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# Isomorphisms from the Space of Multilinear Operators

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**Summary.** In this article, using the Mizar system [5], [2], the isomorphisms from the space of multilinear operators are discussed. In the first chapter, two isomorphisms are formalized. The former isomorphism shows the correspondence between the space of multilinear operators and the space of bilinear operators.

The latter shows the correspondence between the space of multilinear operators and the space of the composition of linear operators. In the last chapter, the above isomorphisms are extended to isometric mappings between the normed spaces. We referred to [6], [11], [9], [3], [10] in this formalization.

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## 1. Plain Isomorphisms from the Space of Multilinear Operators

From now on $X$, $Y$, $Z$, $E$, $F$, $G$, $S$, $T$ denote real linear spaces.

Let $G$ be a real linear space sequence. Note that $\prod G$ is constituted finite sequences. Now we state the propositions:

(1) Let us consider an element $s$ of $\prod\langle E, F\rangle$, an element $i$ of $\operatorname{dom}\langle E, F\rangle$, and an object $x_1$. Then $\operatorname{len}(s +\cdot (i, x_1)) = 2$.

(2) Let us consider a real linear space sequence $G$, an element $i$ of $\operatorname{dom} G$, an element $x$ of $\prod G$, and an element $r$ of $G(i)$. Then $(\operatorname{reproj}(i, x))(r) = x +\cdot (i, r)$.




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Let $X$, $Y$ be real linear spaces. The functor IsoCPRLSP$(X,Y)$ yielding a linear operator from $X \times Y$ into $\prod\langle X,Y\rangle$ is defined by

(Def. 1) for every point $x$ of $X$ and for every point $y$ of $Y$, $it(x,y) = \langle x,y\rangle$.

Now we state the proposition:

(3) Let us consider real linear spaces $X$, $Y$. Then $0_{\prod\langle X,Y\rangle} = (\mathrm{IsoCPRLSP}(X, Y))(0_{X\times Y})$.

Let $X$, $Y$ be real linear spaces. One can check that IsoCPRLSP$(X,Y)$ is bijective and there exists a linear operator from $X \times Y$ into $\prod\langle X,Y\rangle$ which is bijective. Now we state the proposition:

(4) Let us consider a linear operator $I$ from $S$ into $T$. Suppose $I$ is bijective. Then there exists a linear operator $J$ from $T$ into $S$ such that

(i) $J = I^{-1}$, and

(ii) $J$ is bijective.

PROOF: Reconsider $J = I^{-1}$ as a function from $T$ into $S$. For every points $v$, $w$ of $T$, $J(v+w) = J(v) + J(w)$. For every point $v$ of $T$ and for every real number $r$, $J(r \cdot v) = r \cdot J(v)$. □

Let $X$, $Y$ be real linear spaces and $f$ be a bijective linear operator from $X \times Y$ into $\prod\langle X,Y\rangle$. One can verify that the functor $f^{-1}$ yields a linear operator from $\prod\langle X,Y\rangle$ into $X\times Y$. One can check that $f^{-1}$ is bijective as a linear operator from $\prod\langle X,Y\rangle$ into $X \times Y$ and there exists a linear operator from $\prod\langle X,Y\rangle$ into $X \times Y$ which is bijective. Now we state the propositions:

(5) Let us consider real linear spaces $X$, $Y$, a point $x$ of $X$, and a point $y$ of $Y$. Then $((\mathrm{IsoCPRLSP}(X,Y))^{-1})(\langle x,y\rangle) = \langle\!\langle x,\, y\rangle\!\rangle$.

(6) Let us consider real linear spaces $X$, $Y$. Then $((\mathrm{IsoCPRLSP}(X,Y))^{-1})(0_{\prod\langle X,Y\rangle}) = 0_{X\times Y}$. The theorem is a consequence of (3).

