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Concatenation of Finite Sequences

Rafał Ziobro
Department of Carbohydrate Technology
University of Agriculture
Krakow, Poland

Summary. The coexistence of "classical" *finite sequences* [1] and their zero-based equivalents *finite 0-sequences* [6] in Mizar has been regarded as a disadvantage. However the suggested replacement of the former type with the latter [5] has not yet been implemented, despite of several advantages of this form, such as the identity of length and domain operators [4]. On the other hand the number of theorems formalized using *finite sequence* notation is much larger then of those based on *finite 0-sequences*, so such translation would require quite an effort.

The paper addresses this problem with another solution, using the Mizar system [3], [2]. Instead of removing one notation it is possible to introduce operators which would concatenate sequences of various types, and in this way allow utilization of the whole range of formalized theorems. While the operation could replace existing `FS2XFS`, `XFS2FS` commands (by using empty sequences as initial elements) its universal notation (independent on sequences that are concatenated to the initial object) allows to "forget" about the type of sequences that are concatenated on further positions, and thus simplify the proofs.

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1. Preliminaries

Let a be a real number and b be a non negative real number. One can check that $a -' (a + b)$ is zero.

One can check that $a + b -' a$ reduces to b.

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Let n, m be natural numbers. We identify $n \cap m$ with $\min(m, n)$. We identify $\min(m, n)$ with $n \cap m$. We identify $\max(m, n)$ with $n \cup m$. Let n, m be non negative real numbers. Observe that $\min(n + m, n)$ reduces to n and $\max(n + m, n)$ reduces to $n + m$.

Now we state the propositions:

(1) Let us consider a binary relation f, and natural numbers n, m. Then $(f \restriction (n + m)) \restriction n = f \restriction n$.

(2) Let us consider a function f, a natural number n, and a non zero natural number m. Then $(f \restriction (n + m))(n) = f(n)$.

Let D be a non empty set, x be a sequence of D, and n be a natural number. Let us note that $\mathrm{dom}(x \restriction n)$ reduces to n. Observe that $x \restriction n$ is finite and transfinite sequence-like and $x \restriction n$ is n-element.

2. Complex-Valued Sequences

Now we state the proposition:

(3) Let us consider complex-valued functions f, g, and a set X. Then $(f \cdot g) \restriction X = (f \restriction X) \cdot (g \restriction X)$.

Proof: For every object x such that $x \in \mathrm{dom}((f \cdot g) \restriction X)$ holds $((f \cdot g) \restriction X)(x) = ((f \restriction X) \cdot (g \restriction X))(x)$. $\square$

Let D be a non empty set and f, g be sequences of D. Let us note that $f {+}{\cdot} g$ is transfinite sequence-like.

Let f be a constant complex sequence and n be a natural number. Let us note that $f \uparrow n$ is constant and there exists a complex sequence which is empty yielding and there exists a sequence of real numbers which is empty yielding and every complex sequence which is empty yielding is also natural-valued and there exists a complex sequence which is constant and real-valued.

Now we state the proposition:

(4) Let us consider a sequence s of real numbers, and a natural number n. Then $((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum (s \restriction \mathbb{Z}_{n+1})$.

Let c be a complex number. The functor $\{c\}_{n \in \mathbb{N}}$ yielding a complex sequence is defined by the term

(Def. 1) $\mathbb{N} \longmapsto c$.

Let n be a natural number. One can check that $(\{c\}_{n \in \mathbb{N}})(n)$ reduces to c.

Now we state the proposition:

(5) Let us consider complex-valued functions f, g, and a set X. Then $(f + g) \restriction X = f \restriction X + g \restriction X$.

Proof: For every object x such that $x \in \mathrm{dom}((f + g) \restriction X)$ holds $((f + g) \restriction X)(x) = (f \restriction X + g \restriction X)(x)$. $\square$

Let f be a 1-element finite sequence. One can verify that $\langle f(1)\rangle$ reduces to f.

Let f be a 2-element finite sequence. Let us note that $\langle f(1), f(2)\rangle$ reduces to f.

Let f be a 3-element finite sequence. Let us note that $\langle f(1), f(2), f(3)\rangle$ reduces to f.

Now we state the propositions:

(6) Let us consider a complex-valued finite sequence f. Then $\sum f = f(1) + \sum f_{\downharpoonright 1}$.

(7) Let us consider a non empty, complex-valued finite sequence f. Then $\prod f = f(1) \cdot (\prod f_{\downharpoonright 1})$.

(8) Let us consider a natural number n, a non zero natural number m, and an $(n+m)$-element finite sequence f. Then $f \upharpoonright (n+1) = (f \upharpoonright n) \frown \langle f(n+1)\rangle$.

(9) Let us consider a complex-valued finite sequence f, and a natural number n. Then $\prod f = \prod (f \upharpoonright n) \cdot \prod f_{\downharpoonright n}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \prod f = (\prod (f \upharpoonright \$_1)) \cdot (\prod f_{\downharpoonright \$_1})$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [8, (35)], (7). For every natural number x, $\mathcal{P}[x]$. □

(10) Let us consider complex-valued finite sequences f, g. Then $\prod (f \frown g) = (\prod f) \cdot (\prod g)$. The theorem is a consequence of (9).

3. On Product and Sum of Complex Sequences

Let s be a complex sequence. The partial product of s yielding a complex sequence is defined by

(Def. 2) $it(0) = s(0)$ and for every natural number n, $it(n+1) = it(n) \cdot s(n+1)$.

Now we state the propositions:

(11) Let us consider a complex sequence f, and a natural number n. Suppose $f(n) = 0$. Then (the partial product of f)$(n) = 0$.

(12) Let us consider a complex sequence f, and natural numbers n, m. Suppose $f(n) = 0$. Then (the partial product of f)$(n+m) = 0$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial product of f)$(n+\$_1) = 0$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. □

Let c be a complex number and n be a non zero natural number. Observe that the functor c^n is defined by the term

(Def. 3) (the partial product of $\{c\}_{n \in \mathbb{N}}$)$(n-1)$.

Now we state the proposition:

(13) Let us consider a natural number n. Then (the partial product of $\{0_{\mathbb{C}}\}_{n\in\mathbb{N}})(n) = 0$. The theorem is a consequence of (12).

Let k be a natural number. Let us note that (the partial product of $\{0\}_{n\in\mathbb{N}})(k)$ reduces to 0.

One can verify that every complex sequence which is empty yielding is also absolutely summable and every sequence of real numbers which is empty yielding is also absolutely summable.

Observe that $(\sum_{\alpha=0}^{\kappa}(\mathbb{N} \longmapsto 0)(\alpha))_{\kappa\in\mathbb{N}}$ reduces to $\mathbb{N} \longmapsto 0$ and the partial product of $\{0\}_{n\in\mathbb{N}}$ reduces to $\{0\}_{n\in\mathbb{N}}$. One can verify that every complex sequence is transfinite sequence-like and there exists a sequence of $\mathbb{C}$ which is summable.

Let s_1 be an empty yielding complex sequence. One can check that $\sum s_1$ is zero.

Let s_1 be an empty yielding sequence of real numbers. Let us note that $\sum s_1$ is zero.

4. Finite 0-sequences

Let c be a complex number. Observe that $\langle c \rangle$ is complex-valued.

One can verify that $\sum\langle c \rangle$ reduces to c.

Let n be a natural number. One can verify that there exists a natural-valued finite 0-sequence which is n-element.

Let k be an object. One can check that $n \longmapsto k$ is n-element and there exists a finite 0-sequence which is n-element.

Let f be an n-element finite 0-sequence. Let us note that $f{\restriction}n$ reduces to f.

Let n, m be natural numbers. One can check that $f{\restriction}(n+m)$ reduces to f.

Let f be a 1-element finite 0-sequence. Let us note that $\langle f(0) \rangle$ reduces to f.

Let f be a 2-element finite 0-sequence. Let us note that $\langle f(0), f(1) \rangle$ reduces to f.

Let f be a 3-element finite 0-sequence. One can verify that $\langle f(0), f(1), f(2) \rangle$ reduces to f.

Now we state the propositions:

(14) Let us consider natural numbers n, k. If $k \in \mathbb{Z}_{n+1}$, then $n-k$ is a natural number.

(15) Let us consider complex numbers a, b, and natural numbers n, k. Suppose $k \in \mathbb{Z}_{n+1}$. Then there exists an object c and there exists a natural number l such that $l = n - k$ and $c = a^l \cdot (b^k)$. The theorem is a consequence of (14).

5. SHIFTING SEQUENCES

Let f be a complex-valued finite 0-sequence and s_1 be a complex sequence. The functor $f \frown s_1$ yielding a complex sequence is defined by the term

(Def. 4) $f \cup \mathrm{Shift}(s_1, \mathrm{len}\, f)$.

Let f be a function. The functor $s_1 \frown f$ yielding a sequence of $\mathbb{C}$ is defined by the term

(Def. 5) s_1.

Now we state the propositions:

(16) Let us consider an object x. Then x is a real-valued complex sequence if and only if x is a sequence of real numbers.

(17) Let us consider a sequence r_1 of real numbers, and a complex sequence c_1. Suppose $c_1 = r_1$. Then the partial product of r_1 = the partial product of c_1.

Let f be a complex-valued finite 0-sequence and s_1 be a sequence of real numbers. The functor $f \frown s_1$ yielding a complex sequence is defined by the term

(Def. 6) $f \cup \mathrm{Shift}(s_1, \mathrm{len}\, f)$.

Now we state the proposition:

(18) Let us consider a sequence r_1 of real numbers. Then $\langle\rangle_{\mathbb{R}} \frown r_1$ is a real-valued complex sequence.

Let f be a sequence of real numbers and g be a function. The functor $f \frown g$ yielding a real-valued sequence of $\mathbb{C}$ is defined by the term

(Def. 7) f.

Let f be a complex-valued finite 0-sequence and s_1 be a complex sequence. Let us observe that $(f \frown s_1) \restriction \mathrm{dom}\, f$ reduces to f.

Let s_1 be a sequence of real numbers. Let us note that $(f \frown s_1) \restriction \mathrm{dom}\, f$ reduces to f.

Now we state the propositions:

(19) Let us consider a complex-valued finite 0-sequence f, and a natural number x. Then $(f \frown \{0\}_{n \in \mathbb{N}})(x) = f(x)$.

(20) Let us consider a sequence f of real numbers. Then $f \frown f$ is a real-valued complex sequence.

Let f be a real-valued complex sequence. Note that $\Im(f)$ is empty yielding. One can check that $\Re(f)$ reduces to f.

Let us observe that there exists a sequence of real numbers which is empty yielding and every sequence of real numbers is transfinite sequence-like.

Let r be a real number. Let us note that $\Re(r \cdot (i))$ is zero.

One can check that $\Im(r \cdot (i))$ reduces to r.

Let f be a complex-valued finite 0-sequence. Let us note that $\Re(f)$ is real-valued, finite, and transfinite sequence-like and $\Im(f)$ is real-valued, finite, and transfinite sequence-like and $\Re(f)$ is $(\operatorname{len} f)$-element and $\Im(f)$ is $(\operatorname{len} f)$-element.

Let f be a complex-valued finite sequence. Note that $\Re(f)$ is real-valued and finite sequence-like and $\Im(f)$ is real-valued and finite sequence-like.

Let f be a complex-valued function. Let us observe that $\Re(\Re(f))$ reduces to $\Re(f)$ and $\Re(\Im(f))$ reduces to $\Im(f)$. Let us note that $\Im(\Re(f))$ is empty yielding and $\Im(\Im(f))$ is empty yielding.

One can check that $\Re(\Re(f)+i\cdot\Im(f))$ reduces to $\Re(f)$ and $\Im(\Re(f)+i\cdot\Im(f))$ reduces to $\Im(f)$ and $\Re(f)+i\cdot\Im(f)$ reduces to f.

Let n be a natural number. One can check that there exists a finite function which is n-element.

Let f be a finite, complex-valued transfinite sequence. Note that $\operatorname{Shift}(f,n)$ is finite and $\operatorname{Shift}(f,n)$ is $(\operatorname{len} f)$-element and $\{0\}_{n\in\mathbb{N}}$ is empty yielding.

6. Converting Complex 0-sequences into Ordinary Ones

Let f be a complex-valued finite 0-sequence. The functor $\operatorname{Sequel} f$ yielding a complex sequence is defined by the term

(Def. 8) $(\mathbb{N} \longmapsto 0) {+\cdot} f$.

Now we state the propositions:

(21) Let us consider a complex-valued finite 0-sequence f, and a natural number x. Then $(\operatorname{Sequel} f)(x) = f(x)$.

(22) Let us consider a complex-valued finite 0-sequence f. Then $\operatorname{Sequel} f = f \mathbin{^\frown} \{0\}_{n\in\mathbb{N}}$.
PROOF: $\operatorname{dom}(\operatorname{Sequel} f) = \operatorname{dom}(f \mathbin{^\frown} \{0\}_{n\in\mathbb{N}})$. For every natural number x, $(\operatorname{Sequel} f)(x) = (f \mathbin{^\frown} \{0\}_{n\in\mathbb{N}})(x)$. □

(23) Let us consider a complex sequence s_1. Then $s_1 = \Re(s_1) + i \cdot \Im(s_1)$.

Let us consider a complex-valued finite 0-sequence f. Now we state the propositions:

(24) $\Re(\operatorname{Sequel} f) = \operatorname{Sequel} \Re(f)$. The theorem is a consequence of (21).

(25) $\Im(\operatorname{Sequel} f) = \operatorname{Sequel} \Im(f)$. The theorem is a consequence of (21).

Now we state the propositions:

(26) Let us consider a complex number c. Then $0 \cdot (\mathbb{N} \longmapsto c) = \mathbb{N} \longmapsto 0$.

(27) Let us consider a complex sequence s_1, and a natural number x. Suppose for every natural number k such that $k \geqslant x$ holds $s_1(k) = 0$. Then s_1 is summable.

(28) Let us consider a sequence s_1 of real numbers, and a natural number x. Suppose for every natural number k such that $k \geqslant x$ holds $s_1(k) = 0$. Then s_1 is summable.

Let f be a complex-valued finite 0-sequence. One can check that Sequel f is summable.

7. Properties of Concatenation

Let f be a finite 0-sequence and g be a finite sequence. The functor $f \frown g$ yielding a finite 0-sequence is defined by

(Def. 9) $\operatorname{dom} it = \operatorname{len} f + \operatorname{len} g$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $it(k) = f(k)$ and for every natural number k such that $k \in \operatorname{dom} g$ holds $it(\operatorname{len} f + k - 1) = g(k)$.

Let f be a finite sequence and g be a finite 0-sequence. The functor $f \frown g$ yielding a finite sequence is defined by

(Def. 10) $\operatorname{dom} it = \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $it(k) = f(k)$ and for every natural number k such that $k \in \operatorname{dom} g$ holds $it(\operatorname{len} f + k + 1) = g(k)$.

Now we state the proposition:

(29) Let us consider a finite 0-sequence f, and a finite sequence g. Then

(i) $\operatorname{len}(f \frown g) = \operatorname{len} f + \operatorname{len} g$, and

(ii) $\operatorname{len}(g \frown f) = \operatorname{len} f + \operatorname{len} g$.

Let n, m be natural numbers, f be an n-element finite 0-sequence, and g be an m-element finite sequence. Let us note that $f \frown g$ is $(n + m)$-element and $g \frown f$ is $(n + m)$-element.

Now we state the propositions:

(30) Let us consider a finite 0-sequence f, a finite sequence g, and a natural number x. Then $x \in \operatorname{dom}(f \frown g)$ if and only if $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$.

Proof: If $x \in \operatorname{dom}(f \frown g)$, then $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$. If $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$, then $x \in \operatorname{dom}(f \frown g)$. □

(31) Let us consider a finite sequence f, a finite 0-sequence g, and a natural number x. Then $x \in \operatorname{dom}(f \frown g)$ if and only if $x \in \operatorname{dom} f$ or $x - (\operatorname{len} f + 1) \in \operatorname{dom} g$.

Proof: If $x \in \operatorname{dom}(f \frown g)$, then $x \in \operatorname{dom} f$ or $x - (\operatorname{len} f + 1) \in \operatorname{dom} g$. □

(32) Let us consider a finite sequence f, and a finite 0-sequence g. Then

(i) $\operatorname{rng}(f \frown g) = \operatorname{rng} f \cup \operatorname{rng} g$, and

(ii) $\operatorname{rng}(g \frown f) = \operatorname{rng} f \cup \operatorname{rng} g$.

PROOF: $\operatorname{rng}(f \frown g) \subseteq \operatorname{rng} f \cup \operatorname{rng} g$. $\operatorname{rng} f \cup \operatorname{rng} g \subseteq \operatorname{rng}(f \frown g)$. $\operatorname{rng}(g \frown f) \subseteq \operatorname{rng} f \cup \operatorname{rng} g$. $\operatorname{rng} f \cup \operatorname{rng} g \subseteq \operatorname{rng}(g \frown f)$. □

(33) Let us consider a non empty finite 0-sequence f, and a finite sequence g. Then $\operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f - 1)) = \mathbb{Z}_{\operatorname{len} f + \operatorname{len} g}$.
PROOF: For every object x, $x \in \operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f - 1))$ iff $x \in \mathbb{Z}_{\operatorname{len} f + \operatorname{len} g}$. □

(34) Let us consider a finite sequence f, and a finite 0-sequence g. Then $\operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f + 1)) = \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$.
PROOF: For every object x, $x \in \operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f + 1))$ iff $x \in \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$. □

Let f be a complex-valued finite sequence. One can verify that $\langle\rangle_{\mathbb{C}} \frown f$ is complex-valued.

Let f be a complex-valued finite 0-sequence. Let us note that $\varepsilon_{\mathbb{C}} \frown f$ is complex-valued.

Let f be a finite 0-sequence and g be a finite sequence. One can verify that $(f \frown g) \restriction \operatorname{len} f$ reduces to f and $(g \frown f) \restriction \operatorname{len} g$ reduces to g.

Now we state the propositions:

(35) Let us consider a set D, a finite 0-sequence f, and a finite sequence g of elements of D. Then $(f \frown g)_{\lfloor \operatorname{len} f} = \operatorname{FS2XFS}(g)$.
PROOF: For every natural number i such that $i \in \operatorname{dom}((f \frown g)_{\lfloor \operatorname{len} f})$ holds $((f \frown g)_{\lfloor \operatorname{len} f})(i) = (\operatorname{FS2XFS}(g))(i)$. □

(36) Every finite 0-sequence is a finite 0-sequence of $\operatorname{rng} f \cup \{1\}$.

(37) Let us consider a set D, a finite sequence f, and a finite 0-sequence g of D. Then $(f \frown g)_{\lfloor \operatorname{len} f} = \operatorname{XFS2FS}(g)$.
PROOF: $\operatorname{len} f \leqslant \operatorname{len}(f \frown g)$. For every natural number i such that $i \in \operatorname{dom}((f \frown g)_{\lfloor \operatorname{len} f})$ holds $((f \frown g)_{\lfloor \operatorname{len} f})(i) = (\operatorname{XFS2FS}(g))(i)$. □

Let D be a set and f be a finite 0-sequence of D. One can verify that the functor $\operatorname{XFS2FS}(f)$ is defined by the term

(Def. 11) $\varepsilon_D \frown f$.

Now we state the proposition:

(38) Let us consider a set D, and a finite 0-sequence f of D. Then $\operatorname{dom}(\operatorname{Shift}(f, 1)) = \operatorname{Seg} \operatorname{len} f$.
PROOF: For every object x such that $x \in \operatorname{Seg} \operatorname{len} f$ holds $x \in \operatorname{dom}(\operatorname{Shift}(f, 1))$. For every object x such that $x \in \operatorname{dom}(\operatorname{Shift}(f, 1))$ holds $x \in \operatorname{Seg} \operatorname{len} f$ by [7, (106)]. □

Let D be a set and f be a finite 0-sequence of D. One can verify that the functor $\operatorname{XFS2FS}(f)$ is defined by the term

(Def. 12) Shift$(f, 1)$.

Let f be a finite sequence of elements of D. One can check that the functor FS2XFS(f) is defined by the term

(Def. 13) $\langle\rangle_D \frown f$.

Now we state the propositions:

(39) Let us consider a set D, and finite 0-sequences f, g of D. Then $f \frown g = f \frown \mathrm{XFS2FS}(g)$.

PROOF: For every natural number k such that $k \in \mathrm{dom}(f \frown g)$ holds $(f \frown g)(k) = (f \frown \mathrm{XFS2FS}(g))(k)$. □

(40) Let us consider a set D, and finite sequences f, g of elements of D. Then $f \frown g = f \frown \mathrm{FS2XFS}(g)$.

PROOF: For every natural number k such that $k \in \mathrm{dom}(f \frown g)$ holds $(f \frown g)(k) = (f \frown \mathrm{FS2XFS}(g))(k)$. □

Let f be a finite 0-sequence of $\mathbb{R}$. Let us observe that Sequel $f \restriction \mathrm{dom}\, f$ reduces to f. One can check that Shift$(f, 1)$ is finite sequence-like and Sequel $f \uparrow \mathrm{dom}\, f$ is empty yielding.

Now we state the propositions:

(41) Let us consider a set D, a finite sequence f of elements of D, and a finite 0-sequence g of D. Then $f \frown g = f \frown \mathrm{XFS2FS}(g)$. The theorem is a consequence of (40).

(42) Let us consider a set D, a finite 0-sequence f of D, and a finite sequence g of elements of D. Then $f \frown g = f \frown \mathrm{FS2XFS}(g)$. The theorem is a consequence of (39).

(43) Let us consider a set D, and finite sequences f, g of elements of D. Then $\mathrm{FS2XFS}(f \frown g) = \mathrm{FS2XFS}(f) \frown \mathrm{FS2XFS}(g)$.

PROOF: For every natural number x such that $x \in \mathrm{dom}(\mathrm{FS2XFS}(f \frown g))$ holds $(\mathrm{FS2XFS}(f \frown g))(x) = (\mathrm{FS2XFS}(f) \frown \mathrm{FS2XFS}(g))(x)$. □

Let D be a set, f be a finite sequence of elements of D, and g be a finite 0-sequence of D. Note that the functor $f \frown g$ yields a finite sequence of elements of D. Now we state the propositions:

(44) Let us consider a set D, a finite sequence f of elements of D, and a finite 0-sequence g of D. Then $\mathrm{FS2XFS}(f \frown g) = \mathrm{FS2XFS}(f) \frown g$. The theorem is a consequence of (43) and (40).

(45) Let us consider a set D, and finite 0-sequences f, g of D. Then $\mathrm{XFS2FS}(f \frown g) = \mathrm{XFS2FS}(f) \frown \mathrm{XFS2FS}(g)$.

PROOF: For every natural number x such that $x \in \mathrm{dom}(\mathrm{XFS2FS}(f \frown g))$ holds $(\mathrm{XFS2FS}(f \frown g))(x) = (\mathrm{XFS2FS}(f) \frown \mathrm{XFS2FS}(g))(x)$. □

Let D be a set, f be a finite 0-sequence of D, and g be a finite sequence of elements of D. One can check that the functor $f \frown g$ yields a finite 0-sequence of D. Now we state the propositions:

(46) Let us consider a set D, a finite 0-sequence f of D, and a finite sequence g of elements of D. Then $\mathrm{XFS2FS}(f \frown g) = \mathrm{XFS2FS}(f) \frown g$. The theorem is a consequence of (45) and (39).

