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About Supergraphs. Part II

Sebastian Koch
Johannes Gutenberg University
Mainz, Germany[1]

Summary. In the previous article [5] supergraphs and several specializations to formalize the process of drawing graphs were introduced. In this paper another such operation is formalized in Mizar [1], [2]: drawing a vertex and then immediately drawing edges connecting this vertex with a subset of the other vertices of the graph. In case the new vertex is joined with all vertices of a given graph G, this is known as the join of G and the trivial loopless graph K_1. While the join of two graphs is known and found in standard literature (like [9], [4], [8] and [3]), the operation discribed in this article is not.

Alongside the new operation a mode to reverse the directions of a subset of the edges of a graph is introduced. When all edge directions of a graph are reversed, this is commonly known as the converse of a (directed) graph.

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1. Reversing Edge Directions

From now on G, G_2 denote graphs, V, E denote sets, and v denotes an object.

Let us consider G and E.

A graph given by reversing directions of the edges E of G is a graph defined by

(Def. 1) (i) the vertices of *it* = the vertices of G and the edges of *it* = the edges of G and the source of *it* = (the source of G)+·(the target of G)$\restriction E$ and the target of *it* = (the target of G)+·(the source of G)$\restriction E$, **if** $E \subseteq$ the edges of G,

[1] mailto: skoch02@students.uni-mainz.de

(ii) *it* $\approx G$, **otherwise**.

A graph given by reversing directions of the edges of G is a graph given by reversing directions of the edges of G of G. Now we state the propositions:

(1) Let us consider graphs G_1, G_2 given by reversing directions of the edges E of G. Then $G_1 \approx G_2$.

(2) Let us consider a graph G_1 given by reversing directions of the edges E of G. Suppose $G_1 \approx G_2$. Then G_2 is a graph given by reversing directions of the edges E of G.

Let us consider G_2, E, and a graph G_1 given by reversing directions of the edges E of G_2. Now we state the propositions:

(3) G_2 is a graph given by reversing directions of the edges E of G_1.

(4) (i) the vertices of G_1 = the vertices of G_2, and

(ii) the edges of G_1 = the edges of G_2.

(5) Let us consider a graph G_1 given by reversing directions of the edges of G_2. Then G_2 is a graph given by reversing directions of the edges of G_1. The theorem is a consequence of (4) and (3).

(6) Let us consider a trivial graph G_2, a set E, and a graph G_1. Then $G_1 \approx G_2$ if and only if G_1 is a graph given by reversing directions of the edges E of G_2.

Let us consider G_2, E, a graph G_1 given by reversing directions of the edges E of G_2, and objects v_1, e, v_2. Now we state the propositions:

(7) If $E \subseteq$ the edges of G_2 and $e \in E$, then e joins v_1 to v_2 in G_2 iff e joins v_2 to v_1 in G_1. The theorem is a consequence of (3) and (4).

(8) If $E \subseteq$ the edges of G_2 and $e \notin E$, then e joins v_1 to v_2 in G_2 iff e joins v_1 to v_2 in G_1. The theorem is a consequence of (3) and (4).

(9) e joins v_1 and v_2 in G_2 if and only if e joins v_1 and v_2 in G_1. The theorem is a consequence of (3).

(10) Let us consider a graph G_1 given by reversing directions of the edges E of G_2. Then v is a vertex of G_1 if and only if v is a vertex of G_2.

Let us consider G_2, E, V, and a graph G_1 given by reversing directions of the edges E of G_2. Now we state the propositions:

(11) G_1.edgesBetween$(V) = G_2$.edgesBetween(V).

PROOF:
For every object e, $e \in G_1$.edgesBetween(V) iff $e \in G_2$.edgesBetween(V).
□

(12) G_1.edgesInOut$(V) = G_2$.edgesInOut(V).

PROOF: For every object e, $e \in G_1$.edgesInOut(V) iff $e \in G_2$.edgesInOut(V).
□

(13) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1.edgesInOut() $=$ v_2.edgesInOut(). The theorem is a consequence of (12).

Let us consider G_2, E, and a graph G_1 given by reversing directions of the edges E of G_2. Now we state the propositions:

(14) Every walk of G_2 is a walk of G_1. The theorem is a consequence of (4) and (9).

