \mathrm{PreNorms}(L_3)$ holds $t \leqslant \|g\| \cdot \|f\|$ by [11, (16)]. □

(3) Let us consider real normed spaces E, F. Then every Lipschitzian linear operator from E into F is Lipschitzian on the carrier of E and uniformly continuous on the carrier of E.

PROOF: Consider K being a real number such that $0 \leqslant K$ and for every vector x of E, $\|L(x)\| \leqslant K \cdot \|x\|$. Set $r = K + 1$. Set $E_0 =$ the carrier of E. For every points x_1, x_2 of E such that $x_1, x_2 \in E_0$ holds $\|L_{/x_1} - L_{/x_2}\| \leqslant r \cdot \|x_1 - x_2\|$. □

(4) Let us consider real normed spaces E, F, a subreal normal space E_1 of E, and a point f of the real norm space of bounded linear operators from E_1 into F. Suppose F is complete and there exists a subset E_0 of E such that $E_0 =$ the carrier of E_1 and E_0 is dense. Then

(i) there exists a point g of the real norm space of bounded linear operators from E into F such that $\mathrm{dom}\, g =$ the carrier of E and $g \restriction$(the carrier of E_1) $= f$ and $\|g\| = \|f\|$ and there exists a partial function L_1 from E to F such that $L_1 = f$ and for every point x of E, there exists a sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq$ the carrier of E_1 and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $g(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\mathrm{rng}\, s_0 \subseteq$ the carrier of E_1 and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $g(x) = \lim(L_{1*}s_0)$, and

(ii) for every points g_1, g_2 of the real norm space of bounded linear operators from E into F such that $g_1 \restriction$(the carrier of E_1) $= f$ and $g_2 \restriction$(the carrier of E_1) $= f$ holds $g_1 = g_2$.

PROOF: Consider E_0 being a subset of E such that $E_0 =$ the carrier of E_1 and E_0 is dense. Reconsider $L = f$ as a Lipschitzian linear operator from E_1 into F. Reconsider $L_1 = L$ as a partial function from E to F. Consider K being a real number such that $0 \leqslant K$ and for every vector x of E_1, $\|L(x)\| \leqslant K \cdot \|x\|$. Set $r = K + 1$. For every points x_1, x_2 of E such that $x_1, x_2 \in E_0$ holds $\|L_{1/x_1} - L_{1/x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

There exists a function P_3 from E into F such that $P_3 \restriction E_0 = L_1$ and P_3 is uniformly continuous on the carrier of E and for every point

x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every functions P_1, P_2 from E into F such that $P_1 \restriction E_0 = L_1$ and P_1 is continuous on the carrier of E and $P_2 \restriction E_0 = L_1$ and P_2 is continuous on the carrier of E holds $P_1 = P_2$.

Consider P_3 being a function from E into F such that $P_3 \restriction E_0 = L_1$ and P_3 is uniformly continuous on the carrier of E and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E, there exists a sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ and $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$ and for every point x of E and for every sequence s_0 of E such that $\operatorname{rng} s_0 \subseteq E_0$ and s_0 is convergent and $\lim s_0 = x$ holds $L_{1*}s_0$ is convergent and $P_3(x) = \lim(L_{1*}s_0)$. For every points x, y of E, $P_3(x + y) = P_3(x) + P_3(y)$. For every point x of E and for every real number a, $P_3(a \cdot x) = a \cdot P_3(x)$.

Reconsider $g = P_3$ as a point of the real norm space of bounded linear operators from E into F. For every real number t such that $t \in \operatorname{PreNorms}(L)$ holds $t \leqslant \|g\|$. For every real number t such that $t \in \operatorname{PreNorms}(P_3)$ holds $t \leqslant \|f\|$. For every points g_1, g_2 of the real norm space of bounded linear operators from E into F such that $g_1 \restriction$(the carrier of E_1) $= f$ and $g_2 \restriction$(the carrier of E_1) $= f$ holds $g_1 = g_2$ by (3), [8, (7)], (1). □

2. Basic Properties of Currying

Now we state the propositions:

(5) Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and an object x. If $x \in E$, then $(\operatorname{curry} f)(x)$ is a function from F into G.

(6) Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and an object y. If $y \in F$, then $(\operatorname{curry}' f)(y)$ is a function from E into G.