(47) Let us consider a set D, finite 0-sequences f, g of D, and a finite sequence h of elements of D. Then

(i) $(f \frown g) \frown h = f \frown (g \frown h)$, and

(ii) $(f \frown h) \frown g = f \frown (h \frown g)$, and

(iii) $(h \frown f) \frown g = h \frown (f \frown g)$.

The theorem is a consequence of (42), (39), (43), (41), and (45).

8. Sum of Finite 0-sequences

Now we state the proposition:

(48) Let us consider a complex-valued finite 0-sequence f. Then $\sum(f \restriction 1) = f(0)$.

Let n, m be natural numbers and f be an $(n+m)$-element finite 0-sequence. One can verify that $f \restriction n$ is n-element. Let n be a natural number and p be an n-element, complex-valued finite 0-sequence. Let us observe that $-p$ is n-element and p^{-1} is n-element and $p^{\mathbf{2}}$ is n-element and $|p|$ is n-element and $\mathrm{Rev}(p)$ is n-element.

Let m be a natural number and q be an $(n+m)$-element, complex-valued finite 0-sequence. Let us observe that $\mathrm{dom}\, p \cap \mathrm{dom}\, q$ reduces to $\mathrm{dom}\, p$. Note that $p+q$ is n-element and $p-q$ is n-element and $p \cdot q$ is n-element and p/q is n-element. Let p, q be n-element, complex-valued finite 0-sequences. Note that $p+q$ is n-element and $p-q$ is n-element and $p \cdot q$ is n-element and p/q is n-element. Now we state the propositions:

(49) Let us consider a natural number n, and n-element, complex-valued finite 0-sequences f_1, f_2. Then $\sum(f_1+f_2) = \sum f_1 + \sum f_2$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every $\$_1$-element, complex-valued finite 0-sequences f_1, f_2, $\sum(f_1+f_2) = \sum f_1 + \sum f_2$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(50) Let us consider a complex number c. Then $\mathrm{XFS2FS}(\langle c \rangle) = \langle c \rangle$.
Proof: For every natural number k such that $k \in \mathrm{dom}\langle c \rangle$ holds $(\mathrm{XFS2FS}(\langle c \rangle))(k) = \langle c \rangle(k)$. □

(51) Let us consider a finite 0-sequence f of $\mathbb{R}$. Then $\sum \mathrm{XFS2FS}(f) = \sum f$. The theorem is a consequence of (16).

(52) Let us consider a complex-valued finite 0-sequence f. Then $\sum f = \sum \Re(f) + (i) \cdot (\sum \Im(f))$. The theorem is a consequence of (49).

(53) Let us consider a complex-valued transfinite sequence f, and a natural number n. Then

(i) $\Re(\mathrm{Shift}(f, n)) = \mathrm{Shift}(\Re(f), n)$, and

(ii) $\Im(\mathrm{Shift}(f, n)) = \mathrm{Shift}(\Im(f), n)$.

Let us consider a complex-valued finite 0-sequence f.

(54) (i) $\mathrm{XFS2FS}(\Re(f)) = \Re(\mathrm{XFS2FS}(f))$, and

(ii) $\mathrm{XFS2FS}(\Im(f)) = \Im(\mathrm{XFS2FS}(f))$.

(55) $\sum \mathrm{XFS2FS}(f) = \sum f$. The theorem is a consequence of (52), (51), and (53).

(56) Let us consider a finite sequence f of elements of $\mathbb{C}$. Then $\sum \mathrm{FS2XFS}(f) = \sum f$. The theorem is a consequence of (55).

(57) Let us consider a real-valued finite 0-sequence f. Then $\sum f = \sum \mathrm{Sequel}\, f$.

Note that there exists a real-valued complex sequence which is summable.

Let f be a summable complex sequence. The functors: $\Re(f)$ and $\Im(f)$ yield summable, real-valued complex sequences. Now we state the propositions:

(58) Let us consider a complex-valued finite 0-sequence f. Then $\sum f = \sum \mathrm{Sequel}\, f$. The theorem is a consequence of (57), (24), (25), and (52).

(59) Let us consider a finite 0-sequence f of $\mathbb{C}$, and a finite sequence g of elements of $\mathbb{C}$. Then

(i) $\sum(f \frown g) = \sum f + \sum g$, and

(ii) $\sum(g \frown f) = \sum g + \sum f$.

The theorem is a consequence of (39), (56), (40), and (55).

9. Product of Finite 0-sequences

Let f be a finite 0-sequence. The functor $\prod f$ yielding an element of $\mathbb{C}$ is defined by the term

(Def. 14) $\cdot_{\mathbb{C}} \odot f$.

Now we state the proposition:

(60) Let us consider an empty finite 0-sequence f. Then $\prod f = 1$.

Let c be a complex number. One can check that $\prod \langle c \rangle$ reduces to c.

(61) Let us consider a natural number n, and a complex-valued finite 0-sequence f. Suppose $n \in \operatorname{dom} f$. Then $\prod(f{\upharpoonright}n) \cdot f(n) = \prod(f{\upharpoonright}(n+1))$.

(62) Let us consider a natural number n, and a complex sequence f. Then $\prod(f{\upharpoonright}n) \cdot f(n) = \prod(f{\upharpoonright}(n+1))$. The theorem is a consequence of (61).

(63) Let us consider a non empty, complex-valued finite 0-sequence f. Then $\prod(f{\upharpoonright}1) = f(0)$.

(64) Let us consider a natural number n, and n-element, complex-valued finite 0-sequences f_1, f_2. Then $\prod(f_1 \cdot f_2) = (\prod f_1) \cdot (\prod f_2)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every $\$_1$-element, complex-valued finite 0-sequences f_1, f_2, $\prod(f_1 \cdot f_2) = (\prod f_1) \cdot (\prod f_2)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(65) Let us consider a complex sequence f, and a natural number n. Then $\prod(f{\upharpoonright}(n+1)) =$ (the partial product of f)(n).
PROOF: Define $\mathcal{P}$[natural number] $\equiv \prod(f{\upharpoonright}(\$_1+1)) =$ (the partial product of f)$(\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. $\square$

(66) Let us consider a complex-valued finite 0-sequence f. Then $\prod \mathrm{XFS2FS}(f) = \prod f$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \prod \mathrm{XFS2FS}(f{\upharpoonright}\$_1) = \prod(f{\upharpoonright}\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. $\square$

(67) Let us consider a finite sequence f of elements of $\mathbb{C}$. Then $\prod \mathrm{FS2XFS}(f) = \prod f$. The theorem is a consequence of (66).

(68) Let us consider a finite 0-sequence f of $\mathbb{C}$, and a finite sequence g of elements of $\mathbb{C}$. Then

(i) $\prod(f \frown g) = (\prod f) \cdot (\prod g)$, and

(ii) $\prod(g \frown f) = (\prod g) \cdot (\prod f)$.

The theorem is a consequence of (66), (46), (10), and (40).

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Bilinear Operators on Normed Linear Spaces

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Summary. The main aim of this article is proving properties of bilinear operators on normed linear spaces formalized by means of Mizar [1]. In the first two chapters, algebraic structures [3] of bilinear operators on linear spaces are discussed. Especially, the space of bounded bilinear operators on normed linear spaces is developed here. In the third chapter, it is remarked that the algebraic structure of bounded bilinear operators to a certain Banach space also constitutes a Banach space.

In the last chapter, the correspondence between the space of bilinear operators and the space of composition of linear opearators is shown. We referred to [4], [11], [2], [7] and [8] in this formalization.

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1. Real Vector Space of Bilinear Operators

Let X, Y, Z be real linear spaces. The functor $\mathrm{BilinOpers}(X, Y, Z)$ yielding a subset of RealVectSpace((the carrier of $X \times Y$), Z) is defined by

(Def. 1) for every set x, $x \in it$ iff x is a bilinear operator from $X \times Y$ into Z.

Let us observe that $\mathrm{BilinOpers}(X, Y, Z)$ is non empty and functional and $\mathrm{BilinOpers}(X, Y, Z)$ is linearly closed.

The functor $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$ yielding a strict RLS structure is defined by the term

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(Def. 2) $\langle \mathrm{BilinOpers}(X,Y,Z), \mathrm{Zero}(\mathrm{BilinOpers}(X,Y,Z), \mathrm{RealVectSpace}((\text{the carrier of } X \times Y), Z)), \mathrm{Add}(\mathrm{BilinOpers}(X,Y,Z), \mathrm{RealVectSpace}((\text{the carrier of } X \times Y), Z)), \mathrm{Mult}(\mathrm{BilinOpers}(X,Y,Z), \mathrm{RealVectSpace}((\text{the carrier of } X \times Y), Z))\rangle$.

Let us note that $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$ is non empty and $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$ is constituted functions.

Now we state the proposition:

(1) Let us consider real linear spaces X, Y, Z. Then $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$ is a subspace of $\mathrm{RealVectSpace}((\text{the carrier of } X \times Y), Z)$.

Let X, Y, Z be real linear spaces, f be an element of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$, v be a vector of X, and w be a vector of Y. Let us note that the functor $f(v,w)$ yields a vector of Z. Now we state the propositions:

(2) Let us consider real linear spaces X, Y, Z, and vectors f, g, h of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$. Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x,y) = f(x,y) + g(x,y)$.

(3) Let us consider real linear spaces X, Y, Z, vectors f, h of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x,y) = a \cdot f(x,y)$.

Let us consider real linear spaces X, Y, Z. Now we state the propositions:

(4) $0_{\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X,Y,Z)} = (\text{the carrier of } X \times Y) \longmapsto 0_Z$.

(5) (The carrier of $X \times Y$) $\longmapsto 0_Z$ is a bilinear operator from $X \times Y$ into Z.

2. Real Normed Linear Space of Bounded Bilinear Operators

Let X, Y, Z be real normed spaces and I_1 be a bilinear operator from $X \times Y$ into Z. We say that I_1 is Lipschitzian if and only if

(Def. 3) there exists a real number K such that $0 \leqslant K$ and for every vector x of X and for every vector y of Y, $\|I_1(x,y)\| \leqslant K \cdot \|x\| \cdot \|y\|$.

Now we state the propositions:

(6) Let us consider real normed spaces X, Y, Z, and a bilinear operator f from $X \times Y$ into Z. Suppose for every vector x of X for every vector y of Y, $f(x,y) = 0_Z$. Then f is Lipschitzian.

(7) Let us consider real normed spaces X, Y, Z. Then (the carrier of $X \times Y$) $\longmapsto 0_Z$ is a bilinear operator from $X \times Y$ into Z.

Let X, Y, Z be real normed spaces. Let us observe that there exists a bilinear operator from $X \times Y$ into Z which is Lipschitzian.

Now we state the proposition:

(8) Let us consider real normed spaces X, Y, Z, and an object z. Then $z \in \mathrm{BilinOpers}(X, Y, Z)$ if and only if z is a bilinear operator from $X \times Y$ into Z.

Let X, Y, Z be real normed spaces. The functor BoundedBilinOpers(X, Y, Z) yielding a subset of VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is defined by

(Def. 4) for every set x, $x \in$ *it* iff x is a Lipschitzian bilinear operator from $X \times Y$ into Z.

Note that BoundedBilinOpers(X, Y, Z) is non empty and BoundedBilinOpers (X, Y, Z) is linearly closed.

The functor VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ yielding a strict RLS structure is defined by the term

(Def. 5) $\langle$BoundedBilinOpers(X, Y, Z), Zero(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$), Add(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$), Mult(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$)$\rangle$.

Now we state the proposition:

(9) Let us consider real normed spaces X, Y, Z. Then VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is a subspace of VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$.

Let X, Y, Z be real normed spaces. Note that VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is non empty and VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is constituted functions.

Let f be an element of VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, v be a vector of X, and w be a vector of Y. One can verify that the functor $f(v, w)$ yields a vector of Z. Now we state the propositions:

(10) Let us consider real normed spaces X, Y, Z, and vectors f, g, h of VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$. Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) + g(x, y)$. The theorem is a consequence of (2).

(11) Let us consider real normed spaces X, Y, Z, vectors f, h of VectorSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = a \cdot f(x, y)$. The theorem is a consequence of (3).

(12) Let us consider real normed spaces X, Y, Z.

Then $0_{\text{VectorSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)} = (\text{the carrier of } X \times Y) \longmapsto 0_Z$. The theorem is a consequence of (4).

Let X, Y, Z be real normed spaces and f be an object. Assume $f \in$ BoundedBilinOpers(X, Y, Z). The functor modetrans(f, X, Y, Z) yielding a Lipschitzian bilinear operator from $X \times Y$ into Z is defined by the term

(Def. 6) f.

Let u be a bilinear operator from $X \times Y$ into Z. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 7) $\{\|u(t,s)\|$, where t is a vector of X, s is a vector of Y : $\|t\| \leqslant 1$ and $\|s\| \leqslant 1\}$.

Let g be a Lipschitzian bilinear operator from $X \times Y$ into Z. Observe that PreNorms(g) is upper bounded.

Now we state the proposition:

(13) Let us consider real normed spaces X, Y, Z, and a bilinear operator g from $X \times Y$ into Z. Then g is Lipschitzian if and only if PreNorms(g) is upper bounded.

Let X, Y, Z be real normed spaces. The functor BoundedBilinOpersNorm(X, Y, Z) yielding a function from BoundedBilinOpers(X, Y, Z) into $\mathbb{R}$ is defined by

(Def. 8) for every object x such that $x \in$ BoundedBilinOpers(X, Y, Z) holds $it(x) = \sup \text{PreNorms}(\text{modetrans}(x, X, Y, Z))$.

Let f be a Lipschitzian bilinear operator from $X \times Y$ into Z. Let us note that modetrans(f, X, Y, Z) reduces to f.

Now we state the proposition:

(14) Let us consider real normed spaces X, Y, Z, and a Lipschitzian bilinear operator f from $X \times Y$ into Z. Then $(\text{BoundedBilinOpersNorm}(X, Y, Z))(f) = \sup \text{PreNorms}(f)$.

Let X, Y, Z be real normed spaces. The functor NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ yielding a non empty normed structure is defined by the term

(Def. 9) $\langle$BoundedBilinOpers(X, Y, Z), Zero(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$), Add(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$), Mult(BoundedBilinOpers(X, Y, Z), VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y, Z)$), BoundedBilinOpersNorm$(X, Y, Z)\rangle$.

Now we state the propositions:

(15) Let us consider real normed spaces X, Y, Z. Then $(\text{the carrier of } X \times Y) \longmapsto 0_Z = 0_{\text{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)}$. The theorem is a consequence of (12).

(16) Let us consider real normed spaces X, Y, Z, a point f of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, and a Lipschitzian bilinear operator g from $X \times Y$ into Z. Suppose $g = f$. Let us consider a vector t of X, and a vector s of Y. Then $\|g(t, s)\| \leqslant \|f\| \cdot \|t\| \cdot \|s\|$. The theorem is a consequence of (14).

Let us consider real normed spaces X, Y, Z and a point f of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$. Now we state the propositions:

(17) $0 \leqslant \|f\|$. The theorem is a consequence of (14).

(18) If $f = 0_{\text{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)}$, then $0 = \|f\|$. The theorem is a consequence of (15) and (14).

Let X, Y, Z be real normed spaces. One can verify that every element of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is function-like and relation-like.

Let f be an element of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, v be a vector of X, and w be a vector of Y. Observe that the functor $f(v, w)$ yields a vector of Z. Now we state the propositions:

(19) Let us consider real normed spaces X, Y, Z, and points f, g, h of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$. Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) + g(x, y)$. The theorem is a consequence of (10).

(20) Let us consider real normed spaces X, Y, Z, points f, h of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = a \cdot f(x, y)$. The theorem is a consequence of (11).

(21) Let us consider real normed spaces X, Y, Z, points f, g of NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$, and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

PROOF: $\|f + g\| \leqslant \|f\| + \|g\|$. $\|a \cdot f\| = |a| \cdot \|f\|$. □

Let X, Y, Z be real normed spaces. Observe that NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is non empty and NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is reflexive, discernible, and real normed space-like.

Now we state the proposition:

(22) Let us consider real normed spaces X, Y, Z. Then NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is a real normed space.

Let X, Y, Z be real normed spaces. Let us note that NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(23) Let us consider real normed spaces X, Y, Z, and points f, g, h of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)$. Then $h = f - g$ if and only if for every vector x of X and for every vector y of Y, $h(x,y) = f(x,y) - g(x,y)$. The theorem is a consequence of (19).

3. Real Banach Space of Bounded Bilinear Operators

Now we state the propositions:

(24) Let us consider real normed spaces X, Y, Z. Suppose Z is complete. Let us consider a sequence s_1 of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)$. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

Proof: Define $\mathcal{P}[\mathrm{set},\mathrm{set}] \equiv$ there exists a sequence x_3 of Z such that for every natural number n, $x_3(n) = vseq(n)(\$_1)$ and x_3 is convergent and $\$_2 = \lim x_3$. For every element x_4 of $X \times Y$, there exists an element z of Z such that $\mathcal{P}[x_4, z]$. Consider f being a function from the carrier of $X \times Y$ into the carrier of Z such that for every element z of $X \times Y$, $\mathcal{P}[z, f(z)]$. Reconsider $t_1 = f$ as a function from $X \times Y$ into Z. For every points x_1, x_2 of X and for every point y of Y, $t_1(x_1+x_2, y) = t_1(x_1, y)+t_1(x_2, y)$. For every point x of X and for every point y of Y and for every real number a, $t_1(a \cdot x, y) = a \cdot t_1(x, y)$. For every point x of X and for every points y_1, y_2 of Y, $t_1(x, y_1 + y_2) = t_1(x, y_1) + t_1(x, y_2)$.

For every point x of X and for every point y of Y and for every real number a, $t_1(x, a \cdot y) = a \cdot t_1(x, y)$. t_1 is Lipschitzian by [6, (18)], [9, (20)], (16). For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ for every point x of X for every point y of Y, $\|vseq(n)(x,y) - t_1(x,y)\| \leqslant e \cdot \|x\| \cdot \|y\|$ by [10, (8)], (23). Reconsider $t_2 = t_1$ as a point of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)$. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ holds $\|vseq(n) - t_2\| \leqslant e$. For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\|vseq(n) - t_2\| < e$. □

(25) Let us consider real normed spaces X, Y, and a real Banach space Z. Then $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X,Y,Z)$ is a real Banach space. The theorem is a consequence of (24).

Let X, Y be real normed spaces and Z be a real Banach space. Let us note that NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ is complete.

4. Isomorphisms between the Space of Bilinear Operators and the Space of Composition of Linear Operators

From now on X, Y, Z denote real linear spaces.

Now we state the proposition:

(26) There exists a linear operator I from VectorSpaceOfLinearOpers$_{\mathbb{R}}(X,$ VectorSpaceOfLinearOpers$_{\mathbb{R}}(Y, Z))$ into VectorSpaceOfBilinOpers$_{\mathbb{R}}(X, Y,$ $Z)$ such that

(i) I is bijective, and

(ii) for every point u of VectorSpaceOfLinearOpers$_{\mathbb{R}}(X,$ VectorSpaceOf-LinearOpers$_{\mathbb{R}}(Y, Z))$ and for every point x of X and for every point y of Y, $I(u)(x, y) = u(x)(y)$.

Proof: Set $X_1 =$ the carrier of X. Set $Y_1 =$ the carrier of Y. Set $Z_1 =$ the carrier of Z. Consider I_0 being a function from $({Z_1}^{Y_1})^{X_1}$ into ${Z_1}^{X_1 \times Y_1}$ such that I_0 is bijective and for every function f from X_1 into ${Z_1}^{Y_1}$ and for every objects d, e such that $d \in X_1$ and $e \in Y_1$ holds $I_0(f)(d, e) = f(d)(e)$. Set $L_1 =$ the carrier of VectorSpaceOfLinearOpers$_{\mathbb{R}}(X,$ VectorSpaceOf-LinearOpers$_{\mathbb{R}}(Y, Z))$. Set $B =$ the carrier of VectorSpaceOfBilinOpers$_{\mathbb{R}}(X,$ $Y, Z)$. Reconsider $I = I_0 \restriction L_1$ as a function from L_1 into ${Z_1}^{X_1 \times Y_1}$.

For every element x of L_1, for every point p of X and for every point q of Y, there exists a linear operator G from Y into Z such that $G = x(p)$ and $I(x)(p, q) = G(q)$ and $I(x) \in B$. For every elements x_1, x_2 of L_1, $I(x_1 + x_2) = I(x_1) + I(x_2)$. For every element x of L_1 and for every real number a, $I(a \cdot x) = a \cdot I(x)$. For every point u of VectorSpaceOfLinearOpers$_{\mathbb{R}}(X,$ VectorSpaceOfLinearOpers$_{\mathbb{R}}(Y, Z))$ and for every point x of X and for every point y of Y, $I(u)(x, y) = u(x)(y)$. For every object y such that $y \in B$ there exists an object x such that $x \in L_1$ and $y = I(x)$. $\square$

In the sequel X, Y, Z denote real normed spaces.

(27) There exists a linear operator I from the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z into NormSpaceOfBoundedBilinOpers$_{\mathbb{R}}(X, Y, Z)$ such that

(i) I is bijective, and

(ii) for every point u of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z, $\|u\| = \|I(u)\|$ and for every point x of X and for every point y of Y, $I(u)(x, y) = u(x)(y)$.

Proof: Set X_1 = the carrier of X. Set Y_1 = the carrier of Y. Set Z_1 = the carrier of Z. Consider I_0 being a function from $({Z_1}^{Y_1})^{X_1}$ into ${Z_1}^{X_1 \times Y_1}$ such that I_0 is bijective and for every function f from X_1 into ${Z_1}^{Y_1}$ and for every objects d, e such that $d \in X_1$ and $e \in Y_1$ holds $I_0(f)(d, e) = f(d)(e)$. Set L_1 = the carrier of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z. Set B = the carrier of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$. Set L_2 = the carrier of the real norm space of bounded linear operators from Y into Z. ${L_2}^{X_1} \subseteq ({Z_1}^{Y_1})^{X_1}$. Reconsider $I = I_0 \restriction L_1$ as a function from L_1 into ${Z_1}^{X_1 \times Y_1}$.

For every element x of L_1, for every point p of X and for every point q of Y, there exists a Lipschitzian linear operator G from Y into Z such that $G = x(p)$ and $I(x)(p, q) = G(q)$ and $I(x)$ is a Lipschitzian bilinear operator from $X \times Y$ into Z and $I(x) \in B$ and there exists a point I_2 of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$ such that $I_2 = I(x)$ and $\|x\| = \|I_2\|$. For every elements x_1, x_2 of L_1, $I(x_1 + x_2) = I(x_1) + I(x_2)$. For every element x of L_1 and for every real number a, $I(a \cdot x) = a \cdot I(x)$. For every point u of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z, $\|u\| = \|I(u)\|$ and for every point x of X and for every point y of Y, $I(u)(x, y) = u(x)(y)$. For every object y such that $y \in B$ there exists an object x such that $x \in L_1$ and $y = I(x)$ by [5, (12)]. □

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A Simple Example for Linear Partial Differential Equations and Its Solution Using the Method of Separation of Variables

Sora Otsuki
Shinshu University
Nagano, Japan

Pauline N. Kawamoto
Shinshu University
Nagano, Japan

Hiroshi Yamazaki
Shinshu University
Nagano, Japan

Summary. In this article, we formalized in Mizar [4], [1] simple partial differential equations. In the first section, we formalized partial differentiability and partial derivative. The next section contains the method of separation of variables for one-dimensional wave equation. In the last section, we formalized the superposition principle. We referred to [6], [3], [5] and [9] in this formalization.