(15) Every walk of G_1 is a walk of G_2. The theorem is a consequence of (3) and (14).

(16) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a walk W_2 of G_2, and a walk W_1 of G_1. Suppose $E \subseteq$ the edges of G_2 and $W_1 = W_2$ and W_2.edges() $\subseteq E$. Then W_1 is directed if and only if W_2.reverse() is directed.

Proof: For every odd element n of $\mathbb{N}$ such that $n < \operatorname{len} W_1$ holds $W_1(n+1)$ joins $W_1(n)$ to $W_1(n+2)$ in G_1 by [6, (1)], [7, (12)]. □

(17) Let us consider a graph G_1 given by reversing directions of the edges of G_2, a walk W_2 of G_2, and a walk W_1 of G_1. Suppose $W_1 = W_2$. Then W_1 is directed if and only if W_2.reverse() is directed. The theorem is a consequence of (16).

(18) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a walk W_2 of G_2, and a walk W_1 of G_1. If $W_1 = W_2$, then W_1 is chordal iff W_2 is chordal. The theorem is a consequence of (3).

(19) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, and objects v_1, v_2. Then there exists a walk W_1 of G_1 such that W_1 is walk from v_1 to v_2 if and only if there exists a walk W_2 of G_2 such that W_2 is walk from v_1 to v_2. The theorem is a consequence of (15) and (14).

(20) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2). The theorem is a consequence of (19).

(21) Let us consider a graph G_1 given by reversing directions of the edges E of G_2. Then

(i) G_1.componentSet() $=$ G_2.componentSet(), and

(ii) G_1.numComponents() $=$ G_2.numComponents().

The theorem is a consequence of (10) and (20).

Let G be a trivial graph and E be a set. Observe that every graph given by reversing directions of the edges E of G is trivial.

Let G be a non trivial graph. Let us observe that every graph given by reversing directions of the edges E of G is non trivial.

Now we state the propositions:

(22) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a set v, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a graph given by reversing directions of the edges $E \setminus G_1.\mathrm{edgesInOut}(\{v\})$ of G_3. The theorem is a consequence of (11), (2), (3), and (6).

(23) Let us consider a graph G_1 given by reversing directions of the edges E of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_1 is isolated iff v_2 is isolated, and

(ii) v_1 is endvertex iff v_2 is endvertex, and

(iii) v_1 is cut-vertex iff v_2 is cut-vertex.

The theorem is a consequence of (3).

Let us consider G_2, E, and a graph G_1 given by reversing directions of the edges E of G_2. Now we state the propositions:

(24) (i) $G_1.\mathrm{order}() = G_2.\mathrm{order}()$, and

(ii) $G_1.\mathrm{size}() = G_2.\mathrm{size}()$.

The theorem is a consequence of (4).

(25) Suppose $E \subseteq$ the edges of G_2 and G_2 is non-directed-multi and for every objects e_1, e_2, v_1, v_2 such that e_1 joins v_1 and v_2 in G_2 and e_2 joins v_1 and v_2 in G_2 holds $e_1, e_2 \in E$ or $e_1 \notin E$ and $e_2 \notin E$. Then G_1 is non-directed-multi.

PROOF: For every objects e_1, e_2, v_1, v_2 such that e_1 joins v_1 to v_2 in G_1 and e_2 joins v_1 to v_2 in G_1 holds $e_1 = e_2$. □

Let G be a non-directed-multi graph. Let us note that every graph given by reversing directions of the edges of G is non-directed-multi.

Let G be a non non-directed-multi graph. Observe that every graph given by reversing directions of the edges of G is non non-directed-multi.

Let G be a non-multi graph and E be a set. One can verify that every graph given by reversing directions of the edges E of G is non-multi.

Let G be a non non-multi graph. Let us note that every graph given by reversing directions of the edges E of G is non non-multi.

Let G be a loopless graph. One can check that every graph given by reversing directions of the edges E of G is loopless.

Let G be a non loopless graph. One can check that every graph given by reversing directions of the edges E of G is non loopless.

Let G be a connected graph. Let us observe that every graph given by reversing directions of the edges E of G is connected.

Let G be a non connected graph. Observe that every graph given by reversing directions of the edges E of G is non connected.