Let us consider non empty sets E, F, G, a function f from $E \times F$ into G, and objects x, y. Now we state the propositions:

(7) If $x \in E$ and $y \in F$, then $(\mathrm{curry}\, f)(x)(y) = f(x, y)$.

(8) If $x \in E$ and $y \in F$, then $(\mathrm{curry}'\, f)(y)(x) = f(x, y)$.

Let E, F, G be real linear spaces and f be a function from (the carrier of E) $\times$ (the carrier of F) into the carrier of G. We say that f is bilinear if and only if

(Def. 1) for every point v of E such that $v \in \mathrm{dom}(\mathrm{curry}\, f)$ holds $(\mathrm{curry}\, f)(v)$ is an additive, homogeneous function from F into G and for every point v of F such that $v \in \mathrm{dom}(\mathrm{curry}'\, f)$ holds $(\mathrm{curry}'\, f)(v)$ is an additive, homogeneous function from E into G.

3. Equivalence of Some Definitions of Continuity of Bilinear Operators

Now we state the proposition:

(9) Let us consider real linear spaces E, F, G. Then (the carrier of E) $\times$ (the carrier of F) $\longmapsto 0_G$ is bilinear.

Proof: Set $f =$ (the carrier of E) $\times$ (the carrier of F) $\longmapsto 0_G$. For every point x of E, $(\mathrm{curry}\, f)(x)$ is an additive, homogeneous function from F into G. For every point x of F such that $x \in \mathrm{dom}(\mathrm{curry}'\, f)$ holds $(\mathrm{curry}'\, f)(x)$ is an additive, homogeneous function from E into G. □

Let E, F, G be real linear spaces. Observe that there exists a function from (the carrier of E) $\times$ (the carrier of F) into the carrier of G which is bilinear.

Now we state the proposition:

(10) Let us consider real linear spaces E, F, G, and a function L from (the carrier of E) $\times$ (the carrier of F) into the carrier of G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$. The theorem is a consequence of (8) and (7).

Let E, F, G be real linear spaces and f be a function from $E \times F$ into G. We say that f is bilinear if and only if

(Def. 2) there exists a function g from (the carrier of E) $\times$ (the carrier of F) into the carrier of G such that $f = g$ and g is bilinear.

One can verify that there exists a function from $E \times F$ into G which is bilinear.

Let f be a function from $E \times F$ into G, x be a point of E, and y be a point of F. Note that the functor $f(x, y)$ yields a point of G. Now we state the proposition:

(11) Let us consider real linear spaces E, F, G, and a function L from $E \times F$ into G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$.

Let E, F, G be real linear spaces.

A bilinear operator from $E \times F$ into G is a bilinear function from $E \times F$ into G. Let E, F, G be real normed spaces and f be a function from $E \times F$ into G. We say that f is bilinear if and only if

(Def. 3) there exists a function g from (the carrier of E) $\times$ (the carrier of F) into the carrier of G such that $f = g$ and g is bilinear.

Let us note that there exists a function from $E \times F$ into G which is bilinear.

A bilinear operator from $E \times F$ into G is a bilinear function from $E \times F$ into G. From now on E, F, G denote real normed spaces, L denotes a bilinear operator from $E \times F$ into G, x denotes an element of E, and y denotes an element of F.

Let E, F, G be real normed spaces, f be a function from $E \times F$ into G, x be a point of E, and y be a point of F. Note that the functor $f(x, y)$ yields a point of G. Now we state the propositions:

(12) Let us consider real normed spaces E, F, G, and a function L from $E \times F$ into G. Then L is bilinear if and only if for every points x_1, x_2 of E and for every point y of F, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$ and for every point x of E and for every point y of F and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$ and for every point x of E and for every points y_1, y_2 of F, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$ and for every point x of E and for every point y of F and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$.

(13) Let us consider real normed spaces E, F, G, and a bilinear operator f from $E \times F$ into G. Then

(i) f is continuous on the carrier of $E \times F$ iff f is continuous in $0_{E \times F}$, and

(ii) f is continuous on the carrier of $E \times F$ iff there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y of F, $\|f(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$.

Proof: If f is continuous in $0_{E \times F}$, then there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y of F, $\|f(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$ by [9, (7)], [6, (22)], [10, (18)]. If there exists a real number K such that $0 \leqslant K$ and for every point x of E and for every point y of F, $\|f(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$, then f is continuous on the carrier of $E \times F$. □

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