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1. Preliminaries

From now on m, n denote non zero elements of $\mathbb{N}$, i, j, k denote elements of $\mathbb{N}$, Z denotes a subset of $\mathcal{R}^2$, c denotes a real number, I denotes a non empty finite sequence of elements of $\mathbb{N}$, and d_1, d_2 denote elements of $\mathbb{R}$.

Now we state the proposition:

(1) Let us consider a non zero element m of $\mathbb{N}$, a subset X of $\mathcal{R}^m$, a non empty finite sequence I of elements of $\mathbb{N}$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partially differentiable on X w.r.t. I. Then $\mathrm{dom}(f{\restriction}^I X) = X$.

Let us note that $\Omega_{\mathbb{R}}$ is open and $\Omega_{\mathcal{R}^2}$ is open.

Now we state the proposition:

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(2) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a subset Z of $\mathbb{R}$, and a real number x_0. Suppose Z is open and $x_0 \in Z$. Then

(i) f is differentiable in x_0 iff $f \restriction Z$ is differentiable in x_0, and

(ii) if f is differentiable in x_0, then $f'(x_0) = (f \restriction Z)'(x_0)$.

PROOF: f is differentiable in x_0 iff $f \restriction Z$ is differentiable in x_0. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a subset X of $\mathbb{R}$. Now we state the propositions:

(3) If X is open and $X \subseteq \operatorname{dom} f$, then f is differentiable on X iff $f \restriction X$ is differentiable on X. The theorem is a consequence of (2).

(4) If X is open and $X \subseteq \operatorname{dom} f$ and f is differentiable on X, then $(f \restriction X)'_{\restriction X} = f'_{\restriction X}$. The theorem is a consequence of (3) and (2).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a subset Z of $\mathbb{R}$. Now we state the propositions:

(5) If $Z \subseteq \operatorname{dom} f$ and Z is open and f is differentiable 1 times on Z, then f is differentiable on Z and $(f'(Z))(1) = f'_{\restriction Z}$. The theorem is a consequence of (3) and (4).

(6) Suppose $Z \subseteq \operatorname{dom} f$ and Z is open and f is differentiable 2 times on Z. Then

(i) f is differentiable on Z, and

(ii) $(f'(Z))(1) = f'_{\restriction Z}$, and

(iii) $f'_{\restriction Z}$ is differentiable on Z, and

(iv) $(f'(Z))(2) = (f'_{\restriction Z})'_{\restriction Z}$.

The theorem is a consequence of (5).

(7) Let us consider subsets X, T of $\mathbb{R}$, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a partial function g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$. Then there exists a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$ such that

(i) $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exist real numbers x, t such that $x \in X$ and $t \in T$ and $\$_1 = \langle x, t\rangle$ and $\$_2 = f_{/x} \cdot (g_{/t})$. For every objects z, w_1, w_2 such that $z \in \mathcal{R}^2$ and $\mathcal{Q}[z, w_1]$ and $\mathcal{Q}[z, w_2]$ holds $w_1 = w_2$. Consider u being a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ such that for every object z, $z \in \operatorname{dom} u$ iff $z \in \mathcal{R}^2$ and there exists an object w such that $\mathcal{Q}[z, w]$ and for every object z such that $z \in \operatorname{dom} u$ holds $\mathcal{Q}[z, u(z)]$. For every object z,

$z \in \operatorname{dom} u$ iff $z \in \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$. Consider x_1, t_1 being real numbers such that $x_1 \in X$ and $t_1 \in T$ and $\langle x, t\rangle = \langle x_1, t_1\rangle$ and $u(\langle x, t\rangle) = f_{/x_1} \cdot (g_{/t_1})$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a partial function g from $\mathbb{R}$ to $\mathbb{R}$, a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$, real numbers x_0, t_0, and an element z of $\mathcal{R}^2$. Now we state the propositions:

(8) Suppose $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$ and $z = \langle x_0, t_0\rangle$ and $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$. Then

(i) $u \cdot (\mathrm{reproj}(1, z)) = g_{/t_0} \cdot f$, and

(ii) $u \cdot (\mathrm{reproj}(2, z)) = f_{/x_0} \cdot g$.

PROOF: For every object s, $s \in \operatorname{dom}(u \cdot (\mathrm{reproj}(1, z)))$ iff $s \in \operatorname{dom} f$. For every object s, $s \in \operatorname{dom}(u \cdot (\mathrm{reproj}(2, z)))$ iff $s \in \operatorname{dom} g$. For every object s such that $s \in \operatorname{dom}(u \cdot (\mathrm{reproj}(1, z)))$ holds $(u \cdot (\mathrm{reproj}(1, z)))(s) = (g_{/t_0} \cdot f)(s)$. For every object s such that $s \in \operatorname{dom}(u \cdot (\mathrm{reproj}(2, z)))$ holds $(u \cdot (\mathrm{reproj}(2, z)))(s) = (f_{/x_0} \cdot g)(s)$ by [7, (14)]. □

(9) Suppose $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$ and $z = \langle x_0, t_0\rangle$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and f is differentiable in x_0 and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable in z w.r.t. 1, and

(ii) $\mathrm{partdiff}(u, z, 1) = f'(x_0) \cdot (g_{/t_0})$.

The theorem is a consequence of (8).

(10) Suppose $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$ and $z = \langle x_0, t_0\rangle$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and g is differentiable in t_0 and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable in z w.r.t. 2, and

(ii) $\mathrm{partdiff}(u, z, 2) = f_{/x_0} \cdot (g'(t_0))$.

The theorem is a consequence of (8).

Let us consider subsets X, T of $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a partial function g from $\mathbb{R}$ to $\mathbb{R}$, and a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$. Now we state the propositions:

(11) Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$ and X is open and T is open and Z is open and $Z = \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and

$t \in \operatorname{dom} g\}$ and f is differentiable on X and g is differentiable on T and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on Z w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 1\rangle} Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$, and

(iii) u is partially differentiable on Z w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 2\rangle} Z)_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$.

PROOF: $Z \subseteq \operatorname{dom} u$. For every element z of $\mathcal{R}^2$ such that $z \in Z$ holds u is partially differentiable in z w.r.t. 1. For every real numbers x, t and for every element z of $\mathcal{R}^2$ such that $x \in X$ and $t \in T$ and $z = \langle x, t\rangle$ holds $\operatorname{partdiff}(u, z, 1) = f'(x) \cdot (g_{/t})$. For every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 1\rangle} Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$. For every element z of $\mathcal{R}^2$ such that $z \in Z$ holds u is partially differentiable in z w.r.t. 2. For every real numbers x, t and for every element z of $\mathcal{R}^2$ such that $x \in X$ and $t \in T$ and $z = \langle x, t\rangle$ holds $\operatorname{partdiff}(u, z, 2) = f_{/x} \cdot (g'(t))$. □

(12) Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$ and X is open and T is open and Z is open and $Z = \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and f is differentiable 2 times on X and g is differentiable 2 times on T and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on Z w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle} Z)_{/\langle x,t\rangle} = (f'(X))(2)_{/x} \cdot (g_{/t})$, and

(iii) u is partially differentiable on Z w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(iv) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle} Z)_{/\langle x,t\rangle} = f_{/x} \cdot ((g'(T))(2)_{/t})$.

PROOF: u is partially differentiable on Z w.r.t. $\langle 1\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 1\rangle} Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$ and u is partially differentiable on Z w.r.t. $\langle 2\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction}^{\langle 2\rangle} Z)_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$. u is partially differentiable on Z w.r.t. 1. For every real numbers x, t such that $x \in \operatorname{dom}(f'_{\restriction X})$ and $t \in \operatorname{dom}(g{\restriction}T)$ holds $(u{\restriction}^{\langle 1\rangle} Z)_{/\langle x,t\rangle} = (f'_{\restriction X})_{/x} \cdot ((g{\restriction}T)_{/t})$. $u{\restriction}^{\langle 1\rangle} Z$ is partially differentiable on Z w.r.t. $\langle 1\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $((u{\restriction}^{\langle 1\rangle} Z){\restriction}^{\langle 1\rangle} Z)_{/\langle x,t\rangle} =$

$(f'_{\restriction X})'(x) \cdot ((g \restriction T)_{/t})$. For every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u \restriction^{\langle 1 \rangle ^\frown \langle 1 \rangle} Z)_{/\langle x,t \rangle} = (f'(X))(2)_{/x} \cdot (g_{/t})$. u is partially differentiable on Z w.r.t. 2. For every real numbers x, t such that $x \in \mathrm{dom}(f \restriction X)$ and $t \in \mathrm{dom}(g'_{\restriction T})$ holds $(u \restriction^{\langle 2 \rangle} Z)_{/\langle x,t \rangle} = (f \restriction X)_{/x} \cdot ((g'_{\restriction T})_{/t})$. $u \restriction^{\langle 2 \rangle} Z$ is partially differentiable on Z w.r.t. $\langle 2 \rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $((u \restriction^{\langle 2 \rangle} Z) \restriction^{\langle 2 \rangle} Z)_{/\langle x,t \rangle} = (f \restriction X)_{/x} \cdot ((g'_{\restriction T})'(t))$. □

(13) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$, and a real number c. Suppose f is differentiable 2 times on $\Omega_{\mathbb{R}}$ and g is differentiable 2 times on $\Omega_{\mathbb{R}}$ and $\mathrm{dom}\, u = \Omega_{\mathcal{R}^2}$ and for every real numbers x, t, $u_{/\langle x,t \rangle} = f_{/x} \cdot (g_{/t})$ and for every real numbers x, t, $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1 \rangle ^\frown \langle 1 \rangle$, and

(ii) for every real numbers x, t such that x, $t \in \Omega_{\mathbb{R}}$ holds $(u \restriction^{\langle 1 \rangle ^\frown \langle 1 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = (f'(\Omega_{\mathbb{R}}))(2)_{/x} \cdot (g_{/t})$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2 \rangle ^\frown \langle 2 \rangle$, and

(iv) for every real numbers x, t such that x, $t \in \Omega_{\mathbb{R}}$ holds $(u \restriction^{\langle 2 \rangle ^\frown \langle 2 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t})$, and

(v) for every real numbers x, t, $(u \restriction^{\langle 2 \rangle ^\frown \langle 2 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = c^{\mathbf{2}} \cdot ((u \restriction^{\langle 1 \rangle ^\frown \langle 1 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle})$.

The theorem is a consequence of (12).

(14) Let us consider real numbers A, B, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x)$. Then

(i) f is differentiable on $\Omega_{\mathbb{R}}$, and

(ii) for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$.

PROOF: Reconsider $f_1 = A \cdot (\text{the function cos}) \cdot (e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}})$, $f_2 = B \cdot (\text{the function sin}) \cdot (e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}})$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. Reconsider $Z = \Omega_{\mathbb{R}}$ as an open subset of $\mathbb{R}$. Reconsider $E = e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}}$ as a function from $\mathbb{R}$ into $\mathbb{R}$. For every real number x such that $x \in Z$ holds $E(x) = e \cdot x$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = f_1(x) + f_2(x)$. For every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$. □

2. The Method of Separation of Variables for One-dimensional Wave Equation

Now we state the propositions:

(15) Let us consider real numbers A, B, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot$ (the function cos)$(e \cdot x) + B \cdot$ (the function sin)$(e \cdot x)$. Then

(i) f is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(ii) for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot$ (the function sin)$(e \cdot x)$ $-B \cdot$ (the function cos)$(e \cdot x))$ and $((f'_{\restriction \Omega_{\mathbb{R}}})'_{\restriction \Omega_{\mathbb{R}}})(x) = -e^{\mathbf{2}} \cdot (A \cdot$ (the function cos)$(e \cdot x) + B \cdot$ (the function sin)$(e \cdot x))$ and $(f'(\Omega_{\mathbb{R}}))(2)_{/x} + e^{\mathbf{2}} \cdot (f_{/x}) = 0$.

PROOF: f is differentiable on $\Omega_{\mathbb{R}}$ and for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot$ (the function sin)$(e \cdot x) - B \cdot$ (the function cos)$(e \cdot x))$. For every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = e \cdot B \cdot$ (the function cos)$(e \cdot x) + (-e \cdot A) \cdot$ (the function sin)$(e \cdot x)$. For every natural number i such that $i \leqslant 2 - 1$ holds $(f'(\Omega_{\mathbb{R}}))(i)$ is differentiable on $\Omega_{\mathbb{R}}$. □

(16) Let us consider real numbers A, B, e. Then there exists a function f from $\mathbb{R}$ into $\mathbb{R}$ such that for every real number x, $f(x) = A \cdot$ (the function cos)$(e \cdot x) + B \cdot$ (the function sin)$(e \cdot x)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a real number t such that $\$_1 = t$ and $\$_2 = A \cdot$ (the function cos)$(e \cdot t) + B \cdot$ (the function sin)$(e \cdot t)$. For every object x such that $x \in \mathbb{R}$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$. Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every object x such that $x \in \mathbb{R}$ holds $\mathcal{P}[x, f(x)]$. □

(17) Let us consider real numbers A, B, C, d, c, e, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot$ (the function cos)$(e \cdot x) + B \cdot$ (the function sin)$(e \cdot x)$ and for every real number t, $g(t) = C \cdot$ (the function cos)$(e \cdot c \cdot t) + d \cdot$ (the function sin)$(e \cdot c \cdot t)$. Let us consider real numbers x, t. Then $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$. The theorem is a consequence of (15).

(18) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose f is differentiable 2 times on $\Omega_{\mathbb{R}}$ and g is differentiable 2 times on $\Omega_{\mathbb{R}}$ and for every real numbers x, t, $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$ and for every real numbers x, t, $u_{/\langle x,t \rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1 \rangle$, and

(ii) for every real numbers x, t, $(u{\restriction^{\langle 1 \rangle}}\Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = f'(x) \cdot (g_{/t})$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$, and

(v) f is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(vi) g is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(vii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(viii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (f'(\Omega_{\mathbb{R}}))(2)_{/x} \cdot (g_{/t})$, and

(ix) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(x) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t})$, and

(xi) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (11) and (13).

(19) Let us consider real numbers A, B, C, d, e, c, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose for every real numbers x, t, $u_{/\langle x,t\rangle} = (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (-A \cdot e \cdot (\text{the function } \sin)(e \cdot x) + B \cdot e \cdot (\text{the function } \cos)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (-C \cdot (e \cdot c) \cdot (\text{the function } \sin)(e \cdot c \cdot t) + d \cdot (e \cdot c) \cdot (\text{the function } \cos)(e \cdot c \cdot t))$, and

(v) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(vi) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -e^{\mathbf{2}} \cdot (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$ and u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$ and for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(e \cdot c)^{\mathbf{2}} \cdot (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$, and

(vii) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (16), (15), (17), (18), and (6).

(20) Let us consider a real number c. Then there exists a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$ such that

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$ and partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (16), (7), (15), (17), and (18).

3. The Superposition Principle

Now we state the propositions:

(21) Let us consider real numbers C, d, c, a natural number n, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose for every real numbers x, t, $u_{/\langle x,t\rangle} =$ (the function sin)$(n \cdot \pi \cdot x) \cdot (C \cdot$(the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$(the function sin)$(n \cdot \pi \cdot c \cdot t))$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = n \cdot \pi \cdot$ (the function cos)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} =$ (the function sin)$(n \cdot \pi \cdot x) \cdot (-C \cdot (n \cdot \pi \cdot c) \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t) + d \cdot (n \cdot \pi \cdot c) \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t))$, and

(v) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(vi) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(n \cdot \pi)^{\mathbf{2}} \cdot$ (the function sin)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$ and u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$ and for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(n \cdot \pi \cdot c)^{\mathbf{2}} \cdot$ (the function sin)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$, and

(vii) for every real number t, $u_{/\langle 0,t\rangle} = 0$ and $u_{/\langle 1,t\rangle} = 0$, and

(viii) for every real numbers x, t, $(u{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

PROOF: Set $e = n \cdot \pi$. For every real numbers x, t, $(u{\upharpoonright}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = e \cdot$ (the function cos)$(e \cdot x) \cdot (C \cdot$ (the function cos)$(e \cdot c \cdot t) + d \cdot$ (the function sin)$(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\upharpoonright}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} =$ (the function sin)$(e \cdot x) \cdot (-C \cdot (e \cdot c) \cdot$ (the function sin)$(e \cdot c \cdot t) + d \cdot (e \cdot c) \cdot$ (the function cos)$(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -e^{\mathbf{2}} \cdot$ (the function sin)$(e \cdot x) \cdot (C \cdot$ (the function cos)$(e \cdot c \cdot t) + d \cdot$ (the function sin)$(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(e \cdot c)^{\mathbf{2}} \cdot$ (the function sin)$(e \cdot x) \cdot (C \cdot$ (the function cos)$(e \cdot c \cdot t) + d \cdot$ (the function sin)$(e \cdot c \cdot t))$. For every real number t, $u_{/\langle 0,t\rangle} = 0$ and $u_{/\langle 1,t\rangle} = 0$ by [8, (30)]. □

(22) Let us consider partial functions u, v from $\mathcal{R}^2$ to $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, and a real number c. Suppose Z is open and $Z \subseteq \operatorname{dom} u$ and $Z \subseteq \operatorname{dom} v$ and u is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$ and v is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(v{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((v{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$. Then

(i) $Z \subseteq \operatorname{dom}(u + v)$, and

(ii) $u + v$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$, and

(iii) for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u + v{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u + v{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$.

PROOF: For every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u + v{\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u + v{\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$ by (1), [2, (75)]. □

(23) Let us consider a sequence u of partial functions from $\mathcal{R}^2$ into $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, and a real number c. Suppose Z is open and for every natural number i, $Z \subseteq \operatorname{dom}(u(i))$ and $\operatorname{dom}(u(i)) = \operatorname{dom}(u(0))$ and $u(i)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u(i){\upharpoonright}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u(i){\upharpoonright}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$. Let us consider a natural number i. Then

(i) $Z \subseteq \operatorname{dom}(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa \in \mathbb{N}})(i))$, and

(ii) $((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa \in \mathbb{N}})(i)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$, and

(iii) for every real numbers x, t such that $\langle x, t\rangle \in Z$ holds $(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(i)\restriction^{\langle 2\rangle ^\frown \langle 2\rangle} Z)_{/\langle x,t\rangle} =$ $c^{\mathbf{2}} \cdot ((((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(i)\restriction^{\langle 1\rangle ^\frown \langle 1\rangle} Z)_{/\langle x,t\rangle})$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv Z \subseteq \mathrm{dom}(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1))$ and $((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x, t\rangle \in Z$ holds $(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1)\restriction^{\langle 2\rangle ^\frown \langle 2\rangle} Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1)\restriction^{\langle 1\rangle ^\frown \langle 1\rangle} Z)_{/\langle x,t\rangle})$. For every natural number i such that $\mathcal{X}[i]$ holds $\mathcal{X}[i+1]$. For every natural number n, $\mathcal{X}[n]$. □

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Multilinear Operator and Its Basic Properties

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Summary. In the first chapter, the notion of multilinear operator on real linear spaces is discussed. The algebraic structure [2] of multilinear operators is introduced here. In the second chapter, the results of the first chapter are extended to the case of the normed spaces. This chapter shows that bounded multilinear operators on normed linear spaces constitute the algebraic structure. We referred to [3], [7], [5], [6] in this formalization.

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1. Multilinear Operator on Real Linear Spaces

Let X be a non empty, non-empty finite sequence, i be an object, and x be an element of $\prod X$. The functor $\mathrm{reproj}(i, x)$ yielding a function from $X(i)$ into $\prod X$ is defined by

(Def. 1) for every object r such that $r \in X(i)$ holds $it(r) = x +\cdot (i, r)$.

Now we state the propositions:

(1) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, an element i of $\mathrm{dom}\, X$, and an object r. If $r \in X(i)$, then $(\mathrm{reproj}(i, x))(r)(i) = r$.

(2) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, elements i, j of $\operatorname{dom} X$, and an object r. If $r \in X(i)$ and $i \neq j$, then $(\operatorname{reproj}(i, x))(r)(j) = x(j)$.

(3) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, and an element i of $\operatorname{dom} X$. Then $(\operatorname{reproj}(i, x))(x(i)) = x$.

Let X be a real linear space sequence, i be an element of $\operatorname{dom} X$, and x be an element of $\prod X$. The functor $\operatorname{reproj}(i, x)$ yielding a function from $X(i)$ into $\prod X$ is defined by

(Def. 2) there exists an element x_0 of $\prod \overline{X}$ such that $x_0 = x$ and $it = \operatorname{reproj}(i, x_0)$.

Now we state the propositions:

(4) Let us consider a real linear space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and a function F. If $F = (\operatorname{reproj}(i, x))(r)$, then $F(i) = r$. The theorem is a consequence of (1).

(5) Let us consider a real linear space sequence X, elements i, j of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and functions F, s. If $F = (\operatorname{reproj}(i, x))(r)$ and $x = s$ and $i \neq j$, then $F(j) = s(j)$. The theorem is a consequence of (2).

(6) Let us consider a real linear space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, and a function s. If $x = s$, then $(\operatorname{reproj}(i, x))(s(i)) = x$. The theorem is a consequence of (3).

Let X be a real linear space sequence, Y be a real linear space, and f be a function from $\prod X$ into Y. We say that f is multilinear if and only if

(Def. 3) for every element i of $\operatorname{dom} X$ and for every element x of $\prod X$, $f \cdot (\operatorname{reproj}(i, x))$ is a linear operator from $X(i)$ into Y.

One can verify that there exists a function from $\prod X$ into Y which is multilinear.

A multilinear operator from X into Y is a multilinear function from $\prod X$ into Y. Now we state the propositions:

(7) Let us consider real linear spaces X, Y, and a linear operator f from X into Y. Then $0_Y = f(0_X)$.

(8) Let us consider a real linear space sequence X, a real linear space Y, a multilinear operator g from X into Y, a point t of $\prod X$, and an element s of $\prod \overline{X}$. Suppose $s = t$ and there exists an element i of $\operatorname{dom} X$ such that $s(i) = 0_{X(i)}$. Then $g(t) = 0_Y$. The theorem is a consequence of (17) and (7).