Let G be an acyclic graph. Note that every graph given by reversing directions of the edges E of G is acyclic.

Let G be a non acyclic graph. One can verify that every graph given by reversing directions of the edges E of G is non acyclic.

Let G be a complete graph. Observe that every graph given by reversing directions of the edges E of G is complete.

Let G be a non complete graph. Observe that every graph given by reversing directions of the edges E of G is non complete.

Let G be a chordal graph. Note that every graph given by reversing directions of the edges E of G is chordal.

Let G be a finite graph. Let us note that every graph given by reversing directions of the edges E of G is finite.

Let G be a non finite graph. One can verify that every graph given by reversing directions of the edges E of G is non finite.

Now we state the propositions:

(26) Let us consider a graph G_1 given by reversing directions of the edges of G_2. Then

(i) the source of G_1 = the target of G_2, and

(ii) the target of G_1 = the source of G_2.

(27) Let us consider a graph G_1 given by reversing directions of the edges of G_2, and objects v_1, e, v_2. Then e joins v_1 to v_2 in G_2 if and only if e joins v_2 to v_1 in G_1. The theorem is a consequence of (26).

2. Adding a Vertex and Several Edges to a Graph

Let us consider G, v, and V.

A supergraph of G extended by vertex v and edges from v to V of G is a supergraph of G defined by

(Def. 2) (i) the vertices of *it* = (the vertices of G) $\cup \{v\}$ and the edges of *it* = (the edges of G) $\cup$ ($V \longmapsto$ (the edges of G)) and the source of *it* = (the source of G)$+\cdot$(($V \longmapsto$ (the edges of G)) $\longmapsto v$) and the target of *it* = (the target of G)$+\cdot\pi_1(V \boxtimes \{$the edges of $G\})$, **if** $V \subseteq$ the vertices of G and $v \notin$ the vertices of G,

(ii) $it \approx G$, **otherwise**.

A supergraph of G extended by vertex v and edges from V of G to v is a supergraph of G defined by

(Def. 3) (i) the vertices of it = (the vertices of G) $\cup$ $\{v\}$ and the edges of it = (the edges of G) $\cup$ ($V \longmapsto$ (the edges of G)) and the source of it = (the source of G)$+\cdot\pi_1(V \boxtimes \{$the edges of $G\})$ and the target of it = (the target of G)$+\cdot((V \longmapsto$ (the edges of G)) $\longmapsto v)$, **if** $V \subseteq$ the vertices of G and $v \notin$ the vertices of G,

(ii) $it \approx G$, **otherwise**.

A supergraph of G extended by vertex v and edges from v to the vertices of G is a supergraph of G extended by vertex v and edges from v to the vertices of G of G.

A supergraph of G extended by vertex v and edges from the vertices of G to v is a supergraph of G extended by vertex v and edges from the vertices of G of G to v. Now we state the propositions:

(28) Let us consider supergraphs G_1, G_2 of G extended by vertex v and edges from v to V of G. Then $G_1 \approx G_2$.

(29) Let us consider supergraphs G_1, G_2 of G extended by vertex v and edges from V of G to v. Then $G_1 \approx G_2$.

(30) Let us consider a supergraph G_1 of G extended by vertex v and edges from v to V of G. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by vertex v and edges from v to V of G.

(31) Let us consider a supergraph G_1 of G extended by vertex v and edges from V of G to v. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by vertex v and edges from V of G to v.

(32) Let us consider a supergraph G_1 of G extended by vertex v and edges from v to V of G, and a supergraph G_2 of G extended by vertex v and edges from V of G to v. Then

(i) the vertices of G_1 = the vertices of G_2, and

(ii) the edges of G_1 = the edges of G_2.

(33) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from v to V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_1.edgesOutOf($\{v\}$) $= V \longmapsto$ (the edges of G_2).

PROOF: For every object e, $e \in G_1$.edgesOutOf($\{v\}$) iff $e \in V \longmapsto$ (the edges of G_2). $\square$

(34) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from V of G_2 to v. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_1.edgesInto($\{v\}$) $= V \longmapsto$ (the edges of G_2).