(7) Let us consider a multilinear operator $u$ from $\langle E,F\rangle$ into $G$. Then $u \cdot (\mathrm{IsoCPRLSP}(E,F))$ is a bilinear operator from $E \times F$ into $G$.
PROOF: Reconsider $L = u \cdot (\mathrm{IsoCPRLSP}(E,F))$ as a function from $E\times F$ into $G$. For every points $x_1$, $x_2$ of $E$ and for every point $y$ of $F$, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$. For every point $x$ of $E$ and for every point $y$ of $F$ and for every real number $a$, $L(a \cdot x, y) = a \cdot L(x,y)$. For every point $x$ of $E$ and for every points $y_1$, $y_2$ of $F$, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$. For every point $x$ of $E$ and for every point $y$ of $F$ and for every real number $a$, $L(x, a \cdot y) = a \cdot L(x,y)$ by [1, (31)]. □

(8) Let us consider a bilinear operator $u$ from $E \times F$ into $G$. Then $u \cdot ((\mathrm{IsoCPRLSP}(E,F))^{-1})$ is a multilinear operator from $\langle E,F\rangle$ into $G$.
PROOF: Reconsider $M = u \cdot ((\mathrm{IsoCPRLSP}(E,F))^{-1})$ as a function from $\prod\langle E,F\rangle$ into $G$. For every element $i$ of $\mathrm{dom}\langle E,F\rangle$ and for every element

$s$ of $\prod\langle E, F\rangle$, $M \cdot (\mathrm{reproj}(i, s))$ is a linear operator from $\langle E, F\rangle(i)$ into $G$. $\square$

(9) There exists a linear operator $I$ from $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$ into $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$ such that

(i) $I$ is bijective, and

(ii) for every point $u$ of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $I(u) = u \cdot ((\mathrm{IsoCPRLSP}(X, Y))^{-1})$.

Proof: Set $F_1 =$ the carrier of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$. Set $F_2 =$ the carrier of $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$.
Define $\mathcal{P}[\mathrm{function}, \mathrm{function}] \equiv \$_2 = \$_1 \cdot ((\mathrm{IsoCPRLSP}(X, Y))^{-1})$. For every element $x$ of $F_1$, there exists an element $y$ of $F_2$ such that $\mathcal{P}[x, y]$. Consider $I$ being a function from $F_1$ into $F_2$ such that for every element $x$ of $F_1$, $\mathcal{P}[x, I(x)]$. For every objects $x_1$, $x_2$ such that $x_1$, $x_2 \in F_1$ and $I(x_1) = I(x_2)$ holds $x_1 = x_2$. For every object $y$ such that $y \in F_2$ there exists an object $x$ such that $x \in F_1$ and $y = I(x)$. For every points $x$, $y$ of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $I(x + y) = I(x) + I(y)$. For every point $x$ of $\mathrm{VectorSpaceOfBilinOpers}_{\mathbb{R}}(X, Y, Z)$ and for every real number $a$, $I(a \cdot x) = a \cdot I(x)$. $\square$

(10) There exists a linear operator $I$ from $\mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(X, \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(Y, Z))$ into $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$ such that

(i) $I$ is bijective, and

(ii) for every point $u$ of $\mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(X, \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(Y, Z))$ and for every point $x$ of $X$ and for every point $y$ of $Y$, $I(u)(\langle x, y\rangle) = u(x)(y)$.

The theorem is a consequence of (9) and (5).

## 2. Extensions to Isometric Isomorphism from the Normed Space of Multilinear Operators

In the sequel $X$, $Y$, $Z$, $E$, $F$, $G$ denote real normed spaces and $S$, $T$ denote real norm space sequences. Now we state the propositions:

(11) Let us consider a point $s$ of $\prod\langle E, F\rangle$, an element $i$ of $\mathrm{dom}\langle E, F\rangle$, and an object $x_1$. Then $\mathrm{len}(s +\cdot (i, x_1)) = 2$.

(12) Let us consider a Lipschitzian multilinear operator $u$ from $\langle E, F\rangle$ into $G$. Then $u \cdot (\mathrm{IsoCPNrSP}(E, F))$ is a Lipschitzian bilinear operator from $E \times F$ into $G$.

Proof: Reconsider $L = u \cdot (\mathrm{IsoCPNrSP}(E, F))$ as a function from $E \times F$ into $G$. For every points $x_1$, $x_2$ of $E$ and for every point $y$ of $F$, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$. For every point $x$ of $E$ and for every point $y$ of $F$ and for every real number $a$, $L(a \cdot x, y) = a \cdot L(x, y)$. For every point $x$ of $E$ and for every points $y_1$, $y_2$ of $F$, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$. For every point $x$ of $E$ and for every point $y$ of $F$ and for every real number $a$, $L(x, a \cdot y) = a \cdot L(x, y)$. There exists a real number $K$ such that $0 \leqslant K$ and for every vector $x$ of $E$ and for every vector $y$ of $F$, $\|L(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$. $\square$