(9) Let us consider a real linear space sequence X, a real linear space Y, a multilinear operator g from X into Y, and a finite sequence a of elements of $\mathbb{R}$. Suppose $\operatorname{dom} a = \operatorname{dom} X$. Let us consider points t, t_1 of $\prod X$, and

elements s, s_1 of $\prod \overline{X}$. Suppose $t = s$ and $t_1 = s_1$ and for every element i of $\operatorname{dom} X$, $s_1(i) = a_{/i} \cdot s(i)$. Then $g(t_1) = (\prod a) \cdot g(t)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every points t, t_1 of $\prod X$ for every elements s, s_1 of $\prod \overline{X}$ for every finite sequence b of elements of $\mathbb{R}$ such that $t = s$ and $t_1 = s_1$ and $b = a \restriction \$_1$ and $\$_1 \leqslant \operatorname{len} a$ and for every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \$_1$, then $s_1(i) = a_{/i} \cdot s(i)$ and if $i \notin \operatorname{Seg} \$_1$, then $s_1(i) = s(i)$ holds $g(t_1) = (\prod b) \cdot g(t)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. For every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \operatorname{len} a$, then $s_1(i) = a_{/i} \cdot s(i)$ and if $i \notin \operatorname{Seg} \operatorname{len} a$, then $s_1(i) = s(i)$. □

Let X be a real linear space sequence and Y be a real linear space. The functor MultOpers(X, Y) yielding a subset of RealVectSpace((the carrier of $\prod X), Y)$ is defined by

(Def. 4) for every set x, $x \in \mathit{it}$ iff x is a multilinear operator from X into Y.

One can check that MultOpers(X, Y) is non empty and functional and MultOpers(X, Y) is linearly closed.

The functor VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$ yielding a strict RLS structure is defined by the term

(Def. 5) $\langle$MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$), Mult(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$)$\rangle$.

Now we state the proposition:

(10) Let us consider a real linear space sequence X, and a real linear space Y. Then $\langle$MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X)$, $Y)$), Mult(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$)$\rangle$ is a subspace of RealVectSpace((the carrier of $\prod X), Y)$.

Let X be a real linear space sequence and Y be a real linear space. One can verify that VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$ is non empty and VectorSpaceOf MultOpers$_{\mathbb{R}}(X, Y)$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$ is constituted functions.

Let f be an element of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$ and v be a vector of $\prod X$. Let us note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(11) Let us consider a real linear space sequence X, a real linear space Y, and vectors f, g, h of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$.

(12) Let us consider a real linear space sequence X, a real linear space Y, vectors f, h of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

Let us consider a real linear space sequence X and a real linear space Y. Now we state the propositions:

(13) $0_{\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)} = (\text{the carrier of } \prod X) \longmapsto 0_Y$.

(14) (The carrier of $\prod X) \longmapsto 0_Y$ is a multilinear operator from X into Y.

2. Bounded Multilinear Operator on Normed Linear Spaces

Now we state the propositions:

(15) Let us consider a real norm space sequence X, an element i of dom X, an element x of $\prod X$, an element r of $X(i)$, and a function F. If $F = (\mathrm{reproj}(i, x))(r)$, then $F(i) = r$. The theorem is a consequence of (1).

(16) Let us consider a real norm space sequence X, elements i, j of dom X, an element x of $\prod X$, an element r of $X(i)$, and functions F, s. If $F = (\mathrm{reproj}(i, x))(r)$ and $x = s$ and $i \neq j$, then $F(j) = s(j)$. The theorem is a consequence of (2).

(17) Let us consider a real norm space sequence X, an element i of dom X, an element x of $\prod X$, and a function s. If $x = s$, then $(\mathrm{reproj}(i, x))(s(i)) = x$. The theorem is a consequence of (3).

Let X be a real norm space sequence, Y be a real normed space, and f be a function from $\prod X$ into Y. We say that f is multilinear if and only if

(Def. 6) for every element i of dom X and for every element x of $\prod X$, $f \cdot (\mathrm{reproj}(i, x))$ is a linear operator from $X(i)$ into Y.

One can verify that there exists a function from $\prod X$ into Y which is multilinear.

A multilinear operator from X into Y is a multilinear function from $\prod X$ into Y. The functor MultOpers(X, Y) yielding a subset of RealVectSpace((the carrier of $\prod X), Y)$ is defined by

(Def. 7) for every set x, $x \in \mathit{it}$ iff x is a multilinear operator from X into Y.

Note that MultOpers(X, Y) is non empty and functional and MultOpers(X, Y) is linearly closed.

Now we state the proposition:

(18) Let us consider a real norm space sequence X, and a real normed space Y. Then $\langle$MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X)$,

Y)), Mult(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y))⟩ is a subspace of RealVectSpace((the carrier of $\prod X$), Y).

Let X be a real norm space sequence and Y be a real normed space. Note that ⟨MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y)), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y)), Mult(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y))⟩ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The functor VectorSpaceOfMultOpers$_{\mathbb{R}}$(X, Y) yielding a strict real linear space is defined by the term

(Def. 8) ⟨MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y)), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y)), Mult(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X$), Y))⟩.

One can check that VectorSpaceOfMultOpers$_{\mathbb{R}}$(X, Y) is constituted functions.

Let f be an element of VectorSpaceOfMultOpers$_{\mathbb{R}}$(X, Y) and v be a vector of $\prod X$. One can check that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(19) Let us consider a real norm space sequence X, a real normed space Y, and vectors f, g, h of VectorSpaceOfMultOpers$_{\mathbb{R}}$(X, Y). Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$.

(20) Let us consider a real norm space sequence X, a real normed space Y, vectors f, h of VectorSpaceOfMultOpers$_{\mathbb{R}}$(X, Y), and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

Let us consider a real norm space sequence X and a real normed space Y. Now we state the propositions:

(21) $0_{\text{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)}$ = (the carrier of $\prod X$) $\longmapsto 0_Y$.

(22) (The carrier of $\prod X$) $\longmapsto 0_Y$ is a multilinear operator from X into Y.

Let X be a real norm space sequence, Y be a real normed space, I be a multilinear operator from X into Y, and x be a vector of $\prod X$. Let us observe that the functor $I(x)$ yields a point of Y. Note that $\prod X$ is constituted functions.

Let x be a point of $\prod X$ and i be an element of dom X. One can check that the functor $x(i)$ yields a point of $X(i)$. Now we state the propositions:

(23) Let us consider a real norm space sequence G, and points p, q, r of $\prod G$. Then $p+q = r$ if and only if for every element i of dom G, $r(i) = p(i)+q(i)$.

(24) Let us consider a real norm space sequence G, points p, r of $\prod G$, and a real number a. Then $a \cdot p = r$ if and only if for every element i of dom G, $r(i) = a \cdot p(i)$.

(25) Let us consider a real norm space sequence G, and a point p of $\prod G$. Then $0_{\prod G} = p$ if and only if for every element i of $\operatorname{dom} G$, $p(i) = 0_{G(i)}$.

(26) Let us consider a real norm space sequence G, and points p, q, r of $\prod G$. Then $p - q = r$ if and only if for every element i of $\operatorname{dom} G$, $r(i) = p(i) - q(i)$. The theorem is a consequence of (23) and (24).

Let X be a real norm space sequence and x be a point of $\prod X$. The functor $\mathrm{NrProduct}\, x$ yielding a non negative real number is defined by

(Def. 9) there exists a finite sequence N of elements of $\mathbb{R}$ such that $\operatorname{dom} N = \operatorname{dom} X$ and for every element i of $\operatorname{dom} X$, $N(i) = \|x(i)\|$ and $it = \prod N$.

Now we state the proposition:

(27) Let us consider a real norm space sequence X, and a point x of $\prod X$. Then

 (i) there exists an element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$ iff $\mathrm{NrProduct}\, x = 0$, and

 (ii) if there exists no element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$, then $0 < \mathrm{NrProduct}\, x$.

PROOF: Consider N being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} N = \operatorname{dom} X$ and for every element i of $\operatorname{dom} X$, $N(i) = \|x(i)\|$ and $\mathrm{NrProduct}\, x = \prod N$. There exists an element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$ iff $\mathrm{NrProduct}\, x = 0$ by [1, (103)]. If there exists no element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$, then $0 < \mathrm{NrProduct}\, x$ by [4, (42)]. □

Let X be a real norm space sequence, Y be a real normed space, and I be a multilinear operator from X into Y. We say that I is Lipschitzian if and only if

(Def. 10) there exists a real number K such that $0 \leqslant K$ and for every point x of $\prod X$, $\|I(x)\| \leqslant K \cdot (\mathrm{NrProduct}\, x)$.

Now we state the proposition:

(28) Let us consider a real norm space sequence X, a real normed space Y, and a multilinear operator f from X into Y. If for every vector x of $\prod X$, $f(x) = 0_Y$, then f is Lipschitzian.

Let X be a real norm space sequence and Y be a real normed space. One can check that there exists a multilinear operator from X into Y which is Lipschitzian.

The functor $\mathrm{BoundedMultOpers}(X, Y)$ yielding a subset of $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ is defined by

(Def. 11) for every set x, $x \in it$ iff x is a Lipschitzian multilinear operator from X into Y.

Note that BoundedMultOpers(X, Y) is non empty and BoundedMultOpers(X, Y) is linearly closed.

Now we state the proposition:

(29) Let us consider a real norm space sequence X, and a real normed space Y. Then $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$ is a subspace of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$.

Let X be a real norm space sequence and Y be a real normed space. Observe that $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The functor VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ yielding a strict real linear space is defined by the term

(Def. 12) $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$.

Let us note that every element of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ is function-like and relation-like.

Let f be an element of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ and v be a vector of $\prod X$. Note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(30) Let us consider a real norm space sequence X, a real normed space Y, and vectors f, g, h of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$. The theorem is a consequence of (19).

(31) Let us consider a real norm space sequence X, a real normed space Y, vectors f, h of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$. The theorem is a consequence of (20).

(32) Let us consider a real norm space sequence X, and a real normed space Y. Then $0_{\mathrm{VectorSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)} =$ (the carrier of $\prod X$) $\longmapsto 0_Y$. The theorem is a consequence of (21).

Let X be a real norm space sequence, Y be a real normed space, and f be an object. Assume $f \in \mathrm{BoundedMultOpers}(X, Y)$. The functor $\mathrm{PartFuncs}(f, X, Y)$ yielding a Lipschitzian multilinear operator from X into Y is defined by the term

(Def. 13) f.

Let u be a multilinear operator from X into Y. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 14) $\{\|u(t)\|$, where t is a vector of $\prod X$: for every element i of $\mathrm{dom}\, X$, $\|t(i)\| \leqslant 1\}$.

Now we state the propositions:

(33) Let us consider a real norm space sequence X, and an element s of $\prod X$. Then there exists a finite sequence F of elements of $\mathbb{R}$ such that

(i) $\mathrm{dom}\, F = \mathrm{dom}\, X$, and

(ii) for every element i of $\mathrm{dom}\, X$, $F(i) = \|s(i)\|$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\mathrm{dom}\, X$ such that $\$_1 = i$ and $\$_2 = \|s(i)\|$. For every natural number n such that $n \in \mathrm{Seg}\, \mathrm{len}\, X$ there exists an element d of $\mathbb{R}$ such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, X$ and for every natural number n such that $n \in \mathrm{Seg}\, \mathrm{len}\, X$ holds $\mathcal{Q}[n, F_{/n}]$. For every element i of $\mathrm{dom}\, X$, $F(i) = \|s(i)\|$. $\square$

(34) Let us consider a finite sequence F of elements of $\mathbb{R}$. Suppose for every element i of $\mathrm{dom}\, F$, $0 \leqslant F(i) \leqslant 1$. Then $0 \leqslant \prod F \leqslant 1$.

(35) Let us consider a real norm space sequence X, and a point x of $\prod X$. Suppose for every element i of $\mathrm{dom}\, X$, $\|x(i)\| \leqslant 1$. Then $0 \leqslant \mathrm{NrProduct}\, x \leqslant 1$. The theorem is a consequence of (34).

(36) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator g from X into Y, and a point t of $\prod X$. Suppose there exists an element i of $\mathrm{dom}\, X$ such that $t(i) = 0_{X(i)}$. Then $g(t) = 0_Y$. The theorem is a consequence of (17).

(37) Let us consider a real norm space sequence X, and a point x of $\prod X$. Then there exists a finite sequence d of elements of $\mathbb{R}$ such that

(i) $\mathrm{dom}\, d = \mathrm{dom}\, X$, and

(ii) for every element i of $\mathrm{dom}\, X$, $d(i) = \|x(i)\|^{-1}$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\mathrm{dom}\, X$ such that $\$_1 = i$ and $\$_2 = \|x(i)\|^{-1}$. For every natural number n such that $n \in \mathrm{Seg}\, \mathrm{len}\, X$ there exists an element d of $\mathbb{R}$ such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, X$ and for every natural number n such that $n \in \mathrm{Seg}\, \mathrm{len}\, X$ holds $\mathcal{Q}[n, F_{/n}]$. For every element i of $\mathrm{dom}\, X$, $F(i) = \|x(i)\|^{-1}$. $\square$

(38) Let us consider a real norm space sequence X, a point s of $\prod X$, and a finite sequence a of elements of $\mathbb{R}$. Then there exists a point s_1 of $\prod X$ such that for every element i of $\operatorname{dom} X$, $s_1(i) = a_{/i} \cdot s(i)$.
Proof: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\operatorname{dom} X$ such that $\$_1 = i$ and $\$_2 = a_{/i} \cdot x(i)$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an object d such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} X$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ holds $\mathcal{Q}[n, F(n)]$. For every object y such that $y \in \operatorname{dom} \overline{X}$ holds $F(y) \in \overline{X}(y)$. For every element i of $\operatorname{dom} X$, $F(i) = a_{/i} \cdot x(i)$. □

(39) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator g from X into Y, and a finite sequence a of elements of $\mathbb{R}$. Suppose $\operatorname{dom} a = \operatorname{dom} X$. Let us consider points t, t_1 of $\prod X$. Suppose for every element i of $\operatorname{dom} X$, $t_1(i) = a_{/i} \cdot t(i)$. Then $g(t_1) = (\prod a) \cdot g(t)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every points t, t_1 of $\prod X$ for every finite sequence b of elements of $\mathbb{R}$ such that $b = a{\restriction}\$_1$ and $\$_1 \leqslant \operatorname{len} a$ and for every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \$_1$, then $t_1(i) = a_{/i} \cdot t(i)$ and if $i \notin \operatorname{Seg} \$_1$, then $t_1(i) = t(i)$ holds $g(t_1) = (\prod b) \cdot g(t)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. For every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \operatorname{len} a$, then $t_1(i) = a_{/i} \cdot t(i)$ and if $i \notin \operatorname{Seg} \operatorname{len} a$, then $t_1(i) = t(i)$. □

(40) Let us consider finite sequences F, G of elements of $\mathbb{R}$. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every element i of $\operatorname{dom} F$, $G(i) = F(i)^{-1}$. Then $\prod G = (\prod F)^{-1}$.

(41) Let us consider a real norm space sequence X, a real normed space Y, and a Lipschitzian multilinear operator g from X into Y. Then $\operatorname{PreNorms}(g)$ is upper bounded. The theorem is a consequence of (35).

(42) Let us consider a real norm space sequence X, a real normed space Y, and a multilinear operator g from X into Y. Then g is Lipschitzian if and only if $\operatorname{PreNorms}(g)$ is upper bounded. The theorem is a consequence of (36), (37), (38), (39), (40), and (41).

Let X be a real norm space sequence and Y be a real normed space. The functor $\operatorname{BoundedMultOpersNorm}(X, Y)$ yielding a function from $\operatorname{BoundedMultOpers}(X, Y)$ into $\mathbb{R}$ is defined by

(Def. 15) for every object x such that $x \in \operatorname{BoundedMultOpers}(X, Y)$ holds $it(x) = \sup \operatorname{PreNorms}(\operatorname{PartFuncs}(x, X, Y))$.

Let f be a Lipschitzian multilinear operator from X into Y. One can verify that $\operatorname{PartFuncs}(f, X, Y)$ reduces to f.

Now we state the proposition:

(43) Let us consider a real norm space sequence X, a real normed space Y, and a Lipschitzian multilinear operator f from X into Y. Then $(\text{BoundedMultOpersNorm}(X, Y))(f) = \sup \text{PreNorms}(f)$.

Let X be a real norm space sequence and Y be a real normed space. The functor $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ yielding a non empty, strict normed structure is defined by the term

(Def. 16) $\langle \text{BoundedMultOpers}(X, Y), \text{Zero}(\text{BoundedMultOpers}(X, Y), \text{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)), \text{Add}(\text{BoundedMultOpers}(X, Y), \text{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)), \text{Mult}(\text{BoundedMultOpers}(X, Y), \text{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)), \text{BoundedMultOpersNorm}(X, Y) \rangle$.

Now we state the propositions:

(44) Let us consider a real norm space sequence X, and a real normed space Y. Then (the carrier of $\prod X) \longmapsto 0_Y = 0_{\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$. The theorem is a consequence of (32).

(45) Let us consider a real norm space sequence X, a real normed space Y, a point f of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$, and a Lipschitzian multilinear operator g from X into Y. Suppose $g = f$. Let us consider a vector t of $\prod X$. Then $\|g(t)\| \leqslant \|f\| \cdot (\text{NrProduct}\, t)$. The theorem is a consequence of (41), (36), (37), (38), (39), (40), and (43).

Let us consider a real norm space sequence X, a real normed space Y, and a point f of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. Now we state the propositions:

(46) $0 \leqslant \|f\|$. The theorem is a consequence of (41) and (43).

(47) If $f = 0_{\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$, then $0 = \|f\|$. The theorem is a consequence of (41), (44), and (43).

Let X be a real norm space sequence and Y be a real normed space. Let us note that every element of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is function-like and relation-like.

Let f be an element of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ and v be a vector of $\prod X$. Note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(48) Let us consider a real norm space sequence X, a real normed space Y, and points f, g, h of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$. The theorem is a consequence of (30).

(49) Let us consider a real norm space sequence X, a real normed space Y, points f, h of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

The theorem is a consequence of (31).

(50) Let us consider a real norm space sequence X, a real normed space Y, points f, g of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$, and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

Proof: $\|f + g\| \leqslant \|f\| + \|g\|$. $\|a \cdot f\| = |a| \cdot \|f\|$. □

(51) Let us consider a real norm space sequence X, and a real normed space Y. Then $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is a real normed space.

Let X be a real norm space sequence and Y be a real normed space. Let us note that $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(52) Let us consider a real norm space sequence X, a real normed space Y, and points f, g, h of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. Then $h = f - g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) - g(x)$. The theorem is a consequence of (48).

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Cross-Ratio in Real Vector Space

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. Using Mizar [1], in the context of a real vector space, we introduce the concept of affine ratio of three aligned points (see [5]).

It is also equivalent to the notion of "Mesure algébrique"[1], to the opposite of the notion of Teilverhältnis[2] or to the opposite of the ordered length-ratio [9].

In the second part, we introduce the classic notion of "cross-ratio" of 4 points aligned in a real vector space.

Finally, we show that if the real vector space is the real line, the notion corresponds to the classical notion[3] [9]:

> The cross-ratio of a quadruple of distinct points on the real line with coordinates x_1, x_2, x_3, x_4 is given by:

$$(x_1, x_2; x_3, x_4) = \frac{x_3 - x_1}{x_3 - x_2} . \frac{x_4 - x_2}{x_4 - x_1}$$

In the Mizar Mathematical Library, the vector spaces were first defined by Kusak, Leończuk and Muzalewski in the article [6], while the actual real vector space was defined by Trybulec [10] and the complex vector space was defined by Endou [4]. Nakasho and Shidama have developed a solution to explore the notions introduced by different authors[4] [7]. The definitions can be directly linked in the HTMLized version of the Mizar library[5].

The study of the cross-ratio will continue within the framework of the Klein-Beltrami model [2], [3]. For a generalized cross-ratio, see Papadopoulos [8].

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Keywords: affine ratio; cross-ratio; real vector space; geometry

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[1] https://fr.wikipedia.org/wiki/Mesure_algébrique
[2] https://de.wikipedia.org/wiki/Teilverhältnis
[3] https://en.wikipedia.org/wiki/Cross-ratio
[4] http://webmizar.cs.shinshu-u.ac.jp/mmlfe/current
[5] Example: RealLinearSpace http://mizar.org/version/current/html/rlvect_1.html#NM2

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1. Preliminaries

Let a, b, c, d be objects. Observe that $\langle a, b, c, d\rangle(1)$ reduces to a and $\langle a, b, c, d\rangle(2)$ reduces to b and $\langle a, b, c, d\rangle(3)$ reduces to c and $\langle a, b, c, d\rangle(4)$ reduces to d.

Now we state the proposition:

(1) Let us consider objects a, b, c, d, a', b', c', d'. Suppose $\langle a, b, c, d\rangle = \langle a', b', c', d'\rangle$. Then

(i) $a = a'$, and

(ii) $b = b'$, and

(iii) $c = c'$, and

(iv) $d = d'$.

Let r be a real number. We say that r is unit if and only if

(Def. 1) $r = 1$.

Let us observe that there exists a non zero real number which is non unit.

Let r be a non unit, non zero real number. The functor op1(r) yielding a non unit, non zero real number is defined by the term

(Def. 2) $\frac{1}{r}$.

One can check that the functor is involutive.

The functor op2(r) yielding a non unit, non zero real number is defined by the term

(Def. 3) $1 - r$.

Let us observe that the functor is involutive.

From now on a, b, r denote non unit, non zero real numbers.

Now we state the propositions:

(2) (i) $\mathrm{op2}(\mathrm{op1}(r)) = \frac{r-1}{r}$, and

(ii) $\mathrm{op1}(\mathrm{op2}(r)) = \frac{1}{1-r}$, and

(iii) $\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(r))) = \frac{r}{r-1}$, and

(iv) $\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(r))) = \frac{r}{r-1}$.

(3) (i) $\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(r)))) = \mathrm{op1}(\mathrm{op2}(r))$, and

(ii) $\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(r)))) = \mathrm{op2}(\mathrm{op1}(r))$.

The theorem is a consequence of (2).

(4) $\frac{\mathrm{op1}(a)}{\mathrm{op1}(b)} = \frac{b}{a}$.

In the sequel X denotes a non empty set and x denotes a 4-tuple of X.

Now we state the propositions:

(5) $X^4 =$ the set of all $\langle d_1, d_2, d_3, d_4\rangle$ where d_1, d_2, d_3, d_4 are elements of X.

(6) Let us consider objects a, b, c, d. Suppose ($a = x(1)$ or $a = x(2)$ or $a = x(3)$ or $a = x(4)$) and ($b = x(1)$ or $b = x(2)$ or $b = x(3)$ or $b = x(4)$) and ($c = x(1)$ or $c = x(2)$ or $c = x(3)$ or $c = x(4)$) and ($d = x(1)$ or $d = x(2)$ or $d = x(3)$ or $d = x(4)$). Then $\langle a, b, c, d\rangle$ is a 4-tuple of X. The theorem is a consequence of (5).