Proof: For every object e, $e \in G_1$.edgesInto($\{v\}$) iff $e \in V \longmapsto$ (the edges of G_2). $\square$

(35) Let us consider a supergraph G_1 of G extended by vertex v and edges from v to V of G, and a supergraph G_2 of G extended by vertex v and edges from V of G to v. Suppose $V \subseteq$ the vertices of G and $v \notin$ the vertices of G. Then

(i) G_2 is a graph given by reversing directions of the edges G_1.edgesOutOf($\{v\}$) of G_1, and

(ii) G_1 is a graph given by reversing directions of the edges G_2.edgesInto($\{v\}$) of G_2.

The theorem is a consequence of (33) and (34).

(36) Let us consider a supergraph G_1 of G extended by vertex v and edges from v to V of G, a supergraph G_2 of G extended by vertex v and edges from V of G to v, and objects v_1, e, v_2. Then e joins v_1 and v_2 in G_1 if and only if e joins v_1 and v_2 in G_2. The theorem is a consequence of (35) and (9).

(37) Let us consider a supergraph G_1 of G extended by vertex v and edges from v to V of G, a supergraph G_2 of G extended by vertex v and edges from V of G to v, and an object w. Then w is a vertex of G_1 if and only if w is a vertex of G_2.

(38) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from v to V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Let us consider objects e_1, u. Then

(i) e_1 does not join u to v in G_1, and

(ii) if $u \notin V$, then e_1 does not join v to u in G_1, and

(iii) for every object e_2 such that e_1 joins v to u in G_1 and e_2 joins v to u in G_1 holds $e_1 = e_2$.

Proof: e_1 does not join u to v in G_1. If $u \notin V$, then e_1 does not join v to u in G_1. $e_1 \notin$ the edges of G_2 and $e_2 \notin$ the edges of G_2. Consider x_1, y_1 being objects such that $x_1 \in V$ and $y_1 \in \{$the edges of $G_2\}$ and $e_1 = \langle x_1, y_1 \rangle$. Consider x_2, y_2 being objects such that $x_2 \in V$ and $y_2 \in \{$the edges of $G_2\}$ and $e_2 = \langle x_2, y_2 \rangle$. $\square$

(39) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from V of G_2 to v. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Let us consider objects e_1, u. Then

(i) e_1 does not join v to u in G_1, and

(ii) if $u \notin V$, then e_1 does not join u to v in G_1, and

(iii) for every object e_2 such that e_1 joins u to v in G_1 and e_2 joins u to v in G_1 holds $e_1 = e_2$.

PROOF: e_1 does not join v to u in G_1. If $u \notin V$, then e_1 does not join u to v in G_1. $e_1 \notin$ the edges of G_2 and $e_2 \notin$ the edges of G_2. Consider x_1, y_1 being objects such that $x_1 \in V$ and $y_1 \in \{$the edges of $G_2\}$ and $e_1 = \langle x_1, y_1 \rangle$. Consider x_2, y_2 being objects such that $x_2 \in V$ and $y_2 \in \{$the edges of $G_2\}$ and $e_2 = \langle x_2, y_2 \rangle$. □

(40) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from v to V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Let us consider objects e, v_1, v_2. Suppose $v_1 \neq v$. If e joins v_1 to v_2 in G_1, then e joins v_1 to v_2 in G_2.
PROOF: $e \in$ the edges of G_2. □

(41) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from V of G_2 to v. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Let us consider objects e, v_1, v_2. Suppose $v_2 \neq v$. If e joins v_1 to v_2 in G_1, then e joins v_1 to v_2 in G_2.
PROOF: $e \in$ the edges of G_2. □

(42) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from v to V of G_2, and an object v_1. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and $v_1 \in V$. Then $\langle v_1$, the edges of $G_2 \rangle$ joins v to v_1 in G_1.

(43) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges from V of G_2 to v, and an object v_1. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and $v_1 \in V$. Then $\langle v_1$, the edges of $G_2 \rangle$ joins v_1 to v in G_1.

Let us consider G, v, V, a supergraph G_1 of G extended by vertex v and edges from v to V of G, and a supergraph G_2 of G extended by vertex v and edges from V of G to v. Now we state the propositions:

(44) Every walk of G_1 is a walk of G_2. The theorem is a consequence of (35) and (14).

(45) Every walk of G_2 is a walk of G_1. The theorem is a consequence of (35) and (14).