(13) Let us consider a Lipschitzian bilinear operator $u$ from $E \times F$ into $G$. Then $u \cdot ((\mathrm{IsoCPNrSP}(E, F))^{-1})$ is a Lipschitzian multilinear operator from $\langle E, F\rangle$ into $G$.
Proof: Reconsider $M = u \cdot ((\mathrm{IsoCPNrSP}(E, F))^{-1})$ as a function from $\prod\langle E, F\rangle$ into $G$. For every element $i$ of $\mathrm{dom}\langle E, F\rangle$ and for every element $s$ of $\prod\langle E, F\rangle$, $M \cdot (\mathrm{reproj}(i, s))$ is a linear operator from $\langle E, F\rangle(i)$ into $G$. There exists a real number $K$ such that $0 \leqslant K$ and for every point $s$ of $\prod\langle E, F\rangle$, $\|M(s)\| \leqslant K \cdot (\mathrm{NrProduct}\, s)$. $\square$

(14) There exists a linear operator $I$ from $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$ into $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$ such that
   (i) $I$ is bijective and isometric, and
   (ii) for every point $u$ of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $I(u) = u \cdot ((\mathrm{IsoCPNrSP}(X, Y))^{-1})$.

Proof: Set $F_1 =$ the carrier of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$. Set $F_2 =$ the carrier of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = \$_1 \cdot ((\mathrm{IsoCPNrSP}(X, Y))^{-1})$. For every element $x$ of $F_1$, there exists an element $y$ of $F_2$ such that $\mathcal{P}[x, y]$. Consider $I$ being a function from $F_1$ into $F_2$ such that for every element $x$ of $F_1$, $\mathcal{P}[x, I(x)]$. For every objects $x_1$, $x_2$ such that $x_1$, $x_2 \in F_1$ and $I(x_1) = I(x_2)$ holds $x_1 = x_2$. For every object $y$ such that $y \in F_2$ there exists an object $x$ such that $x \in F_1$ and $y = I(x)$. For every points $x$, $y$ of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $I(x + y) = I(x) + I(y)$. For every point $x$ of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$ and for every real number $a$, $I(a \cdot x) = a \cdot I(x)$ by [8, (19)], [4, (18)], [7, (20)]. For every element $u$ of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $\|I(u)\| = \|u\|$. $\square$

(15) There exists a linear operator $I$ from the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$ into $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$ such that

(i) $I$ is bijective and isometric, and

(ii) for every point $u$ of the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$, $\|u\| = \|I(u)\|$ and for every point $x$ of $X$ and for every point $y$ of $Y$, $I(u)(\langle x, y\rangle) = u(x)(y)$.

Proof: Consider $I$ being a linear operator from the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$ into $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$ such that $I$ is bijective and for every point $u$ of the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$, $\|u\| = \|I(u)\|$ and for every point $x$ of $X$ and for every point $y$ of $Y$, $I(u)(x, y) = u(x)(y)$. Consider $J$ being a linear operator from $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$ into $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$ such that $J$ is bijective and isometric and for every point $u$ of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(X, Y, Z)$, $J(u) = u \cdot ((\mathrm{IsoCPNrSP}(X, Y))^{-1})$.

Reconsider $K = J \cdot I$ as a linear operator from the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$ into $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X, Y\rangle, Z)$. For every element $x$ of the real norm space of bounded linear operators from $X$ into the real norm space of bounded linear operators from $Y$ into $Z$, $\|K(x)\| = \|x\|$. $\square$

(16) Let us consider real norm space sequences $X$, $Y$, and a real normed space $Z$. Then there exists a linear operator $I$ from the real norm space of bounded linear operators from $\prod X$ into the real norm space of bounded linear operators from $\prod Y$ into $Z$ into $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle \prod X, \prod Y\rangle, Z)$ such that

(i) $I$ is bijective and isometric, and

(ii) for every point $u$ of the real norm space of bounded linear operators from $\prod X$ into the real norm space of bounded linear operators from $\prod Y$ into $Z$, $\|u\| = \|I(u)\|$ and for every point $x$ of $\prod X$ and for every point $y$ of $\prod Y$, $I(u)(\langle x, y\rangle) = u(x)(y)$.

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