Let X be a non empty set and x be a 4-tuple of X. The functors: $\sigma_{1342}(x)$, $\sigma_{1423}(x)$, $\sigma_{2143}(x)$, $\sigma_{2314}(x)$, and $\sigma_{2341}(x)$ yielding 4-tuples of X are defined by terms

(Def. 4) $\langle x(1), x(3), x(4), x(2)\rangle$,

(Def. 5) $\langle x(1), x(4), x(2), x(3)\rangle$,

(Def. 6) $\langle x(2), x(1), x(4), x(3)\rangle$,

(Def. 7) $\langle x(2), x(3), x(1), x(4)\rangle$,

(Def. 8) $\langle x(2), x(3), x(4), x(1)\rangle$,

respectively. The functors: $\sigma_{2413}(x)$, $\sigma_{2431}(x)$, $\sigma_{3124}(x)$, $\sigma_{3142}(x)$, and $\sigma_{3241}(x)$ yielding 4-tuples of X are defined by terms

(Def. 9) $\langle x(2), x(4), x(1), x(3)\rangle$,

(Def. 10) $\langle x(2), x(4), x(3), x(1)\rangle$,

(Def. 11) $\langle x(3), x(1), x(2), x(4)\rangle$,

(Def. 12) $\langle x(3), x(1), x(4), x(2)\rangle$,

(Def. 13) $\langle x(3), x(2), x(4), x(1)\rangle$,

respectively. The functors: $\sigma_{3412}(x)$, $\sigma_{3421}(x)$, $\sigma_{4123}(x)$, $\sigma_{4132}(x)$, and $\sigma_{4213}(x)$ yielding 4-tuples of X are defined by terms

(Def. 14) $\langle x(3), x(4), x(1), x(2)\rangle$,

(Def. 15) $\langle x(3), x(4), x(2), x(1)\rangle$,

(Def. 16) $\langle x(4), x(1), x(2), x(3)\rangle$,

(Def. 17) $\langle x(4), x(1), x(3), x(2)\rangle$,

(Def. 18) $\langle x(4), x(2), x(1), x(3)\rangle$,

respectively. The functors: $\sigma_{4312}(x)$ and $\sigma_{4321}(x)$ yielding 4-tuples of X are defined by terms

(Def. 19) $\langle x(4), x(3), x(1), x(2)\rangle$,

(Def. 20) $\langle x(4), x(3), x(2), x(1)\rangle$,

respectively. The functors: $\sigma_{\mathrm{id}}(x)$ and $\sigma_{12}(x)$ yielding 4-tuples of X are defined by terms

(Def. 21) $\langle x(1), x(2), x(3), x(4)\rangle$,

(Def. 22) $\langle x(2), x(1), x(3), x(4)\rangle$, respectively. Observe that the functor is involutive.

The functors: $\sigma_{13}(x)$ and $\sigma_{14}(x)$ yielding 4-tuples of X are defined by terms

(Def. 23) $\langle x(3), x(2), x(1), x(4)\rangle$,

(Def. 24) $\langle x(4), x(2), x(3), x(1)\rangle$,

respectively. One can check that the functor is involutive.

The functor $\sigma_{23}(x)$ yielding a 4-tuple of X is defined by the term

(Def. 25) $\langle x(1), x(3), x(2), x(4)\rangle$.

Note that the functor is involutive.

The functors: $\sigma_{24}(x)$ and $\sigma_{34}(x)$ yielding 4-tuples of X are defined by terms

(Def. 26) $\langle x(1), x(4), x(3), x(2)\rangle$,

(Def. 27) $\langle x(1), x(2), x(4), x(3)\rangle$,

respectively. Let us observe that the functor is involutive.

Note that $\sigma_{\rm id}(x)$ reduces to x.

We introduce the notation $\sigma_{1234}(x)$ as a synonym of $\sigma_{\rm id}(x)$ and $\sigma_{2134}(x)$ as a synonym of $\sigma_{12}(x)$ and $\sigma_{3214}(x)$ as a synonym of $\sigma_{13}(x)$. And $\sigma_{4231}(x)$ as a synonym of $\sigma_{14}(x)$ and $\sigma_{1324}(x)$ as a synonym of $\sigma_{23}(x)$ and $\sigma_{1432}(x)$ as a synonym of $\sigma_{24}(x)$ and $\sigma_{1243}(x)$ as a synonym of $\sigma_{34}(x)$.

Now we state the propositions:

(7) (i) $\sigma_{12}(\sigma_{13}(x)) = \sigma_{13}(\sigma_{23}(x))$, and

(ii) $\sigma_{12}(\sigma_{14}(x)) = \sigma_{14}(\sigma_{24}(x))$, and

(iii) $\sigma_{12}(\sigma_{23}(x)) = \sigma_{13}(\sigma_{12}(x))$, and

(iv) $\sigma_{12}(\sigma_{24}(x)) = \sigma_{14}(\sigma_{12}(x))$, and

(v) $\sigma_{12}(\sigma_{34}(x)) = \sigma_{34}(\sigma_{12}(x))$, and

(vi) $\sigma_{13}(\sigma_{12}(x)) = \sigma_{23}(\sigma_{13}(x))$, and

(vii) $\sigma_{13}(\sigma_{14}(x)) = \sigma_{34}(\sigma_{13}(x))$, and

(viii) $\sigma_{13}(\sigma_{23}(x)) = \sigma_{12}(\sigma_{13}(x))$, and

(ix) $\sigma_{13}(\sigma_{24}(x)) = \sigma_{13}(\sigma_{24}(x))$, and

(x) $\sigma_{13}(\sigma_{34}(x)) = \sigma_{14}(\sigma_{13}(x))$, and

(xi) $\sigma_{23}(\sigma_{12}(x)) = \sigma_{13}(\sigma_{23}(x))$, and

(xii) $\sigma_{23}(\sigma_{13}(x)) = \sigma_{12}(\sigma_{23}(x))$, and

(xiii) $\sigma_{23}(\sigma_{14}(x)) = \sigma_{14}(\sigma_{23}(x))$, and

(xiv) $\sigma_{23}(\sigma_{24}(x)) = \sigma_{34}(\sigma_{23}(x))$, and

(xv) $\sigma_{23}(\sigma_{34}(x)) = \sigma_{24}(\sigma_{23}(x))$, and

(xvi) $\sigma_{24}(\sigma_{12}(x)) = \sigma_{14}(\sigma_{24}(x))$, and

(xvii) $\sigma_{24}(\sigma_{13}(x)) = \sigma_{13}(\sigma_{24}(x))$, and

(xviii) $\sigma_{24}(\sigma_{14}(x)) = \sigma_{12}(\sigma_{24}(x))$, and

(xix) $\sigma_{24}(\sigma_{23}(x)) = \sigma_{34}(\sigma_{24}(x))$, and

(xx) $\sigma_{24}(\sigma_{34}(x)) = \sigma_{23}(\sigma_{24}(x))$, and

(xxi) $\sigma_{34}(\sigma_{12}(x)) = \sigma_{12}(\sigma_{34}(x))$, and

(xxii) $\sigma_{34}(\sigma_{13}(x)) = \sigma_{14}(\sigma_{34}(x))$, and

(xxiii) $\sigma_{34}(\sigma_{14}(x)) = \sigma_{13}(\sigma_{34}(x))$, and

(xxiv) $\sigma_{34}(\sigma_{23}(x)) = \sigma_{24}(\sigma_{34}(x))$, and

(xxv) $\sigma_{34}(\sigma_{24}(x)) = \sigma_{23}(\sigma_{34}(x))$.

(8) (i) $\sigma_{1342}(x) = \sigma_{34}(\sigma_{23}(x))$, and

(ii) $\sigma_{1423}(x) = \sigma_{34}(\sigma_{24}(x))$, and

(iii) $\sigma_{2143}(x) = \sigma_{12}(\sigma_{34}(x))$, and

(iv) $\sigma_{2314}(x) = \sigma_{23}(\sigma_{12}(x))$, and

(v) $\sigma_{2341}(x) = \sigma_{34}(\sigma_{23}(\sigma_{12}(x)))$, and

(vi) $\sigma_{2413}(x) = \sigma_{34}(\sigma_{24}(\sigma_{12}(x)))$, and

(vii) $\sigma_{2431}(x) = \sigma_{24}(\sigma_{12}(x))$, and

(viii) $\sigma_{3124}(x) = \sigma_{23}(\sigma_{13}(x))$, and

(ix) $\sigma_{3142}(x) = \sigma_{24}(\sigma_{34}(\sigma_{13}(x)))$, and

(x) $\sigma_{3241}(x) = \sigma_{34}(\sigma_{13}(x))$, and

(xi) $\sigma_{3412}(x) = \sigma_{24}(\sigma_{13}(x))$, and

(xii) $\sigma_{3421}(x) = \sigma_{24}(\sigma_{23}(\sigma_{13}(x)))$, and

(xiii) $\sigma_{4123}(x) = \sigma_{23}(\sigma_{34}(\sigma_{14}(x)))$, and

(xiv) $\sigma_{4132}(x) = \sigma_{24}(\sigma_{14}(x))$, and

(xv) $\sigma_{4213}(x) = \sigma_{34}(\sigma_{14}(x))$, and

(xvi) $\sigma_{4312}(x) = \sigma_{23}(\sigma_{24}(\sigma_{14}(x)))$, and

(xvii) $\sigma_{4321}(x) = \sigma_{23}(\sigma_{14}(x))$.

(9) (i) $\sigma_{13}(\sigma_{23}(\sigma_{13}(x))) = \sigma_{12}(x)$, and

(ii) $\sigma_{12}(\sigma_{34}(\sigma_{23}(\sigma_{13}(x)))) = \sigma_{34}(\sigma_{23}(x))$, and

(iii) $\sigma_{23}(\sigma_{24}(\sigma_{14}(\sigma_{23}(\sigma_{13}(x))))) = \sigma_{14}(x)$.

(10) (i) $\sigma_{23}(\sigma_{14}(\sigma_{34}(x))) = \sigma_{24}(\sigma_{23}(\sigma_{13}(x)))$, and

(ii) $\sigma_{34}(\sigma_{24}(\sigma_{12}(x))) = \sigma_{24}(\sigma_{13}(\sigma_{23}(x)))$, and

(iii) $\sigma_{24}(\sigma_{34}(\sigma_{13}(x))) = \sigma_{12}(\sigma_{34}(\sigma_{23}(x)))$.

2. Affine Ratio

In the sequel V denotes a real linear space and A, B, C, P, Q, R, S denote elements of V.

Now we state the proposition:

(11) P, Q and Q are collinear.

Let V be a real linear space and A, B, C be elements of V. Assume $A \neq C$ and A, B and C are collinear. The functor AffineRatio(A, B, C) yielding a real number is defined by

(Def. 28) $B - A = it \cdot (C - A)$.

Now we state the propositions:

(12) If $A \neq C$ and A, B and C are collinear, then $A - B = (\text{AffineRatio}(A, B, C)) \cdot (A - C)$.

(13) If $A \neq C$ and A, B and C are collinear, then AffineRatio$(A, B, C) = 0$ iff $A = B$.

(14) If $A \neq C$ and A, B and C are collinear, then AffineRatio$(A, B, C) = 1$ iff $B = C$.

(15) Let us consider real numbers a, b. If $P \neq Q$ and $a \cdot (P - Q) = b \cdot (P - Q)$, then $a = b$.

(16) If P, Q and R are collinear and $P \neq R$ and $P \neq Q$, then AffineRatio$(P, R, Q) = \frac{1}{\text{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (15).

(17) Suppose P, Q and R are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$. Then AffineRatio$(Q, P, R) = \frac{\text{AffineRatio}(P,Q,R)}{\text{AffineRatio}(P,Q,R) - 1}$. The theorem is a consequence of (13) and (14).

(18) If P, Q and R are collinear and $P \neq R$, then AffineRatio$(R, Q, P) = 1 - \text{AffineRatio}(P, Q, R)$. The theorem is a consequence of (15).

(19) If P, Q and R are collinear and $P \neq R$ and $P \neq Q$, then AffineRatio$(Q, R, P) = \frac{\text{AffineRatio}(P,Q,R) - 1}{\text{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (13) and (15).

(20) If P, Q and R are collinear and $P \neq R$ and $Q \neq R$, then AffineRatio$(R, P, Q) = \frac{1}{1 - \text{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (14) and (15).

(21) Let us consider a real number r. Suppose P, Q and R are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$ and $r = \text{AffineRatio}(P, Q, R)$. Then

(i) AffineRatio$(P, R, Q) = \frac{1}{r}$, and

(ii) AffineRatio$(Q, P, R) = \frac{r}{r-1}$, and

(iii) AffineRatio$(Q, R, P) = \frac{r-1}{r}$, and

(iv) AffineRatio$(R, P, Q) = \frac{1}{1-r}$, and

(v) AffineRatio$(R, Q, P) = 1 - r$.

(22) Let us consider a non zero real number a. Suppose P, Q and R are collinear and $P \neq R$. Then AffineRatio$(P, Q, R) =$ AffineRatio$(a \cdot P, a \cdot Q, a \cdot R)$.

(23) Let us consider elements x, y of $\mathcal{R}^1$, and 1-tuples p, q of $\mathbb{R}$. If $p = x$ and $q = y$, then $x + y = p + q$.

Let us consider elements x, y of $\mathcal{E}^1_{\rm T}$ and 1-tuples p, q of $\mathbb{R}$. Now we state the propositions:

(24) If $p = x$ and $q = y$, then $x + y = p + q$.

(25) If $p = x$ and $q = y$, then $x - y = p - q$.

(26) Let us consider an element x of $\mathcal{E}^1_{\rm T}$, and a 1-tuple p of $\mathbb{R}$. If $p = x$, then $-x = -p$.

(27) Let us consider a real linear space T, elements x, y of T, and 1-tuples p, q of $\mathbb{R}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$ and $q = y$, then $x + y = p + q$.

(28) Let us consider a 1-tuple p of $\mathbb{R}$. Then $-p$ is a 1-tuple of $\mathbb{R}$.

(29) Let us consider a real linear space T, an element x of T, and a 1-tuple p of $\mathbb{R}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$, then $-p = -x$. The theorem is a consequence of (27).

(30) Let us consider a real linear space T, an element x of T, and an element p of $\mathcal{E}^1_{\rm T}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$, then $-p = -x$. The theorem is a consequence of (29).

(31) Let us consider a real linear space T, elements x, y of T, and 1-tuples p, q of $\mathbb{R}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$ and $q = y$, then $x - y = p - q$. The theorem is a consequence of (28) and (29).

(32) Let us consider a real linear space T, elements x, y of T, and elements p, q of $\mathcal{E}^1_{\rm T}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$ and $q = y$, then $x + y = p + q$. The theorem is a consequence of (27).

(33) Let us consider a set D, and an element d of D. Then $\mathrm{Seg}\,1 \longmapsto d = \langle d \rangle$.

(34) Let us consider real numbers a, r. Then $(\cdot_{\mathbb{R}})^{\circ}(\mathrm{Seg}\,1 \longmapsto a, \langle r \rangle) = \langle a \cdot r \rangle$. The theorem is a consequence of (33).

Let us consider a real number a and a 1-tuple p of $\mathbb{R}$. Now we state the propositions:

(35) $(\cdot_{\mathbb{R}})^{\circ}(\mathrm{dom}\, p \longmapsto a, p) = a \cdot p$. The theorem is a consequence of (34).

(36) $(\cdot_{\mathbb{R}})^{\circ}(\mathrm{dom}\, p \longmapsto a, p) = a \cdot p$.

(37) Let us consider a real linear space T, elements x, y of T, a real number a, and 1-tuples p, q of $\mathbb{R}$. If $T = \mathcal{E}^1_{\rm T}$ and $p = x$ and $q = y$ and $x = a \cdot y$, then $p = a \cdot q$. The theorem is a consequence of (35).

(38) Let us consider a real linear space T, elements x, y of T, a real number a, and elements p, q of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then if $x = a \cdot y$, then $p = a \cdot q$. The theorem is a consequence of (37).

(39) Let us consider a real linear space T, elements x, y of T, and elements p, q of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then $x - y = p - q$. The theorem is a consequence of (30) and (32).

(40) Let us consider 1-tuples p, q of $\mathbb{R}$, and a real number r. Suppose $p = r \cdot q$ and $p \neq \langle 0 \rangle$. Then there exist real numbers a, b such that

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $r = \frac{a}{b}$.

(41) Let us consider elements x, y, z of $\mathcal{E}^1_{\mathrm{T}}$. Then x, y and z are collinear.

Let us consider a real linear space T and elements x, y, z of T. Now we state the propositions:

(42) If $T = \mathcal{E}^1_{\mathrm{T}}$, then x, y and z are collinear.

(43) Suppose $T = \mathcal{E}^1_{\mathrm{T}}$. Then suppose $z \neq x$ and $y \neq x$. Then there exist real numbers a, b, c such that

(i) $x = \langle a \rangle$, and

(ii) $y = \langle b \rangle$, and

(iii) $z = \langle c \rangle$, and

(iv) $\mathrm{AffineRatio}(x, y, z) = \frac{b-a}{c-a}$.

The theorem is a consequence of (31), (41), (37), and (40).

Now we state the propositions:

(44) Let us consider an element x of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers a, r. If $x = \langle a \rangle$, then $r \cdot x = \langle r \cdot a \rangle$.

(45) Let us consider elements x, y of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers a, b, r. If $x = \langle a \rangle$ and $y = \langle b \rangle$, then $x = r \cdot y$ iff $a = r \cdot b$. The theorem is a consequence of (44).

(46) Let us consider elements x, y of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers a, b. If $x = \langle a \rangle$ and $y = \langle b \rangle$, then $x - y = \langle a - b \rangle$.

(47) Let us consider a real linear space V, elements x, y of $\mathbb{R}_{\mathrm{F}}$, and elements x', y' of V. If $V = \mathbb{R}_{\mathrm{F}}$ and $x = x'$ and $y = y'$, then $x + y = x' + y'$.

Let us consider a real linear space V and elements P, Q, R of V. Now we state the propositions:

(48) If P, Q and R are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$, then $\mathrm{AffineRatio}(P, Q, R) \neq 0$ and $\mathrm{AffineRatio}(P, Q, R) \neq 1$.

(49) Suppose P, Q and R are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$. Then there exists a non unit, non zero real number r such that

(i) $r = \text{AffineRatio}(P, Q, R)$, and

(ii) $\text{AffineRatio}(P, R, Q) = \text{op1}(r)$, and

(iii) $\text{AffineRatio}(Q, P, R) = \text{op1}(\text{op2}(\text{op1}(r)))$, and

(iv) $\text{AffineRatio}(Q, R, P) = \text{op2}(\text{op1}(r))$, and

(v) $\text{AffineRatio}(R, P, Q) = \text{op1}(\text{op2}(r))$, and

(vi) $\text{AffineRatio}(R, Q, P) = \text{op2}(r)$.

The theorem is a consequence of (13), (14), (16), (17), (18), (19), (20), and (2).

3. Cross-Ratio

Now we state the propositions:

(50) Let us consider a non empty set X, a 4-tuple x of X, and elements P, Q, R, S of X. Suppose $x = \langle P, Q, R, S \rangle$. Then

(i) $\sigma_{1234}(x) = \langle P, Q, R, S \rangle$, and

(ii) $\sigma_{1243}(x) = \langle P, Q, S, R \rangle$, and

(iii) $\sigma_{1324}(x) = \langle P, R, Q, S \rangle$, and

(iv) $\sigma_{1342}(x) = \langle P, R, S, Q \rangle$, and

(v) $\sigma_{1423}(x) = \langle P, S, Q, R \rangle$, and

(vi) $\sigma_{1432}(x) = \langle P, S, R, Q \rangle$, and

(vii) $\sigma_{2134}(x) = \langle Q, P, R, S \rangle$, and

(viii) $\sigma_{2143}(x) = \langle Q, P, S, R \rangle$, and

(ix) $\sigma_{2314}(x) = \langle Q, R, P, S \rangle$, and

(x) $\sigma_{2341}(x) = \langle Q, R, S, P \rangle$, and

(xi) $\sigma_{2413}(x) = \langle Q, S, P, R \rangle$, and

(xii) $\sigma_{2431}(x) = \langle Q, S, R, P \rangle$, and

(xiii) $\sigma_{3124}(x) = \langle R, P, Q, S \rangle$, and

(xiv) $\sigma_{3142}(x) = \langle R, P, S, Q \rangle$, and

(xv) $\sigma_{3214}(x) = \langle R, Q, P, S \rangle$, and

(xvi) $\sigma_{3241}(x) = \langle R, Q, S, P \rangle$, and

(xvii) $\sigma_{3412}(x) = \langle R, S, P, Q \rangle$, and

(xviii) $\sigma_{3421}(x) = \langle R, S, Q, P \rangle$, and
(xix) $\sigma_{4123}(x) = \langle S, P, Q, R \rangle$, and
(xx) $\sigma_{4132}(x) = \langle S, P, R, Q \rangle$, and
(xxi) $\sigma_{4213}(x) = \langle S, Q, P, R \rangle$, and
(xxii) $\sigma_{4231}(x) = \langle S, Q, R, P \rangle$, and
(xxiii) $\sigma_{4312}(x) = \langle S, R, P, Q \rangle$, and
(xxiv) $\sigma_{4321}(x) = \langle S, R, Q, P \rangle$.

(51) Let us consider a non empty set X, and a 4-tuple x of X. Then
(i) $\sigma_{1324}(\sigma_{1243}(x)) = \sigma_{1423}(x)$, and
(ii) $\sigma_{2143}(\sigma_{1243}(x)) = \sigma_{2134}(x)$, and
(iii) $\sigma_{3412}(\sigma_{1243}(x)) = \sigma_{4312}(x)$, and
(iv) $\sigma_{4321}(\sigma_{1243}(x)) = \sigma_{3421}(x)$, and
(v) $\sigma_{3412}(\sigma_{1324}(x)) = \sigma_{2413}(x)$, and
(vi) $\sigma_{2143}(\sigma_{1324}(x)) = \sigma_{3142}(x)$, and
(vii) $\sigma_{4321}(\sigma_{1324}(x)) = \sigma_{4231}(x)$, and
(viii) $\sigma_{3412}(\sigma_{1423}(x)) = \sigma_{2314}(x)$, and
(ix) $\sigma_{2143}(\sigma_{1423}(x)) = \sigma_{4132}(x)$, and
(x) $\sigma_{4321}(\sigma_{1423}(x)) = \sigma_{3241}(x)$, and
(xi) $\sigma_{1243}(\sigma_{1423}(x)) = \sigma_{1432}(x)$, and
(xii) $\sigma_{4321}(\sigma_{1432}(x)) = \sigma_{2341}(x)$, and
(xiii) $\sigma_{3412}(\sigma_{1432}(x)) = \sigma_{3214}(x)$, and
(xiv) $\sigma_{2143}(\sigma_{1432}(x)) = \sigma_{4123}(x)$, and
(xv) $\sigma_{4321}(\sigma_{3124}(x)) = \sigma_{4213}(x)$, and
(xvi) $\sigma_{3412}(\sigma_{3124}(x)) = \sigma_{2431}(x)$, and
(xvii) $\sigma_{2143}(\sigma_{3124}(x)) = \sigma_{1342}(x)$, and
(xviii) $\sigma_{4312}(\sigma_{3124}(x)) = \sigma_{4231}(x)$, and
(xix) $\sigma_{4321}(\sigma_{3124}(x)) = \sigma_{4213}(x)$.

In the sequel x denotes a 4-tuple of the carrier of V and P', Q', R', S' denote elements of V.