Let us consider G, v, and V.

A supergraph of G extended by vertex v and edges between v and V of G is a supergraph of G defined by

(Def. 4) (i) the vertices of *it* = (the vertices of G) $\cup \{v\}$ and for every object e, e does not join v and v in *it* and for every object v_1, if $v_1 \notin V$, then e does not join v_1 and v in *it* and for every object v_2 such that $v_1 \neq v$

and $v_2 \neq v$ and e joins v_1 to v_2 in *it* holds e joins v_1 to v_2 in G and there exists a set E such that $\overline{\overline{V}} = \overline{\overline{E}}$ and E misses the edges of G and the edges of *it* = (the edges of G) $\cup E$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in *it* and for every object e_2 such that e_2 joins v_1 and v in *it* holds $e_1 = e_2$, **if** $V \subseteq$ the vertices of G and $v \notin$ the vertices of G,

(ii) *it* $\approx G$, **otherwise**.

A supergraph of G extended by vertex v and edges between v and the vertices of G is a supergraph of G extended by vertex v and edges between v and the vertices of G of G.

One can verify that a supergraph of G extended by vertex v and edges from v to V of G is a supergraph of G extended by vertex v and edges between v and V of G.

Note that a supergraph of G extended by vertex v and edges from V of G to v is a supergraph of G extended by vertex v and edges between v and V of G. Now we state the propositions:

(46) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and $\emptyset$ of G_2. Then the edges of G_2 = the edges of G_1.

(47) Let us consider a non empty set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then the edges of $G_1 \neq \emptyset$.

(48) Let us consider a supergraph G_1 of G extended by vertex v and edges between v and V of G. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by vertex v and edges between v and V of G.

(49) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and objects v_1, e, v_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and $v_1 \neq v$ and $v_2 \neq v$ and e joins v_1 and v_2 in G_1. Then e joins v_1 and v_2 in G_2.

(50) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then v is a vertex of G_1.

(51) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a set E, and objects v_1, e, v_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and the edges of G_1 = (the edges of G_2) $\cup E$ and E misses the edges of G_2 and e joins v_1 and v_2 in G_1 and $e \notin$ the edges of G_2. Then

(i) $e \in E$, and

(ii) $v_1 = v$ and $v_2 \in V$ or $v_2 = v$ and $v_1 \in V$.

(52) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and a set E. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and the edges of $G_1 =$ (the edges of G_2) $\cup E$ and E misses the edges of G_2. Then there exist functions f, g from E into $V \cup \{v\}$ such that

(i) the source of $G_1 =$ (the source of G_2)$+\cdot f$, and

(ii) the target of $G_1 =$ (the target of G_2)$+\cdot g$, and

(iii) for every object e such that $e \in E$ holds e joins $f(e)$ to $g(e)$ in G_1 and ($f(e) = v$ iff $g(e) \neq v$).

Proof: Consider E_1 being a set such that $\overline{\overline{V}} = \overline{\overline{E_1}}$ and E_1 misses the edges of G_2 and the edges of $G_1 =$ (the edges of G_2) $\cup E_1$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E_1$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an object v_2 such that $\$_1$ joins $\$_2$ to v_2 in G_1. For every object e such that $e \in E$ there exists an object v_1 such that $v_1 \in V \cup \{v\}$ and $\mathcal{P}[e, v_1]$.

Consider f being a function from E into $V \cup \{v\}$ such that for every object e such that $e \in E$ holds $\mathcal{P}[e, f(e)]$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv \$_1$ joins $f(\$_1)$ to $\$_2$ in G_1. For every object e such that $e \in E$ there exists an object v_2 such that $v_2 \in V \cup \{v\}$ and $\mathcal{Q}[e, v_2]$.

Consider g being a function from E into $V \cup \{v\}$ such that for every object e such that $e \in E$ holds $\mathcal{Q}[e, g(e)]$. For every object e such that $e \in \text{dom}$(the source of G_1) holds (the source of G_1)$(e) =$ ((the source of G_2)$+\cdot f$)(e). For every object e such that $e \in \text{dom}$(the target of G_1) holds (the target of G_1)$(e) =$ ((the target of G_2)$+\cdot g$)(e). □

(53) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then the edges of $G_2 = G_1$.edgesBetween(the vertices of G_2).