Let V be a real linear space and P, Q, R, S be elements of V. The functor CrossRatio(P, Q, R, S) yielding a real number is defined by the term

(Def. 29) $\frac{\text{AffineRatio}(R,P,Q)}{\text{AffineRatio}(S,P,Q)}$.

Now we state the propositions:

(52) If P, Q, R, and S are collinear and $R \neq Q$ and $S \neq Q$ and $S \neq P$, then $R = P$ iff $\text{CrossRatio}(P, Q, R, S) = 0$. The theorem is a consequence of (13).

(53) If $P \neq R$ and $P \neq S$, then $\text{CrossRatio}(P, P, R, S) = 1$. The theorem is a consequence of (11) and (14).

(54) If P, Q, R, and S are collinear and $R \neq Q$ and $S \neq Q$ and $R \neq S$ and $\text{CrossRatio}(P, Q, R, S) = 1$, then $P = Q$. The theorem is a consequence of (15) and (14).

(55) Suppose P, Q, R, and S are collinear and P', Q', R', and S' are collinear and $S \neq P$ and $S \neq Q$ and $S' \neq P'$ and $S' \neq Q'$. Then $\text{CrossRatio}(P, Q, R, S) = \text{CrossRatio}(P', Q', R', S')$ if and only if $\text{AffineRatio}(R, P, Q) \cdot \text{AffineRatio}(S', P', Q') = \text{AffineRatio}(R', P', Q') \cdot \text{AffineRatio}(S, P, Q)$. The theorem is a consequence of (13).

(56) If P, Q, R, and S are collinear and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\text{CrossRatio}(P, Q, R, S) = \text{CrossRatio}(R, S, P, Q)$. The theorem is a consequence of (13).

(57) Let us consider a real linear space V, and elements P, Q, R, S of V. Suppose P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$. Then $\text{CrossRatio}(P, Q, R, S) = \text{CrossRatio}(Q, P, S, R)$. The theorem is a consequence of (11), (14), and (49).

(58) If P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\text{CrossRatio}(P, Q, R, S) = \text{CrossRatio}(S, R, Q, P)$. The theorem is a consequence of (57) and (56).

(59) $\text{CrossRatio}(P, Q, S, R) = \frac{1}{\text{CrossRatio}(P, Q, R, S)}$.

(60) If P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\text{CrossRatio}(Q, P, R, S) = \frac{1}{\text{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (57).

(61) If P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\text{CrossRatio}(R, S, Q, P) = \frac{1}{\text{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (58).

(62) If P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\text{CrossRatio}(S, R, P, Q) = \frac{1}{\text{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (56).

(63) If P, Q, R, and S are collinear and P, Q, R, S are mutually different, then $\text{CrossRatio}(P, R, Q, S) = 1 - \text{CrossRatio}(P, Q, R, S)$. The theorem is a consequence of (17), (20), (14), (13), and (15).

(64) If P, Q, R, and S are collinear and P, Q, R, S are mutually different, then $\text{CrossRatio}(Q, S, P, R) = 1 - \text{CrossRatio}(P, Q, R, S)$. The theorem is

a consequence of (56) and (63).

(65) If P, Q, R, and S are collinear and P, Q, R, S are mutually different, then $\mathrm{CrossRatio}(R,P,S,Q) = 1 - \mathrm{CrossRatio}(P,Q,R,S)$. The theorem is a consequence of (57) and (63).

(66) If P, Q, R, and S are collinear and P, Q, R, S are mutually different, then $\mathrm{CrossRatio}(S,Q,R,P) = 1 - \mathrm{CrossRatio}(P,Q,R,S)$. The theorem is a consequence of (58) and (63).

Let V be a real linear space and x be a 4-tuple of the carrier of V. The functor $\mathrm{CrossRatio}(x)$ yielding a real number is defined by

(Def. 30) there exist elements P, Q, R, S of V such that $P = x(1)$ and $Q = x(2)$ and $R = x(3)$ and $S = x(4)$ and $it = \mathrm{CrossRatio}(P,Q,R,S)$.

Now we state the propositions:

(67) If $x = \langle P,Q,R,S \rangle$, then $\mathrm{CrossRatio}(P,Q,R,S) = \mathrm{CrossRatio}(x)$.

(68) Suppose $x = \langle P,Q,R,S \rangle$ and P, Q, R, and S are collinear and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{3412}(x))$. The theorem is a consequence of (56).

(69) Suppose $x = \langle P,Q,R,S \rangle$ and P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then

(i) $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{2143}(x))$, and

(ii) $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{4321}(x))$.

The theorem is a consequence of (57) and (58).

(70) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$.

(71) Suppose $x = \langle P,Q,R,S \rangle$ and P, Q, R, S are mutually different and P, Q, R, and S are collinear. Then there exists a non unit, non zero real number r such that

(i) $r = \mathrm{CrossRatio}(x)$, and

(ii) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \mathrm{op1}(r)$.

The theorem is a consequence of (54), (52), and (70).

(72) Suppose $x = \langle P,Q,R,S \rangle$ and P, Q, R, and S are collinear and $P \neq R$ and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then

(i) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2134}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3421}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4312}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (69) and (68).

(73) Suppose $x = \langle P, Q, R, S \rangle$ and P, Q, R, S are mutually different and P, Q, R, and S are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1324}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2413}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3142}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4231}(x)) = 1 - \mathrm{CrossRatio}(x)$.

The theorem is a consequence of (68), (69), and (63).

(74) Suppose $x = \langle P, Q, R, S \rangle$ and P, Q, R, S are mutually different and P, Q, R, and S are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{3124}(x)) = \frac{1}{1 - \mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2431}(x)) = \frac{1}{1 - \mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{1342}(x)) = \frac{1}{1 - \mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4213}(x)) = \frac{1}{1 - \mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (70), (73), (68), and (69).

(75) Suppose $x = \langle P, Q, R, S \rangle$ and P, Q, R, S are mutually different and P, Q, R, and S are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1423}(x)) = \frac{\mathrm{CrossRatio}(x) - 1}{\mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2314}(x)) = \frac{\mathrm{CrossRatio}(x) - 1}{\mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{4132}(x)) = \frac{\mathrm{CrossRatio}(x) - 1}{\mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{3241}(x)) = \frac{\mathrm{CrossRatio}(x) - 1}{\mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (52), (67), (73), (72), (68), and (69).

(76) Suppose $x = \langle P, Q, R, S \rangle$ and P, Q, R, S are mutually different and P, Q, R, and S are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1432}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x) - 1}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2341}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x) - 1}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3214}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x) - 1}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4123}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x) - 1}$.

The theorem is a consequence of (70), (75), (69), and (68).

4. Cross-Ratio and the Real Line

Now we state the proposition:

(77) Let us consider elements x_1, x_2, x_3, x_4 of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $x_2 \neq x_3$ and $x_3 \neq x_1$ and $x_2 \neq x_4$ and $x_1 \neq x_4$. Then there exist real numbers a, b, c, d such that

(i) $x_1 = \langle a \rangle$, and

(ii) $x_2 = \langle b \rangle$, and

(iii) $x_3 = \langle c \rangle$, and

(iv) $x_4 = \langle d \rangle$, and

(v) $\mathrm{CrossRatio}(\langle x_1, x_2, x_3, x_4 \rangle) = \frac{c-a}{c-b} \cdot \frac{d-b}{d-a}$.

The theorem is a consequence of (43).

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Continuity of Multilinear Operator on Normed Linear Spaces

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, various definitions of contuity of multilinear operators on normed linear spaces are discussed in the Mizar formalism [4], [1] and [2]. In the first chapter, several basic theorems are prepared to handle the norm of the multilinear operator, and then it is formalized that the linear space of bounded multilinear operators is a complete Banach space.

In the last chapter, the continuity of the multilinear operator on finite normed spaces is addressed. Especially, it is formalized that the continuity at the origin can be extended to the continuity at every point in its whole domain. We referred to [5], [11], [8], [9] in this formalization.

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1. Completeness of the Space of Multilinear Operators

Now we state the propositions:

(1) Let us consider a natural number n, and a real number r. Suppose $0 < r$. Then there exists a real number s such that

(i) $0 < s < r$, and

(ii) $\sqrt{s \cdot s \cdot n} < r$.

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(2) Let us consider finite sequences R_1, R_2 of elements of $\mathbb{R}$, natural numbers n, i, and a real number r. Suppose $i \in \operatorname{dom} R_1$ and $R_1 = n \mapsto (1$ **qua** real number) and $R_2 = R_1 +\cdot (i, r)$. Then $\prod R_2 = r$.

(3) Let us consider a finite sequence F of elements of $\mathbb{R}$. Suppose for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} F$ holds $0 \leqslant F(k)$. Then $0 \leqslant \prod F$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence F of elements of $\mathbb{R}$ such that for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} F$ holds $0 \leqslant F(k)$ and $\operatorname{len} F = \$_1$ holds $0 \leqslant \prod F$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[0]$. For every natural number n, $\mathcal{P}[n]$. $\square$

From now on X, G denote real norm space sequences, Y denotes a real normed space, and f denotes a multilinear operator from X into Y.

Now we state the propositions:

(4) $\operatorname{dom} \overline{X} = \operatorname{dom} X$.

(5) Let us consider an element z of $\prod X$. If $z = 0_{\prod X}$, then for every element i of $\operatorname{dom} X$, $z(i) = 0_{X(i)}$. The theorem is a consequence of (4).

(6) $f(0_{\prod X}) = 0_Y$. The theorem is a consequence of (5).

(7) Let us consider a finite sequence F of elements of $\mathbb{R}$. If for every element i of $\operatorname{dom} F$, $F(i) > 0$, then $\prod F > 0$.

(8) Let us consider a real norm space sequence X, and a real normed space Y. Suppose Y is complete. Let us consider a sequence s_1 of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists a sequence x_1 of Y such that for every natural number n, $x_1(n) = (\text{PartFuncs}(vseq(n), X, Y))(\$_1)$ and x_1 is convergent and $\$_2 = \lim x_1$. For every element x of $\prod X$, there exists an element y of Y such that $\mathcal{P}[x, y]$. Consider f being a function from the carrier of $\prod X$ into the carrier of Y such that for every element x of $\prod X$, $\mathcal{P}[x, f(x)]$. Reconsider $t_1 = f$ as a function from $\prod X$ into Y. For every point u of $\prod X$ and for every element i of $\operatorname{dom} X$ and for every point x of $X(i)$, there exists a sequence x_2 of Y such that for every natural number n, $x_2(n) = ((\text{PartFuncs}(vseq(n), X, Y)) \cdot (\text{reproj}(i, u)))(x)$ and x_2 is convergent and $(t_1 \cdot (\text{reproj}(i, u)))(x) = \lim x_2$. t_1 is Lipschitzian by [10, (20)].

For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ for every point x of $\prod X$, $\|(\text{PartFuncs}(vseq(n), X, Y))(x) - t_1(x)\| \leqslant e \cdot (\text{NrProduct}\, x)$. Reconsider $t_2 = t_1$ as a point of $\text{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ holds $\|vseq(n) -$

$t_2\| \leqslant e$. For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\|vseq(n) - t_2\| < e$. □

(9) Let us consider a real norm space sequence X, and a real Banach space Y. Then $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is a real Banach space. The theorem is a consequence of (8).

Let X be a real norm space sequence and Y be a real Banach space. One can check that $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is complete.

2. Equivalence of Continuity Definitions of Multilinear Operators

Now we state the propositions:

(10) Let us consider a natural number n, an element F of $\mathcal{R}^n$, and a real number s. Suppose for every natural number i such that $i \in \mathrm{dom}\, F$ holds $0 \leqslant F(i) \leqslant s$. Then $|F| \leqslant \sqrt{s \cdot s \cdot (\mathrm{len}\, F)}$.
PROOF: Set $G = \mathrm{len}\, F \mapsto s$. Reconsider $F_0 = F$ as an element of $\mathbb{R}^{\mathrm{len}\, F}$. For every natural number j such that $j \in \mathrm{Seg}\,\mathrm{len}\, F_0$ holds $({}^2F_0)(j) \leqslant ({}^2G)(j)$. □

(11) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator f from X into Y, and a real number K. Suppose $0 \leqslant K$ and for every point x of $\prod X$, $\|f(x)\| \leqslant K \cdot (\mathrm{NrProduct}\, x)$. Let us consider points v_0, v_1 of $\prod X$, finite sequences C_0, C_1, and an element i of $\mathrm{dom}\, X$. Suppose $C_0 = v_0$ and $C_1 = v_1$ and $\|v_1 - v_0\| \leqslant 1$ and for every element j of $\mathrm{dom}\, X$ such that $i \neq j$ holds $C_1(j) = C_0(j)$. Then $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 1)^{\mathrm{len}\, X} \cdot K \cdot \|(v_1 - v_0)(i)\|$.
PROOF: For every object x such that $x \in \mathrm{dom}\, v_1$ holds $v_1(x) = (\mathrm{reproj}(i, v_0))(v_1(i))(x)$. Reconsider $v_3 = (\mathrm{reproj}(i, v_0))(v_1(i) - v_0(i))$ as a point of $\prod X$. Reconsider $R_1 = \mathrm{len}\, X \mapsto (1$ **qua** real number$)$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $N_1 = \|(v_1 - v_0)(i)\|$ as an element of $\mathbb{R}$. Reconsider $R_2 = R_1 +\cdot (i, N_1)$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $R_3 = \mathrm{len}\, X \mapsto (\|v_0\| + 1)$ as a finite sequence of elements of $\mathbb{R}$. Set $R_4 = R_2 \bullet R_3$. $\prod R_2 = \|(v_1 - v_0)(i)\|$. Consider N_2 being a finite sequence of elements of $\mathbb{R}$ such that $\mathrm{dom}\, N_2 = \mathrm{dom}\, X$ and for every element i of $\mathrm{dom}\, X$, $N_2(i) = \|v_3(i)\|$ and $\mathrm{NrProduct}\, v_3 = \prod N_2$. For every element k of $\mathbb{N}$ such that $k \in \mathrm{dom}\, N_2$ holds $N_2(k) \leqslant R_4(k)$ and $0 \leqslant N_2(k)$. □

(12) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator f from X into Y, and a real number K. Suppose

$0 \leqslant K$ and for every point x of $\prod X$, $\|f(x)\| \leqslant K \cdot (\text{NrProduct}\, x)$. Let us consider a point v_0 of $\prod X$. Then there exists a real number M such that

(i) $0 \leqslant M$, and

(ii) for every point v_1 of $\prod X$ such that $\|v_1 - v_0\| \leqslant 1$ there exists a finite sequence F of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{dom} X$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant M \cdot K \cdot (\sum F)$ and for every element i of $\operatorname{dom} X$, $F(i) = \|(v_1 - v_0)(i)\|$.

PROOF: Consider g being a function such that $v_0 = g$ and $\operatorname{dom} g = \operatorname{dom} \overline{X}$ and for every object i such that $i \in \operatorname{dom} \overline{X}$ holds $g(i) \in \overline{X}(i)$. Reconsider $C_0 = v_0$ as a finite sequence. Define $\mathcal{P}[\text{natural number}] \equiv$ for every points v_0, v_1 of $\prod X$ for every finite sequences C_0, C_1 such that $\|v_1 - v_0\| \leqslant 1$ and $v_0 = C_0$ and $v_1 = C_1$ and $\$_1 \leqslant \operatorname{len} X$ and $C_1 \restriction (\operatorname{len} X -' \$_1) = C_0 \restriction (\operatorname{len} X -' \$_1)$ there exists a finite sequence F of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{Seg} \$_1$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 3)^{\operatorname{len} X} \cdot K \cdot (\sum F)$ and for every natural number n such that $n \in \operatorname{Seg} \$_1$ there exists an element i of $\operatorname{dom} X$ such that $i = \operatorname{len} X -' \$_1 + n$ and $F(n) = \|(v_1 - v_0)(i)\|$.

$\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. Consider g being a function such that $v_1 = g$ and $\operatorname{dom} g = \operatorname{dom} \overline{X}$ and for every object i such that $i \in \operatorname{dom} \overline{X}$ holds $g(i) \in \overline{X}(i)$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} X$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 3)^{\operatorname{len} X} \cdot K \cdot (\sum F)$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an element i of $\operatorname{dom} X$ such that $i = \operatorname{len} X -' \operatorname{len} X + n$ and $F(n) = \|(v_1 - v_0)(i)\|$. For every element i of $\operatorname{dom} X$, $F(i) = \|(v_1 - v_0)(i)\|$. □

(13) Let us consider a point x of $\prod X$, and a real number r. Suppose $0 < r$. Then there exists a finite sequence s of elements of $\mathbb{R}$ and there exists a non empty, non-empty finite sequence Y such that $\operatorname{dom} s = \operatorname{dom} X$ and $\operatorname{dom} Y = \operatorname{dom} X$ and $\prod Y \subseteq \operatorname{Ball}(x, r)$ and for every element i of $\operatorname{dom} X$, $0 < s(i) < r$ and $Y(i) = \operatorname{Ball}(x(i), s(i))$.

PROOF: Consider s_0 being a real number such that $0 < s_0 < r$ and $\sqrt{s_0 \cdot s_0 \cdot (\operatorname{len} X)} < r$. Set $C_2 = \operatorname{len} X \mapsto s_0$. For every element i of $\operatorname{dom} X$, $0 < C_2(i) < r$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element i of $\operatorname{dom} X$ such that $\$_1 = i$ and $\$_2 = \operatorname{Ball}(x(i), C_2(i))$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an object d such that $\mathcal{P}[n, d]$. Consider Y being a finite sequence such that $\operatorname{dom} Y = \operatorname{Seg} \operatorname{len} X$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ holds $\mathcal{P}[n, Y(n)]$. $\emptyset \notin \operatorname{rng} Y$ by [6, (14)]. For every element i of $\operatorname{dom} X$, $Y(i) = \operatorname{Ball}(x(i), C_2(i))$. For every object z such that $z \in \prod Y$ holds $z \in \operatorname{Ball}(x, r)$. □

(14) Let us consider a real norm space sequence X, a real normed space Y,

and a multilinear operator f from X into Y. Then

(i) f is continuous on the carrier of $\prod X$ iff f is continuous in $0_{\prod X}$, and

(ii) f is continuous on the carrier of $\prod X$ iff f is Lipschitzian.

Proof: $f_{/0_{\prod X}} = 0_Y$. If f is continuous in $0_{\prod X}$, then f is Lipschitzian by [7, (7)], (13), (4), (5). If f is Lipschitzian, then f is continuous on the carrier of $\prod X$ by (12), [3, (10)]. □

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Fubini's Theorem

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. Fubini theorem is an essential tool for the analysis of high-dimensional space [8], [2], [3], a theorem about the multiple integral and iterated integral. The author has been working on formalizing Fubini's theorem over the past few years [4], [6] in the Mizar system [7], [1]. As a result, Fubini's theorem (30) was proved in complete form by this article.

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1. Preliminaries

From now on X denotes a set.

Now we state the proposition:

(1) Let us consider a subset A of X, and an X-defined binary relation f. Then $f \restriction A^{\rm c} = f \restriction (\operatorname{dom} f \setminus A)$.

Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(2) GTE-dom$(f, +\infty) =$ EQ-dom$(f, +\infty)$.

(3) LEQ-dom$(f, -\infty) =$ EQ-dom$(f, -\infty)$.

(4) Let us consider a partial function f from X to $\overline{\mathbb{R}}$, and an extended real e. Then GTE-dom(f, e) misses LE-dom(f, e).

(5) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Then $\operatorname{dom} f =$ (EQ-dom $(f, -\infty) \cup$ GT-dom$(f, -\infty) \cap$ LE-dom$(f, +\infty)) \cup$ EQ-dom$(f, +\infty)$.

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In the sequel X, X_1, X_2 denote non empty sets.

(6) Let us consider a partial function f from X to $\overline{\mathbb{R}}$, and an element x of X. Then

(i) $(\max_+(f))(x) \leqslant |f|(x)$, and

(ii) $(\max_-(f))(x) \leqslant |f|(x)$.

(7) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(|f|, x) = |\mathrm{ProjPMap1}(f, x)|$, and

(ii) $\mathrm{ProjPMap2}(|f|, y) = |\mathrm{ProjPMap2}(f, y)|$.

2. Markov's Inequality

From now on S denotes a σ-field of subsets of X, S_1 denotes a σ-field of subsets of X_1, S_2 denotes a σ-field of subsets of X_2, M denotes a σ-measure on S, M_1 denotes a σ-measure on S_1, and M_2 denotes a σ-measure on S_2.

Let X be a non empty set, S be a σ-field of subsets of X, and E be an element of S. One can verify that there exists a partial function from X to $\overline{\mathbb{R}}$ which is E-measurable.

Now we state the proposition:

(8) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\mathrm{dom}\, f = E$.
Then $\text{EQ-dom}(f, +\infty)$, $\text{EQ-dom}(f, -\infty) \in S$.

Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$ and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(9) Suppose M_1 is σ-finite and M_2 is σ-finite and $\mathrm{dom}\, f = E$. Then

(i) $\int \mathrm{Integral2}(M_2, |f|)\, \mathrm{d}M_1 = \int |f|\, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$, and

(ii) $\int \mathrm{Integral1}(M_1, |f|)\, \mathrm{d}M_2 = \int |f|\, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$.

(10) Suppose M_1 is σ-finite and M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ if and only if $\int \mathrm{Integral1}(M_1, |f|)\, \mathrm{d}M_2 < +\infty$.

(11) Suppose M_1 is σ-finite and M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ if and only if $\int \mathrm{Integral2}(M_2, |f|)\, \mathrm{d}M_1 < +\infty$.

(12) Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element U of S_1, and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then $\mathrm{Integral2}(M_2, |f|)$ is U-measurable.

(13) Let us consider an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element V of S_2, and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and $E = \text{dom}\, f$. Then $\text{Integral1}(M_1, |f|)$ is V-measurable.

Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(14) Suppose M_2 is σ-finite and f is integrable on $\text{ProdMeas}(M_1, M_2)$. Then

(i) $\int \max_+(\text{Integral2}(M_2, |f|))\, \mathrm{d}M_1 = \int \text{Integral2}(M_2, |f|)\, \mathrm{d}M_1$, and

(ii) $\int \max_-(\text{Integral2}(M_2, |f|))\, \mathrm{d}M_1 = 0$.

The theorem is a consequence of (12).

(15) Suppose M_1 is σ-finite and f is integrable on $\text{ProdMeas}(M_1, M_2)$. Then

(i) $\int \max_+(\text{Integral1}(M_1, |f|))\, \mathrm{d}M_2 = \int \text{Integral1}(M_1, |f|)\, \mathrm{d}M_2$, and

(ii) $\int \max_-(\text{Integral1}(M_1, |f|))\, \mathrm{d}M_2 = 0$.

The theorem is a consequence of (13).

(16) MARKOV'S INEQUALITY:

Let us consider an element E of S, an E-measurable partial function f from X to $\overline{\mathbb{R}}$, and an extended real e. Suppose $\text{dom}\, f = E$ and f is non-negative and $e \geqslant 0$. Then $e \cdot M(\text{GTE-dom}(f, e)) \leqslant \int f\, \mathrm{d}M$.

PROOF: $\text{GTE-dom}(f, +\infty) = \text{EQ-dom}(f, +\infty)$. Reconsider $E_3 = \text{GTE-dom}(f, e)$ as an element of S. For every element x of X such that $x \in \text{dom}(\chi_{e,E_3,X} \restriction E_3)$ holds $(\chi_{e,E_3,X} \restriction E_3)(x) \leqslant (f \restriction E_3)(x)$. $\square$

3. FUBINI'S THEOREM

Now we state the propositions:

(17) Let us consider partial functions f, g from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M. Then

(i) $\int f + g\, \mathrm{d}M = \int f \restriction (\text{dom}\, f \cap \text{dom}\, g)\, \mathrm{d}M + \int g \restriction (\text{dom}\, f \cap \text{dom}\, g)\, \mathrm{d}M$, and

(ii) $\int f - g\, \mathrm{d}M = \int f \restriction (\text{dom}\, f \cap \text{dom}\, g)\, \mathrm{d}M - \int g \restriction (\text{dom}\, f \cap \text{dom}\, g)\, \mathrm{d}M$.

(18) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Then f is integrable on M if and only if $\max_+(f)$ is integrable on M and $\max_-(f)$ is integrable on M.

(19) Let us consider elements A, B of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $B \subseteq A$ and $f \restriction A$ is B-measurable. Then f is B-measurable.

Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is integrable a.e. w.r.t. M if and only if

(Def. 1) there exists an element A of S such that $M(A) = 0$ and $A \subseteq \operatorname{dom} f$ and $f{\restriction}A^{\rm c}$ is integrable on M.

Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(20) If f is integrable a.e. w.r.t. M, then $\operatorname{dom} f \in S$.

(21) If f is integrable on M, then f is integrable a.e. w.r.t. M. The theorem is a consequence of (1).

Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is finite M-a.e. if and only if

(Def. 2) there exists an element A of S such that $M(A) = 0$ and $A \subseteq \operatorname{dom} f$ and $f{\restriction}A^{\rm c}$ is a partial function from X to $\mathbb{R}$.

Now we state the propositions:

(22) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} f = E$. Then f is finite M-a.e. if and only if $M(\text{EQ-dom}(f, +\infty) \cup \text{EQ-dom}(f, -\infty)) = 0$. The theorem is a consequence of (8).

(23) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then

- (i) $M(\text{EQ-dom}(f, +\infty)) = 0$, and
- (ii) $M(\text{EQ-dom}(f, -\infty)) = 0$, and
- (iii) f is finite M-a.e., and
- (iv) for every real number r such that $r > 0$ holds $M(\text{GTE-dom}(|f|, r)) < +\infty$.

The theorem is a consequence of (16).

(24) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\text{ProdMeas}(M_1, M_2)$. Then

- (i) $\text{Integral1}(M_1, \max_+(f))$ is integrable on M_2, and
- (ii) $\text{Integral2}(M_2, \max_+(f))$ is integrable on M_1, and
- (iii) $\text{Integral1}(M_1, \max_-(f))$ is integrable on M_2, and
- (iv) $\text{Integral2}(M_2, \max_-(f))$ is integrable on M_1, and
- (v) $\text{Integral1}(M_1, |f|)$ is integrable on M_2, and
- (vi) $\text{Integral2}(M_2, |f|)$ is integrable on M_1.

(25) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} f \subseteq E$ and f is integrable a.e. w.r.t. M. Then f is integrable on M. The theorem is a consequence of (20) and (1).

(26) Let us consider an element A of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $M(A) = 0$ and $A \subseteq \operatorname{dom} f$ and $f \restriction A^{\mathrm{c}}$ is integrable on M. Then there exists a partial function g from X to $\overline{\mathbb{R}}$ such that

(i) $\operatorname{dom} g = \operatorname{dom} f$, and

(ii) $f \restriction A^{\mathrm{c}} = g \restriction A^{\mathrm{c}}$, and

(iii) g is integrable on M, and

(iv) $\int f \restriction A^{\mathrm{c}} \, \mathrm{d}M = \int g \, \mathrm{d}M$.

PROOF: Consider B being an element of S such that $B = \operatorname{dom}(f \restriction A^{\mathrm{c}})$ and $f \restriction A^{\mathrm{c}}$ is B-measurable. $f \restriction A^{\mathrm{c}} = f \restriction (\operatorname{dom} f \setminus A)$. Define $\mathcal{C}[\text{object}] \equiv \$_1 \in A$. Define $\mathcal{F}(\text{object}) = +\infty$. Define $\mathcal{G}(\text{object}) = f(\$_1)$. Consider g being a function such that $\operatorname{dom} g = \operatorname{dom} f$ and for every object x such that $x \in \operatorname{dom} f$ holds if $\mathcal{C}[x]$, then $g(x) = \mathcal{F}(x)$ and if not $\mathcal{C}[x]$, then $g(x) = \mathcal{G}(x)$. For every real number r, $(A \cup B) \cap \text{LE-dom}(g, r) \in S$. $\int f \restriction A^{\mathrm{c}} \, \mathrm{d}M = \int g \restriction (\operatorname{dom} g \setminus A) \, \mathrm{d}M$. □

(27) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\text{ProdMeas}(M_1, M_2)$. Then

(i) $\int f \, \mathrm{d}\,\text{ProdMeas}(M_1, M_2) =$
$\int \text{Integral1}(M_1, \max_+(f)) \, \mathrm{d}M_2 - \int \text{Integral1}(M_1, \max_-(f)) \, \mathrm{d}M_2$, and

(ii) $\int f \, \mathrm{d}\,\text{ProdMeas}(M_1, M_2) =$
$\int \text{Integral2}(M_2, \max_+(f)) \, \mathrm{d}M_1 - \int \text{Integral2}(M_2, \max_-(f)) \, \mathrm{d}M_1$.

(28) Let us consider an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2. Then

(i) if $M_1(\text{MeasurableYsection}(E, y)) \neq 0$, then
$(\text{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = +\infty$, and

(ii) if $M_1(\text{MeasurableYsection}(E, y)) = 0$, then
$(\text{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = 0$.

(29) Let us consider an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element x of X_1. Then

(i) if $M_2(\text{MeasurableXsection}(E, x)) \neq 0$, then
$(\text{Integral2}(M_2, \overline{\chi}_{E, X_1 \times X_2}))(x) = +\infty$, and

(ii) if $M_2(\text{MeasurableXsection}(E, x)) = 0$, then
$(\text{Integral2}(M_2, \overline{\chi}_{E, X_1 \times X_2}))(x) = 0$.

(30) FUBINI'S THEOREM:

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element S_3 of S_1. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ and $X_1 = S_3$. Then there exists an element U of S_1 such that

(i) $M_1(U) = 0$, and

(ii) for every element x of X_1 such that $x \in U^{\mathrm{c}}$ holds $\mathrm{ProjPMap1}(f, x)$ is integrable on M_2, and

(iii) $\mathrm{Integral2}(M_2, |f|){\restriction}U^{\mathrm{c}}$ is a partial function from X_1 to $\mathbb{R}$, and

(iv) $\mathrm{Integral2}(M_2, f)$ is $(S_3 \setminus U)$-measurable, and

(v) $\mathrm{Integral2}(M_2, f){\restriction}U^{\mathrm{c}}$ is integrable on M_1, and

(vi) $\mathrm{Integral2}(M_2, f){\restriction}U^{\mathrm{c}} \in$ the L^1 functions of M_1, and

(vii) there exists a function g from X_1 into $\overline{\mathbb{R}}$ such that g is integrable on M_1 and $g{\restriction}U^{\mathrm{c}} = \mathrm{Integral2}(M_2, f){\restriction}U^{\mathrm{c}}$ and $\int f \, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int g \, \mathrm{d}M_1$.

PROOF: Consider A being an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$ such that $A = \mathrm{dom}\, f$ and f is A-measurable. $\mathrm{Integral2}(M_2, |f|)$ is integrable on M_1 and $\mathrm{Integral2}(M_2, \max_+(f))$ is integrable on M_1 and $\mathrm{Integral2}(M_2, \max_-(f))$ is integrable on M_1. $\mathrm{Integral2}(M_2, |f|)$ is finite M_1-a.e.. Consider U being an element of S_1 such that $M_1(U) = 0$ and $\mathrm{Integral2}(M_2, |f|){\restriction}U^{\mathrm{c}}$ is a partial function from X_1 to $\mathbb{R}$. For every element x of X_1 such that $x \in U^{\mathrm{c}}$ holds $\mathrm{ProjPMap1}(f, x)$ is integrable on M_2. Consider g_1 being a partial function from X_1 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g_1 = \mathrm{dom}(\mathrm{Integral2}(M_2, \max_+(f)))$ and $g_1{\restriction}U^{\mathrm{c}} = \mathrm{Integral2}(M_2, \max_+(f)){\restriction}U^{\mathrm{c}}$ and g_1 is integrable on M_1 and $\int g_1 \, \mathrm{d}M_1 = \int \mathrm{Integral2}(M_2, \max_+(f)){\restriction}U^{\mathrm{c}} \, \mathrm{d}M_1$.

Consider g_2 being a partial function from X_1 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g_2 = \mathrm{dom}(\mathrm{Integral2}(M_2, \max_-(f)))$ and $g_2{\restriction}U^{\mathrm{c}} = \mathrm{Integral2}(M_2, \max_-(f)){\restriction}U^{\mathrm{c}}$ and g_2 is integrable on M_1 and $\int g_2 \, \mathrm{d}M_1 = \int \mathrm{Integral2}(M_2, \max_-(f)){\restriction}U^{\mathrm{c}} \, \mathrm{d}M_1$. Consider g being a partial function from X_1 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g = \mathrm{dom}(\mathrm{Integral2}(M_2, f))$ and $g{\restriction}U^{\mathrm{c}} = \mathrm{Integral2}(M_2, f){\restriction}U^{\mathrm{c}}$ and g is integrable on M_1 and $\int g \, \mathrm{d}M_1 = \int \mathrm{Integral2}(M_2, f){\restriction}U^{\mathrm{c}} \, \mathrm{d}M_1$. $\int f \, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int g{\restriction}U^{\mathrm{c}} \, \mathrm{d}M_1$. $\square$

(31) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element S_4 of S_2. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ and $X_2 = S_4$. Then there exists an element V of S_2 such that

(i) $M_2(V) = 0$, and

(ii) for every element y of X_2 such that $y \in V^{\rm c}$ holds ProjPMap2(f, y) is integrable on M_1, and

(iii) Integral1$(M_1, |f|) \restriction V^{\rm c}$ is a partial function from X_2 to $\mathbb{R}$, and

(iv) Integral1(M_1, f) is $(S_4 \setminus V)$-measurable, and

(v) Integral1$(M_1, f) \restriction V^{\rm c}$ is integrable on M_2, and

(vi) Integral1$(M_1, f) \restriction V^{\rm c} \in$ the L^1 functions of M_2, and

(vii) there exists a function g from X_2 into $\overline{\mathbb{R}}$ such that g is integrable on M_2 and $g \restriction V^{\rm c} =$ Integral1$(M_1, f) \restriction V^{\rm c}$ and $\int f \, {\rm d}\, \text{ProdMeas}(M_1, M_2) = \int g \, {\rm d} M_2$.

PROOF: Consider A being an element of σ(MeasRect(S_1, S_2)) such that $A = \operatorname{dom} f$ and f is A-measurable. Integral1$(M_1, |f|)$ is integrable on M_2 and Integral1$(M_1, \max_+(f))$ is integrable on M_2 and Integral1$(M_1, \max_-(f))$ is integrable on M_2. Integral1$(M_1, |f|)$ is finite M_2-a.e.. Consider V being an element of S_2 such that $M_2(V) = 0$ and Integral1$(M_1, |f|) \restriction V^{\rm c}$ is a partial function from X_2 to $\mathbb{R}$. For every element y of X_2 such that $y \in V^{\rm c}$ holds ProjPMap2(f, y) is integrable on M_1 by (7), [5, (31)]. Consider g_1 being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\operatorname{dom} g_1 = \operatorname{dom}(\text{Integral1}(M_1, \max_+(f)))$ and $g_1 \restriction V^{\rm c} =$ Integral1$(M_1, \max_+(f)) \restriction V^{\rm c}$ and g_1 is integrable on M_2 and $\int g_1 \, {\rm d} M_2 = \int$ Integral1$(M_1, \max_+(f)) \restriction V^{\rm c}$ ${\rm d} M_2$.

Consider g_2 being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\operatorname{dom} g_2 = \operatorname{dom}(\text{Integral1}(M_1, \max_-(f)))$ and $g_2 \restriction V^{\rm c} =$ Integral1$(M_1, \max_-(f)) \restriction V^{\rm c}$ and g_2 is integrable on M_2 and $\int g_2 \, {\rm d} M_2 = \int$ Integral1$(M_1, \max_-(f)) \restriction V^{\rm c}$ ${\rm d} M_2$. Consider g being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\operatorname{dom} g = \operatorname{dom}(\text{Integral1}(M_1, f))$ and $g \restriction V^{\rm c} =$ Integral1$(M_1, f) \restriction V^{\rm c}$ and g is integrable on M_2 and $\int g \, {\rm d} M_2 = \int$ Integral1$(M_1, f) \restriction V^{\rm c} \, {\rm d} M_2$. $\int f \, {\rm d}\, \text{ProdMeas}(M_1, M_2) = \int g \restriction V^{\rm c} \, {\rm d} M_2$. □

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(32) Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas (M_1, M_2) and for every element x of X_1, $(\text{Integral2}(M_2, |f|))(x) < +\infty$. Then

(i) for every element x of X_1, ProjPMap1(f, x) is integrable on M_2, and

(ii) for every element U of S_1, Integral2(M_2, f) is U-measurable, and

(iii) Integral2(M_2, f) is integrable on M_1, and

(iv) $\int f \,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, f)\,\mathrm{d}M_1$, and

(v) $\mathrm{Integral2}(M_2, f) \in$ the L^1 functions of M_1.

The theorem is a consequence of (7), (24), (6), and (17).

(33) Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas (M_1, M_2) and for every element y of X_2, $(\mathrm{Integral1}(M_1, |f|))(y) < +\infty$. Then

(i) for every element y of X_2, $\mathrm{ProjPMap2}(f, y)$ is integrable on M_1, and

(ii) for every element V of S_2, $\mathrm{Integral1}(M_1, f)$ is V-measurable, and

(iii) $\mathrm{Integral1}(M_1, f)$ is integrable on M_2, and

(iv) $\int f \,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, f)\,\mathrm{d}M_2$, and

(v) $\mathrm{Integral1}(M_1, f) \in$ the L^1 functions of M_2.

The theorem is a consequence of (7), (24), (6), and (17).

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Tarski Geometry Axioms. Part IV – Right Angle

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Adam Grabowski
Institute of Informatics
University of Białystok
Poland

Summary. In the article, we continue [7] the formalization of the work devoted to Tarski's geometry – the book "Metamathematische Methoden in der Geometrie" (SST for short) by W. Schwabhäuser, W. Szmielew, and A. Tarski [14], [9], [10]. We use the Mizar system to systematically formalize Chapter 8 of the SST book.

We define the notion of right angle and prove some of its basic properties, a theory of intersecting lines (including orthogonality). Using the notion of perpendicular foot, we prove the existence of the midpoint (Satz 8.22), which will be used in the form of the Mizar functor (as the uniqueness can be easily shown) in Chapter 10. In the last section we give some lemmas proven by means of Otter during *Tarski Formalization Project* by M. Beeson (the so-called Section 8A of SST).

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0. Introduction

We use the Mizar system [1], [2] to systematically formalize Chapter 8 ("Rechte Winkel – Right angle") of the SST book. The theorems of this chapter are valid in neutral geometry [13].

We start (Def. 1) with the translation of the definition of the "right angle" which in SST reads as follows:

a,*b*,*c* bilden einen *rechten winkel* (mit dem *Scheitel* *b*):

$$Rabc :\longleftrightarrow ac \equiv aS_b(c).$$

In the Mizar formalism (note explicit use of Tarski's axioms):

```
definition
  let S be satisfying_CongruenceIdentity satisfying_CongruenceSymmetry
           satisfying_CongruenceEquivalenceRelation
           satisfying_SegmentConstruction satisfying_SAS
     satisfying_BetweennessIdentity TarskiGeometryStruct;
  let a,b,c be POINT of S;
  pred right_angle a,b,c means
  a,c equiv a,reflection(b,c);
end;
```

where `reflection` is defined in [7].

For the purpose of this presentation, we use the notation $\llcorner(a, b, c)$ instead of Rabc chosen in SST. Section 3 starts with variants of Definition 8.11, while in the next section predicate A, B Is x is defined, and this is Def. 7 in our translation. Section 5 deals with perpendicular foot – Satz 8.18 is Lotsatz, Satz 8.22 states that every segment has a midpoint (Gupta 1965 [11]).

In 2006, the first eight chapters were formalised in Coq in 2006 by Narboux [12] and we are essentially in this place. The entire SST book have been formalized within intuitionistic logic [5]. Note that the definitions in [6][1]:

```
(* Definition 8.1. *)
Definition Per A B C := exists C', Midpoint B C C' /\ Cong A C A C'.
```

and in [4]: ABC is a right angle if there is a point D such that $\mathbf{B}(A, B, D)$ and $AB = DB$ and $AC = DC$:

```
rightangle `RR A B C <=> ?X. BE A B X /\ EE A B X B /\ EE A C X C /\ NE B C`
```

are slightly different than in SST.

Some of the results were obtained by means of other automatic proof assistants, either partially [8], or completely [3].

[1] https://github.com/GeoCoq/GeoCoq/blob/master/Tarski_dev/Definitions.v

1. Preliminaries

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, c', x, y, z, p, q, q' denote points of S.

Let S be a non empty Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b be points of S. Let us note that the functor $\mathrm{Line}(a, b)$ is commutative.

Now we state the proposition:

(1) Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, and the axiom of congruence identity, and points a, b, c, d of S. Suppose $\overline{ab} \cong \overline{cd}$. Then

(i) $\overline{ab} \cong \overline{dc}$, and

(ii) $\overline{ba} \cong \overline{cd}$, and

(iii) $\overline{ba} \cong \overline{dc}$, and

(iv) $\overline{cd} \cong \overline{ab}$, and

(v) $\overline{dc} \cong \overline{ab}$, and

(vi) $\overline{cd} \cong \overline{ba}$, and

(vii) $\overline{dc} \cong \overline{ba}$.

Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, and the axiom of congruence identity and points p, q, a, b, c, d of S. Now we state the propositions:

(2) Suppose ($\overline{pq} \cong \overline{ab}$ or $\overline{pq} \cong \overline{ba}$ or $\overline{qp} \cong \overline{ab}$ or $\overline{qp} \cong \overline{ba}$) and ($\overline{pq} \cong \overline{cd}$ or $\overline{pq} \cong \overline{dc}$ or $\overline{qp} \cong \overline{cd}$ or $\overline{qp} \cong \overline{dc}$). Then

(i) $\overline{ab} \cong \overline{dc}$, and

(ii) $\overline{ba} \cong \overline{cd}$, and

(iii) $\overline{ba} \cong \overline{dc}$, and

(iv) $\overline{cd} \cong \overline{ab}$, and

(v) $\overline{dc} \cong \overline{ab}$, and

(vi) $\overline{cd} \cong \overline{ba}$, and

(vii) $\overline{dc} \cong \overline{ba}$.

The theorem is a consequence of (1).

(3) Suppose ($\overline{pq} \cong \overline{ab}$ or $\overline{pq} \cong \overline{ba}$ or $\overline{qp} \cong \overline{ab}$ or $\overline{qp} \cong \overline{ba}$ or $\overline{ab} \cong \overline{pq}$ or $\overline{ba} \cong \overline{pq}$ or $\overline{ab} \cong \overline{qp}$ or $\overline{ba} \cong \overline{qp}$) and ($\overline{pq} \cong \overline{cd}$ or $\overline{pq} \cong \overline{dc}$ or $\overline{qp} \cong \overline{cd}$ or $\overline{qp} \cong \overline{dc}$ or $\overline{cd} \cong \overline{pq}$ or $\overline{dc} \cong \overline{pq}$ or $\overline{cd} \cong \overline{qp}$ or $\overline{dc} \cong \overline{qp}$). Then

(i) $\overline{ab} \cong \overline{dc}$, and
(ii) $\overline{ba} \cong \overline{cd}$, and
(iii) $\overline{ba} \cong \overline{dc}$, and
(iv) $\overline{cd} \cong \overline{ab}$, and
(v) $\overline{dc} \cong \overline{ab}$, and
(vi) $\overline{cd} \cong \overline{ba}$, and
(vii) $\overline{dc} \cong \overline{ba}$, and
(viii) $\overline{ab} \cong \overline{cd}$.

The theorem is a consequence of (1) and (2).

(4) Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b of S. Then

(i) a, b and b are collinear, and
(ii) b, b and a are collinear, and
(iii) b, a and b are collinear.

(5) Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points p, q, r of S. Suppose $p \neq q$ and $p \neq r$ and (p, q and r are collinear or q, r and p are collinear or r, p and q are collinear or p, r and q are collinear or q, p and r are collinear or r, q and p are collinear). Then

(i) $\text{Line}(p, q) = \text{Line}(p, r)$, and
(ii) $\text{Line}(p, q) = \text{Line}(r, p)$, and
(iii) $\text{Line}(q, p) = \text{Line}(p, r)$, and
(iv) $\text{Line}(q, p) = \text{Line}(r, p)$.

(6) Let us consider a Tarski plane S, and points a, b, c of S. Suppose $\text{Middle}(a, b, c)$ or b lies between a and c. Then a, b and c are collinear.

(7) Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b, c of S. Suppose $\text{Middle}(a, b, c)$ or b lies between a and c. Then

(i) a, b and c are collinear, and
(ii) b, c and a are collinear, and
(iii) c, a and b are collinear, and
(iv) c, b and a are collinear, and

(v) b, a and c are collinear, and

(vi) a, c and b are collinear.

The theorem is a consequence of (6).

(8) EXT1:
Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b, c, d of S. Suppose $a \neq b$ and a, b and c are collinear and a, b and d are collinear. Then a, c and d are collinear. The theorem is a consequence of (4) and (5).

(9) Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b of S. Suppose $\mathrm{Middle}(a, a, b)$ or $\mathrm{Middle}(a, b, b)$ or $\mathrm{Middle}(a, b, a)$. Then $a = b$.

(10) Suppose ($\mathrm{Middle}(a, b, c)$ or $\mathrm{Middle}(c, b, a)$) and ($a \neq b$ or $b \neq c$). Then

(i) $\mathrm{Line}(b, a) = \mathrm{Line}(b, c)$, and

(ii) $\mathrm{Line}(a, b) = \mathrm{Line}(b, c)$, and

(iii) $\mathrm{Line}(a, b) = \mathrm{Line}(c, b)$, and

(iv) $\mathrm{Line}(b, a) = \mathrm{Line}(c, b)$.

The theorem is a consequence of (9).

(11) Suppose $a \neq b$ and $c \neq c'$ and ($c \in \mathrm{Line}(a, b)$ or $c \in \mathrm{Line}(b, a)$) and ($c' \in \mathrm{Line}(a, b)$ or $c' \in \mathrm{Line}(b, a)$). Then

(i) $\mathrm{Line}(c, c') = \mathrm{Line}(a, b)$, and

(ii) $\mathrm{Line}(c, c') = \mathrm{Line}(b, a)$, and

(iii) $\mathrm{Line}(c', c) = \mathrm{Line}(b, a)$, and

(iv) $\mathrm{Line}(c', c) = \mathrm{Line}(a, b)$.