Proof: Set $B = G_1$.edgesBetween(the vertices of G_2). For every object e, $e \in$ the edges of G_2 iff $e \in B$. □

(54) Let us consider a graph G_2, sets v, V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_2 is a subgraph of G_1 with vertex v removed. The theorem is a consequence of (53).

(55) Every supergraph of G_2 extended by vertex v and edges between v and $\emptyset$ of G_2 is a supergraph of G_2 extended by v. The theorem is a consequence of (46).

(56) Let us consider an object v_1, and a supergraph G_1 of G_2 extended by vertex v and edges between v and $\{v_1\}$ of G_2. Suppose $v_1 \in$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then there exists an object e such that

(i) $e \notin$ the edges of G_2, and

(ii) G_1 is supergraph of G_2 extended by vertices v, v_1 and e between them or supergraph of G_2 extended by vertices v_1, v and e between them.

The theorem is a consequence of (52).

(57) Let us consider a subset W of V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then there exists a function f from W into G_1.edgesBetween$(W, \{v\})$ such that

(i) f is one-to-one and onto, and

(ii) for every object w such that $w \in W$ holds $f(w)$ joins w and v in G_1.

Proof: Consider E being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and E misses the edges of G_2 and the edges of $G_1 =$ (the edges of $G_2) \cup E$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2$ joins $\$_1$ and v in G_1. For every object w such that $w \in W$ there exists an object e such that $e \in G_1$.edgesBetween$(W, \{v\})$ and $\mathcal{P}[w, e]$.

Consider f being a function from W into G_1.edgesBetween$(W, \{v\})$ such that for every object w such that $w \in W$ holds $\mathcal{P}[w, f(w)]$. For every objects w_1, w_2 such that $w_1, w_2 \in W$ and $f(w_1) = f(w_2)$ holds $w_1 = w_2$. For every object e such that $e \in G_1$.edgesBetween$(W, \{v\})$ holds $e \in \operatorname{rng} f$. □

(58) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and E misses the edges of G_2 and the edges of $G_1 =$ (the edges of $G_2) \cup E$. Then $E = G_1$.edgesBetween$(V, \{v\})$.

Proof: Consider E_1 being a set such that $\overline{\overline{V}} = \overline{\overline{E_1}}$ and E_1 misses the edges of G_2 and the edges of $G_1 =$ (the edges of $G_2) \cup E_1$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E_1$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. For every object e, $e \in E$ iff $e \in G_1$.edgesBetween$(V, \{v\})$. □

(59) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices

of G_2. Then

(i) G_1.edgesBetween$(V, \{v\})$ misses the edges of G_2, and

(ii) the edges of $G_1 =$ (the edges of G_2) $\cup$ G_1.edgesBetween$(V, \{v\})$.

PROOF: G_1.edgesBetween$(V, \{v\}) \cap$ (the edges of G_2) $= \emptyset$. For every object e such that $e \in$ the edges of G_1 holds $e \in$ (the edges of G_2) $\cup$ G_1.edgesBetween$(V, \{v\})$. □

(60) Let us consider a graph G_3, an object v, sets V_1, V_2, a supergraph G_1 of G_3 extended by vertex v and edges between v and $V_1 \cup V_2$ of G_3, and a subgraph G_2 of G_1 with edges G_1.edgesBetween$(V_2, \{v\})$ removed. Suppose $V_1 \cup V_2 \subseteq$ the vertices of G_3 and $v \notin$ the vertices of G_3 and V_1 misses V_2. Then G_2 is a supergraph of G_3 extended by vertex v and edges between v and V_1 of G_3.

PROOF: Consider E being a set such that $\overline{\overline{V_1 \cup V_2}} = \overline{\overline{E}}$ and E misses the edges of G_3 and the edges of $G_1 =$ (the edges of G_3) $\cup E$ and for every object v_1 such that $v_1 \in V_1 \cup V_2$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. $E = G_1$.edgesBetween$(V_1 \cup V_2, \{v\})$. For every object e such that $e \in$ the edges of G_3 holds $e \in$ (the edges of G_3) $\setminus$ G_1.edgesBetween$(V_2, \{v\})$. G_2 is a supergraph of G_3. □

(61) Let us consider a graph G_3, an object v, a set V, a vertex v_1 of G_3, and a supergraph G_1 of G_3 extended by vertex v and edges between v and $V \cup \{v_1\}$ of G_3. Suppose $V \subseteq$ the vertices of G_3 and $v \notin$ the vertices of G_3 and $v_1 \notin V$.

Then there exists a supergraph G_2 of G_3 extended by vertex v and edges between v and V of G_3 and there exists an object e such that $e \notin$ the edges of G_3 and G_1 is supergraph of G_2 extended by e between vertices v and v_1 or supergraph of G_2 extended by e between vertices v_1 and v.

PROOF: Reconsider $W = \{v_1\}$ as a subset of $V \cup \{v_1\}$. Consider f being a function from W into G_1.edgesBetween$(W, \{v\})$ such that f is one-to-one and onto and for every object w such that $w \in W$ holds $f(w)$ joins w and v in G_1. $f(v_1) \notin$ the edges of G_3. v is a vertex of G_1. □

(62) Let us consider a graph G_3, an object v, a set V, a vertex v_1 of G_3, an object e, and a supergraph G_2 of G_3 extended by vertex v and edges between v and V of G_3. Suppose $V \subseteq$ the vertices of G_3 and $v \notin$ the vertices of G_3 and $v_1 \notin V$ and $e \notin$ the edges of G_2.

Let us consider a graph G_1. Suppose G_1 is supergraph of G_2 extended by e between vertices v_1 and v or supergraph of G_2 extended by e between vertices v and v_1. Then G_1 is a supergraph of G_3 extended by vertex v and edges between v and $V \cup \{v_1\}$ of G_3.

Proof: Consider E being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and E misses the edges of G_3 and the edges of $G_2 =$ (the edges of $G_3) \cup E$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in G_2 and for every object e_2 such that e_2 joins v_1 and v in G_2 holds $e_1 = e_2$. Consider f being a function such that f is one-to-one and $\operatorname{dom} f = E$ and $\operatorname{rng} f = V$. Set $f_1 = f {+\cdot} (e \dot{\longmapsto} v_1)$. $\operatorname{rng} f \cap \operatorname{rng}(e \dot{\longmapsto} v_1) = \emptyset$. For every object w such that $w \in \operatorname{rng} f \cup \operatorname{rng}(e \dot{\longmapsto} v_1)$ holds $w \in \operatorname{rng} f_1$. v is a vertex of G_2 and v_1 is a vertex of G_3. $\square$

Let us consider G_2, v, V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and a walk W of G_1. Now we state the propositions:

(63) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then

(i) if W.edges() $\subseteq$ the edges of G_2 and W is not trivial, then $v \notin W$.vertices(), and

(ii) if $v \notin W$.vertices(), then W.edges() $\subseteq$ the edges of G_2.

Proof: Consider E being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and E misses the edges of G_2 and the edges of $G_1 =$ (the edges of $G_2) \cup E$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. For every object e such that $e \in W$.edges() holds $e \in$ the edges of G_2. $\square$

(64) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and (W.edges() $\subseteq$ the edges of G_2 and W is not trivial or $v \notin W$.vertices()). Then W is a walk of G_2. The theorem is a consequence of (63).

(65) If W.vertices() $\subseteq$ the vertices of G_2, then W.edges() $\subseteq$ the edges of G_2. The theorem is a consequence of (63).

(66) Let us consider supergraphs G_1, G_2 of G extended by vertex v and edges between v and V of G. Then

(i) the vertices of $G_1 =$ the vertices of G_2, and

(ii) every vertex of G_1 is a vertex of G_2.

Proof: The vertices of $G_1 =$ the vertices of G_2. $\square$

(67) Let us consider supergraphs G_1, G_2 of G extended by vertex v and edges between v and V of G, and objects v_1, e_1, v_2. Suppose e_1 joins v_1 and v_2 in G_1. Then there exists an object e_2 such that e_2 joins v_1 and v_2 in G_2.

(68) Let us consider supergraphs G_1, G_2 of G extended by vertex v and edges between v and V of G. Then there exists a function f from the edges of G_1 into the edges of G_2 such that

(i) $f\restriction$(the edges of G) $= \mathrm{id}_\alpha$, and

(ii) f is one-to-one and onto, and

(iii) for every objects v_1, e, v_2 such that e joins v_1 and v_2 in G_1 holds $f(e)$ joins v_1 and v_2 in G_2,

where α is the edges of G. The theorem is a consequence of (67), (47), and (51).