(12) $\mathrm{Middle}(\mathrm{S}_p(c), \mathrm{S}_p(b), \mathrm{S}_p((\mathrm{S}_b(c))))$.

2. RIGHT ANGLE

Let S be Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of SAS and a, b, c be points of S. We say that $\llcorner(a, b, c)$ if and only if

(Def. 1) $\overline{ac} \cong \overline{a\mathrm{S}_b(c)}$.

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, a', b, b', c, c' denote points of S.

Now we state the propositions:

(13) 8.2 Satz:
If $\llcorner(a,b,c)$, then $\llcorner(c,b,a)$.

(14) $\mathrm{S}_a(a) = a$.

(15) 8.3 Satz:
If $\llcorner(a,b,c)$ and $a \neq b$ and b, a and a' are collinear, then $\llcorner(a',b,c)$. The theorem is a consequence of (14).

(16) 8.4 Satz:
If $\llcorner(a,b,c)$, then $\llcorner(a,b,\mathrm{S}_b(c))$.

(17) 8.5 Satz:
$\llcorner(a,b,b)$. The theorem is a consequence of (14).

(18) 8.6 Satz:
If $\llcorner(a,b,c)$ and $\llcorner(a',b,c)$ and c lies between a and a', then $b = c$.

(19) 8.7 Satz:
If $\llcorner(a,b,c)$ and $\llcorner(a,c,b)$, then $b = c$. The theorem is a consequence of (13), (17), (1), (7), (15), and (18).

(20) 8.8 Satz:
If $\llcorner(a,b,a)$, then $a = b$. The theorem is a consequence of (13), (17), and (19).

(21) 8.9 Satz:
If $\llcorner(a,b,c)$ and a, b and c are collinear, then $a = b$ or $c = b$. The theorem is a consequence of (15) and (20).

(22) 8.10 Satz:
If $\llcorner(a,b,c)$ and $\triangle abc \cong \triangle a'b'c'$, then $\llcorner(a',b',c')$. The theorem is a consequence of (17), (1), and (3).

3. Orthogonality

Let S be a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, A' be subsets of S, and x be a point of S. We say that $A \perp_x A'$ if and only if

(Def. 2) A is a line and A' is a line and $x \in A$ and $x \in A'$ and for every points u, v of S such that $u \in A$ and $v \in A'$ holds $\llcorner(u,x,v)$.

We say that $A \perp A'$ if and only if

(Def. 3) there exists a point x of S such that $A \perp_x A'$.

Let A be a subset of S and x, c, d be points of S. We say that $\overline{A,x} \perp \overline{c,d}$ if and only if

(Def. 4) $c \neq d$ and $A \perp_x \mathrm{Line}(c,d)$.

Let a, b, x, c, d be points of S. We say that $\overline{a,b} \perp_x \overline{c,d}$ if and only if

(Def. 5) $a \neq b$ and $c \neq d$ and $\mathrm{Line}(a, b) \perp_x \mathrm{Line}(c, d)$.

Let a, b, c, d be points of S. We say that $\overline{a, b} \perp \overline{c, d}$ if and only if

(Def. 6) $a \neq b$ and $c \neq d$ and $\mathrm{Line}(a, b) \perp \mathrm{Line}(c, d)$.

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, A' denote subsets of S, and x, y, z, a, b, c, c', d, u, p, q, q' denote points of S.

Now we state the propositions:

(23) 8.12 SATZ:
$A \perp_x A'$ if and only if $A' \perp_x A$.

(24) 8.13 SATZ:
$A \perp_x A'$ if and only if A is a line and A' is a line and $x \in A$ and $x \in A'$ and there exist points u, v of S such that $u \in A$ and $v \in A'$ and $u \neq x$ and $v \neq x$ and $\llcorner(u, x, v)$. The theorem is a consequence of (15) and (13).

(25) 8.14 (I) SATZ:
If $A \perp A'$, then $A \neq A'$. The theorem is a consequence of (24) and (21).

4. INTERSECTION OF LINES

Let S be a non empty Tarski plane, A, B be subsets of S, and x be a point of S. We say that A, B intersect at x if and only if

(Def. 7) A is a line and B is a line and $A \neq B$ and $x \in A$ and $x \in B$.

Now we state the propositions:

(26) 8.14 (II) SATZ:
$A \perp_x A'$ if and only if $A \perp A'$ and A, A' intersect at x. The theorem is a consequence of (25).

(27) 8.14 (III) SATZ:
If $A \perp_x A'$ and $A \perp_y A'$, then $x = y$. The theorem is a consequence of (25) and (26).

(28) If a, b and x are collinear and $\overline{a, b} \perp \overline{c, x}$, then $\overline{a, b} \perp_x \overline{c, x}$. The theorem is a consequence of (25) and (26).

(29) 8.15 SATZ:
If $a \neq b$ and a, b and x are collinear, then $\overline{a, b} \perp \overline{c, x}$ iff $\overline{a, b} \perp_x \overline{c, x}$. The theorem is a consequence of (28).

(30) 8.16 SATZ:
Suppose $a \neq b$ and a, b and x are collinear and a, b and u are collinear and $u \neq x$. Then $\overline{a, b} \perp \overline{c, x}$ if and only if a, b and c are not collinear and $\llcorner(c, x, u)$. The theorem is a consequence of (29), (13), (21), and (24).

5. Perpendicular Foot

Let S be a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, x be points of S. We say that x is perpendicular foot of a, b, c if and only if

(Def. 8) a, b and x are collinear and $\overline{a,b} \perp \overline{c,x}$.

Now we state the propositions:

(31) 8.18 Satz – Uniqueness:
If x is perpendicular foot of a, b, c and y is perpendicular foot of a, b, c, then $x = y$. The theorem is a consequence of (29), (13), and (19).

(32) Suppose a, b and c are not collinear and a lies between b and y and $a \neq y$ and y lies between a and z and $\overline{yz} \cong \overline{yp}$ and $y \neq p$ and $q' = \mathrm{S}_z(q)$ and $\mathrm{Middle}(c, x, c')$ and $c \neq y$ and y lies between q' and c' and $\mathrm{Middle}(y, p, c)$ and y lies between p and q and $q \neq q'$. Then $x \neq y$. The theorem is a consequence of (10) and (11).

In the sequel S denotes a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and a, b, c, p, q, x, y, z, t denote points of S.

Now we state the propositions:

(33) 8.18 Satz – Existence:
If a, b and c are not collinear, then there exists x such that x is perpendicular foot of a, b, c.
Proof: Consider y such that a lies between b and y and $\overline{ay} \cong \overline{ac}$. Consider p such that $\mathrm{Middle}(y, p, c)$. Consider z such that y lies between a and z and $\overline{yz} \cong \overline{yp}$. Consider q such that y lies between p and q and $\overline{yq} \cong \overline{ya}$. Set $q' = \mathrm{S}_z(q)$. Consider c' such that y lies between q' and c' and $\overline{yc'} \cong \overline{yc}$. $a \neq y$. $⊾(q, z, y)$. $⊾(y, z, q)$. Consider x such that $\mathrm{Middle}(c, x, c')$. $y \neq p$. $c \neq y$. $q \neq q'$. $c \neq x$. □

(34) 8.20 Lemma:
If $⊾(a, b, c)$ and $\mathrm{Middle}(\mathrm{S}_a(c), p, \mathrm{S}_b(c))$, then $⊾(b, a, p)$ and if $b \neq c$, then $a \neq p$.
Proof: Set $d = \mathrm{S}_b(c)$. Set $b' = \mathrm{S}_a(b)$. Set $c' = \mathrm{S}_a(c)$. Set $d' = \mathrm{S}_a(d)$. Set $p' = \mathrm{S}_a(p)$. $⊾(b', b, c)$. $\overline{b'b} \cong \overline{bb'}$. $\overline{b'c} \cong \overline{bc'}$. $\triangle b'bc \cong \triangle bb'c'$. $⊾(b, b', c')$. $\mathrm{S}_{b'}(c') = d'$. IFS $\left(\begin{smallmatrix} c', & p, & d, & b \\ d', & p', & c, & b \end{smallmatrix}\right)$. If $b \neq c$, then $a \neq p$. □

(35) Suppose a, b and c are not collinear. Then there exists p and there exists t such that $\overline{a,b} \perp \overline{p,a}$ and a, b and t are collinear and t lies between c and p. The theorem is a consequence of (33), (29), (34), and (24).

(36) 8.21 Satz:
If $a \neq b$, then there exists p and there exists t such that $\overline{a,b} \perp \overline{p,a}$ and a, b

and t are collinear and t lies between c and p. The theorem is a consequence of (35).

(37) If $a \neq b$ and $a \neq p$ and $\llcorner(b,a,p)$ and $\llcorner(a,b,q)$, then p, a and q are not collinear. The theorem is a consequence of (13), (15), and (19).

(38) Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b, p, q, t of S. Suppose $a,p \leqslant b,q$ and $\overline{a,b} \perp \overline{q,b}$ and $\overline{a,b} \perp \overline{p,a}$ and a, b and t are collinear and t lies between q and p. Then there exists a point x of S such that Middle(a,x,b).

PROOF: Consider r being a point of S such that r lies between b and q and $\overline{ap} \cong \overline{br}$. Consider x being a point of S such that x lies between t and b and x lies between r and p. a, b and x are collinear. Consider x' being a point of S such that Line$(a,b) \perp_{x'}$ Line(q,b). Consider y being a point of S such that Line$(a,b) \perp_{y}$ Line(p,a). $\llcorner(q,b,a)$ and $q \neq b$ and b, q and r are collinear. $\llcorner(r,b,a)$. b, a and p are not collinear and a, b and q are not collinear. □

(39) 8.22 SATZ:

Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b of S. Then there exists a point x of S such that Middle(a,x,b). The theorem is a consequence of (36) and (38).

(40) 8.24 LEMMA:

Let us consider a non empty Tarski plane S satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, and points a, b, p, q, r, t of S. Suppose $\overline{p,a} \perp \overline{a,b}$ and $\overline{q,b} \perp \overline{a,b}$ and a, b and t are collinear and t lies between p and q and r lies between b and q and $\overline{ap} \cong \overline{br}$. Then there exists a point x of S such that

(i) Middle(a,x,b), and

(ii) Middle(p,x,r).

PROOF: Consider x being a point of S such that x lies between t and b and x lies between r and p. a, b and x are collinear. Consider x' being a point of S such that Line$(a,b) \perp_{x'}$ Line(q,b). Consider y being a point of S such that Line$(a,b) \perp_{y}$ Line(p,a). $\llcorner(q,b,a)$ and $q \neq b$ and b, q and r are collinear. $\llcorner(r,b,a)$. b, a and p are not collinear and a, b and q are not collinear. □

6. Additional Lemmas Needed by Otter: Chapter 8A

Now we state the propositions:

(41) ExtCol2:

Let us consider points a, b, c, d, x, p, q of S. Suppose c, $d \in \mathrm{Line}(a, b)$ and $a \neq b$ and $c \neq d$. Then $\mathrm{Line}(a, b) = \mathrm{Line}(c, d)$.

(42) ExtPerp:

Let us consider points a, b, c, d, x, p, q of S. Suppose c, $d \in \mathrm{Line}(a, b)$ and $c \neq d$ and $\overline{a, b} \perp_x \overline{p, q}$. Then $\overline{c, d} \perp_x \overline{p, q}$. The theorem is a consequence of (11).

(43) ExtPerp2:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a, b)$ and $a \neq b$ and $\overline{p, q} \perp \overline{c, d}$. Then $\overline{a, b} \perp \overline{c, d}$. The theorem is a consequence of (11).

(44) ExtPerp3:

Let us consider points a, b, c, d of S. Suppose $a \neq b$ and $b \neq c$ and $c \neq d$ and $a \neq c$ and $a \neq d$ and $b \neq d$ and $\overline{b, a} \perp \overline{a, c}$ and a, c and d are collinear. Then $\overline{b, a} \perp \overline{a, d}$. The theorem is a consequence of (11).

(45) ExtPerp4:

Let us consider points a, b, p, q of S. If $\overline{a, b} \perp \overline{p, q}$, then $\overline{a, b} \perp \overline{q, p}$.

(46) ExtPerp5:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a, b)$ and $p \neq q$ and $\overline{a, b} \perp \overline{c, d}$. Then $\overline{p, q} \perp \overline{c, d}$. The theorem is a consequence of (11).

(47) ExtPerp5A:

Let us consider points a, b, c, d, p, q of S. Suppose a, b and p are collinear and a, b and q are collinear and $p \neq q$ and $\overline{a, b} \perp \overline{c, d}$. Then $\overline{p, q} \perp \overline{c, d}$. The theorem is a consequence of (46).

(48) ExtPerp6:

Let us consider points a, b, c, d, p, q of S. Suppose p, $q \in \mathrm{Line}(a, b)$ and $p \neq q$ and $a \neq b$ and $\overline{c, d} \perp \overline{p, q}$. Then $\overline{c, d} \perp \overline{a, b}$. The theorem is a consequence of (11).

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Maximum Number of Steps Taken by Modular Exponentiation and Euclidean Algorithm[1]

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Koh-ichi Nagao
Kanto Gakuin University
Kanagawa, Japan

Yuichi Futa
Tokyo University of Technology
Tokyo, Japan

Summary. In this article we formalize in Mizar [1], [2] the maximum number of steps taken by some number theoretical algorithms, "right–to–left binary algorithm" for modular exponentiation and "Euclidean algorithm" [5]. For any natural numbers a, b, n, "right–to–left binary algorithm" can calculate the natural number, see (Def. 2), $\mathrm{Algo}_{\mathrm{BPow}}(a,n,m) := a^b \bmod n$ and for any integers a, b, "Euclidean algorithm" can calculate the non negative integer $\gcd(a,b)$. We have not formalized computational complexity of algorithms yet, though we had already formalize the "Euclidean algorithm" in [7].

For "right-to-left binary algorithm", we formalize the theorem, which says that the required number of the modular squares and modular products in this algorithms are $1+\lfloor\log_2 n\rfloor$ and for "Euclidean algorithm", we formalize the Lamé's theorem [6], which says the required number of the divisions in this algorithm is at most $5\log_{10}\min(|a|,|b|)$. Our aim is to support the implementation of number theoretic tools and evaluating computational complexities of algorithms to prove the security of cryptographic systems.

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1. Right–to–Left Binary Algorithm for Modular Exponentiation

Let F be an element of *Boolean** and x be an object. Let us note that the functor $F(x)$ yields a natural number. Let n, m be natural numbers. Let us note that the functor n^m yields a natural number. Let a, b be objects and c be a natural number. The functor BinBranch(a, b, c) is defined by the term

(Def. 1) $\begin{cases} a, & \textbf{if } c = 0, \\ b, & \textbf{otherwise}. \end{cases}$

Let a, b, c be natural numbers. Let us note that the functor BinBranch(a, b, c) yields a natural number. Let a, n, m be elements of $\mathbb{N}$. The functor $\mathrm{Algo}_{\mathrm{BPow}}(a, n, m)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 2) there exist sequences A, B of $\mathbb{N}$ such that $it = B(\mathrm{LenBinSeq}(n))$ and $A(0) = a \bmod m$ and $B(0) = 1$ and for every natural number i, $A(i + 1) = A(i) \cdot A(i) \bmod m$ and $B(i + 1) = \mathrm{BinBranch}(B(i), B(i) \cdot A(i) \bmod m, (\mathrm{Nat2BinLen})(n)(i + 1))$.

Now we state the propositions:

(1) Let us consider natural numbers a, m, i, and a sequence A of $\mathbb{N}$. Suppose $A(0) = a \bmod m$ and for every natural number j, $A(j + 1) = A(j) \cdot A(j) \bmod m$. Then $A(i) = a^{2^i} \bmod m$.

Proof: Define $\mathcal{P}$[natural number] $\equiv A(\$_1) = a^{2^{\$_1}} \bmod m$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [8, (11)]. For every natural number i, $\mathcal{P}[i]$. □

(2) $\mathrm{LenBinSeq}(0) = 1$.

(3) $\mathrm{LenBinSeq}(1) = 1$.

(4) Let us consider a natural number x. If $2 \leqslant x$, then $1 < \mathrm{LenBinSeq}(x)$.

(5) Let us consider a natural number n. Suppose $0 < n$. Then $\mathrm{LenBinSeq}(n) = \lfloor \log_2 n \rfloor + 1$.

(6) $(\mathrm{Nat2BinLen})(0) = \langle 0 \rangle$.

(7) $(\mathrm{Nat2BinLen})(1) = \langle 1 \rangle$. The theorem is a consequence of (3).

(8) Let us consider an element n of $\mathbb{N}$. If $0 < n$, then $(\mathrm{Nat2BinLen})(n)(\mathrm{LenBinSeq}(n)) = 1$.

Proof: Reconsider $x = (\mathrm{Nat2BinLen})(n)$ as an element of *Boolean**. $x \notin \{y$, where y is an element of *Boolean*$^* : y(\mathrm{len}\, y) = 1\}$. □

(9) $(\mathrm{Nat2BinLen})(2) = \langle 0, 1 \rangle$. The theorem is a consequence of (5).

(10) $(\mathrm{Nat2BinLen})(3) = \langle 1, 1 \rangle$. The theorem is a consequence of (5).

(11) $(\mathrm{Nat2BinLen})(4) = \langle 0, 0, 1 \rangle$. The theorem is a consequence of (5).

(12) Let us consider a natural number n. Then $(\mathrm{Nat2BinLen})(2^n) = \langle \underbrace{0, \ldots, 0}_{n} \rangle ^\frown \langle 1 \rangle$. The theorem is a consequence of (5).

(13) Let us consider an element m of $\mathbb{N}$. Then $\mathrm{Algo}_{\mathrm{BPow}}(0, 0, m) = 1$. The theorem is a consequence of (6).

(14) Let us consider elements n, m of $\mathbb{N}$. If $0 < n$, then $\mathrm{Algo}_{\mathrm{BPow}}(0, n, m) = 0$. The theorem is a consequence of (1) and (8).

Let us consider elements a, n, m of $\mathbb{N}$. Now we state the propositions:

(15) If $0 < n$ and $m \leqslant 1$, then $\mathrm{Algo}_{\mathrm{BPow}}(a, n, m) = 0$. The theorem is a consequence of (8).

(16) If $a \neq 0$ and $1 < m$, then $\mathrm{Algo}_{\mathrm{BPow}}(a, n, m) = a^n \bmod m$.
PROOF: Consider A, B being sequences of $\mathbb{N}$ such that $\mathrm{Algo}_{\mathrm{BPow}}(a, n, m) = B(\mathrm{LenBinSeq}(n))$ and $A(0) = a \bmod m$ and $B(0) = 1$ and for every natural number i, $A(i + 1) = A(i) \cdot A(i) \bmod m$ and $B(i + 1) = \mathrm{BinBranch}(B(i), B(i) \cdot A(i) \bmod m, (\mathrm{Nat2BinLen})(n)(i + 1))$.
Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 < \mathrm{LenBinSeq}(n)$, then there exists a $(\$_1 + 1)$-tuple S of *Boolean* such that $S = (\mathrm{Nat2BinLen})(n){\restriction}(\$_1 + 1)$ and $B(\$_1 + 1) = a^{\mathrm{AbsVal}(S)} \bmod m$. $\mathcal{P}[0]$ by [3, (5)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every natural number i, $\mathcal{P}[i]$. Reconsider $f = \mathrm{LenBinSeq}(n) - 1$ as a natural number. Consider F_1 being an $(f + 1)$-tuple of *Boolean* such that $F_1 = (\mathrm{Nat2BinLen})(n){\restriction}(f + 1)$ and $B(f + 1) = a^{\mathrm{AbsVal}(F_1)} \bmod m$. $\square$

2. LAMÉ'S THEOREM

Now we state the propositions:

(17) $\mathrm{Fib}(5) = 5$.

(18) $1 < \tau$.

(19) $\tau < 2$.

(20) $\log_\tau 10 < 5$. The theorem is a consequence of (17) and (18).

(21) Let us consider a natural number n. If $3 \leqslant n$, then $\tau^{n-2} < \mathrm{Fib}(n)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \tau^{\$_1 - 2} < \mathrm{Fib}(\$_1)$. For every natural number k such that $k \geqslant 3$ holds if for every natural number i such that $i \geqslant 3$ holds if $i < k$, then $\mathcal{P}[i]$, then $\mathcal{P}[k]$ by [4, (22)], (19). For every natural number k such that $k \geqslant 3$ holds $\mathcal{P}[k]$. $\square$

(22) Let us consider elements a, b of $\mathbb{Z}$. Suppose $|a| > |b|$ and $b > 1$. Then there exist sequences A, B of $\mathbb{N}$ and there exists a sequence C of real numbers and there exists an element n of $\mathbb{N}$ such that $A(0) = |a|$ and

$B(0) = |b|$ and for every natural number i, $A(i+1) = B(i)$ and $B(i+1) = A(i) \bmod B(i)$ and $n = \min^*\{i$, where i is a natural number : $B(i) = 0\}$ and $\gcd(a, b) = A(n)$ and $\mathrm{Fib}(n+1) \leqslant |b|$ and $n \leqslant 5 \cdot \lceil \log_{10} |b| \rceil$ and $n \leqslant C(|b|)$ and C is polynomially bounded.

Proof: Consider A, B being sequences of $\mathbb{N}$ such that $A(0) = |a|$ and $B(0) = |b|$ and for every natural number i, $A(i+1) = B(i)$ and $B(i+1) = A(i) \bmod B(i)$ and $\mathrm{Algo_{GCD}}(a, b) = A(\min^*\{i$, where i is a natural number : $B(i) = 0\})$. Consider n being an element of $\mathbb{N}$ such that $n = \min^*\{i$, where i is a natural number : $B(i) = 0\}$ and $\mathrm{Algo_{GCD}}(a, b) = A(n)$. For every elements a, b of $\mathbb{Z}$ and for every sequences A, B of $\mathbb{N}$ such that $A(0) = |a|$ and $B(0) = |b|$ and for every natural number i, $A(i+1) = B(i)$ and $B(i+1) = A(i) \bmod B(i)$ holds $\{i$, where i is a natural number : $B(i) = 0\}$ is a non empty subset of $\mathbb{N}$. $B(n-1) \neq 0$. For every natural number i such that $i < n$ holds $B(i) > 0$. For every natural number i such that $i < n$ holds $B(i+1) \leqslant B(i) - 1$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant n$, then $B(\$_1) \leqslant B(0) - \$_1$.

For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. $n \leqslant B(0)$. For every natural number j such that $j < n$ holds $A(j+1) < A(j)$. If $1 < n$, then $\mathrm{Fib}(3) \leqslant A(n-1)$. For every natural number i such that $0 < i < n$ holds $A(i+2) + A(i+1) \leqslant A(i)$. For every natural number i such that $i < n$ holds $\mathrm{Fib}(i+2) \leqslant A(n-i)$. $n \leqslant 5 \cdot \lceil \log_{10} |b| \rceil$. □

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