Let G be a loopless graph. Let us consider v and V. Observe that every supergraph of G extended by vertex v and edges between v and V of G is loopless.

Let G be a non-directed-multi graph. Let us note that every supergraph of G extended by vertex v and edges between v and V of G is non-directed-multi.

Let G be a non-multi graph. Note that every supergraph of G extended by vertex v and edges between v and V of G is non-multi.

Let G be a directed-simple graph. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is directed-simple.

Let G be a simple graph. Let us observe that every supergraph of G extended by vertex v and edges between v and V of G is simple.

Now we state the proposition:

(69) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a walk W of G_1, and vertices v_1, v_2 of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and W.first() $= v_1$ and W.last() $= v_2$ and $v_2 \notin G_2$.reachableFrom(v_1). Then $v \in W$.vertices(). The theorem is a consequence of (64).

Let us consider G_2, v, V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Now we state the propositions:

(70) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and G_2 is acyclic and for every component G_3 of G_2 and for every vertices w_1, w_2 of G_3 such that $w_1, w_2 \in V$ holds $w_1 = w_2$. Then G_1 is acyclic.

PROOF: Consider E being a set such that $\overline{\overline{V}} = \overline{\overline{E}}$ and E misses the edges of G_2 and the edges of $G_1 =$ (the edges of G_2) $\cup E$ and for every object v_1 such that $v_1 \in V$ there exists an object e_1 such that $e_1 \in E$ and e_1 joins v_1 and v in G_1 and for every object e_2 such that e_2 joins v_1 and v in G_1 holds $e_1 = e_2$. There exists no walk W of G_1 such that W is cycle-like. □

(71) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and (G_2 is not acyclic or there exists a component G_3 of G_2 and there exist vertices w_1, w_2 of G_3 such that $w_1, w_2 \in V$ and $w_1 \neq w_2$). Then G_1 is not acyclic.

(72) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and for every component G_3 of G_2, there exists a vertex w of G_3 such that $w \in V$. Then G_1 is connected.

PROOF: For every vertex u of G_1 such that $u \neq v$ there exists a walk W_1 of G_1 such that W_1 is walk from u to v. For every vertices u, w of G_1, there exists a walk W_1 of G_1 such that W_1 is walk from u to w. □

Let G be a connected graph, v be an object, and V be a non empty set. Note that every supergraph of G extended by vertex v and edges between v and V of G is connected.

Let us consider G_2, v, V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Now we state the propositions:

(73) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and there exists a component G_3 of G_2 such that for every vertex w of G_3, $w \notin V$. Then G_1 is not connected.

PROOF: Consider G_3 being a component of G_2 such that for every vertex w of G_3, $w \notin V$. Set $v_1 =$ the vertex of G_3. There exists no walk W of G_1 such that W is walk from v_1 to v. □

(74) Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and there exists a component G_3 of G_2 such that the vertices of G_3 misses V. Then G_1 is not connected. The theorem is a consequence of (73).

Let G be a non connected graph and v be an object. One can check that every supergraph of G extended by vertex v and edges between v and $\emptyset$ of G is non connected.

(75) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_1 is complete if and only if G_2 is complete and $V =$ the vertices of G_2.

PROOF: For every vertices u, v of G_1 such that $u \neq v$ holds u and v are adjacent. □

Let G be a complete graph. Observe that every supergraph of G extended by vertex the vertices of G and edges between the vertices of G and the vertices of G is complete.

Now we state the propositions:

(76) Let us consider a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then

(i) G_1.order() $= G_2$.order() $+ 1$, and

(ii) G_1.size() $= G_2$.size() $+ \overline{\overline{V}}$.

(77) Let us consider a finite graph G_2, an object v, a set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_1.order() $=$ G_2.order() $+ 1$.

(78) Let us consider a finite graph G_2, an object v, a finite set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then G_1.size() $=$ G_2.size() $+ \overline{\overline{V}}$.

Let G be a finite graph, v be an object, and V be a set. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is finite.

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