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Automatization of Ternary Boolean Algebras

Wojciech Kuśmierowski
Institute of Computer Science
University of Białystok
Ciołkowskiego 1M, 15-245 Białystok
Poland

Adam Grabowski
Institute of Computer Science
University of Białystok
Ciołkowskiego 1M, 15-245 Białystok
Poland

Summary. The main aim of this article is to introduce formally ternary Boolean algebras (TBAs) in terms of an abstract ternary operation, and to show their connection with the ordinary notion of a Boolean algebra, already present in the Mizar Mathematical Library [2]. Essentially, the core of this Mizar [1] formalization is based on the paper of A.A. Grau "Ternary Boolean Algebras" [7]. The main result is the single axiom for this class of lattices [12]. This is the continuation of the articles devoted to various equivalent axiomatizations of Boolean algebras: following Huntington [8] in terms of the binary sum and the complementation useful in the formalization of the Robbins problem [5], in terms of Sheffer stroke [9]. The classical definition ([6], [3]) can be found in [15] and its formalization is described in [4].

Similarly as in the case of recent formalizations of WA-lattices [14] and quasilattices [10], some of the results were proven in the Mizar system with the help of Prover9 [13], [11] proof assistant, so proofs are quite lengthy. They can be subject for subsequent revisions to make them more compact.

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0. Introduction

Ternary Boolean algebras (TBA for short) were introduced in the paper by A.A. Grau [7] in 1947. There the corresponding algebraic structure is

$$\langle T, \mathrm{cmpl}, \mathrm{trn}\rangle,$$

where T is a set, $\mathrm{trn} : T^3 \to T$ is a ternary operation on T, and $\mathrm{cmpl} : T \to T$ plays a role of the complementation operator.

The set of axioms: distributivity, idempotence, and absorption is given by definitions (Def. 3) – (Def. 7) in Sect. 2. The definition of the type "Ternary Boolean algebra" concludes this section.

Section 3 is devoted to formal correspondence between the usual definition of a Boolean algebra and TBAs. It is enough to choose arbitrary element $0 \in T$ and set

$$a \sqcup b = \mathrm{trn}(a, 0, b);$$

$$a \sqcap b = \mathrm{trn}(a, \mathrm{cmpl}(0), b).$$

In order to have all the operations (binary, unary, and ternary) available in the common framework, we introduced `LattTBAStr`. The Mizar functor converting ordinary Boolean algebras into TBAs is given in Sect. 4 (actually, `BA2TBA` in (Def. 13) returns TBA structure and `BA2TBAA` (Def. 14) – merged TBA and lattice structure). The ternary operation and usual binary lattice operations satisfy the equation

$$trn(a, b, c) = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a).$$

We call it the rosetta operation, hence `RosTrn` is used in the Mizar source (see Sect. 5). In Sect. 6 it is proven that the structure obtained in this way satisfy classical lattice axioms and, furthermore `BA2TBAA` is indeed a Boolean algebra (Sect. 7). Section 8 presents the single axiom for TBAs (Def. 15) and concluding cluster registrations show that TBAs defined in Sect. 2 satisfy also this single axiom.

1. Preliminaries

We consider TBA structures which extend ComplStr and are systems

$$\langle \text{a carrier}, \text{a complement operation}, \text{a ternary operation}\rangle$$

where the carrier is a set, the complement operation is a unary operation on the carrier, the ternary operation is a ternary operation on the carrier.

We consider TBA lattice structures which extend TBA structures and lattice structures and are systems

$$\langle \text{a carrier}, \text{a join operation}, \text{a meet operation}, \text{a complement operation},$$
$$\text{a ternary operation}\rangle$$

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, the complement operation is a unary operation on the carrier, the ternary operation is a ternary operation on the carrier.

The functor op3 yielding a ternary operation on $\{0\}$ is defined by

(Def. 1) $it(0,\ 0,\ 0) = 0$.

Let us observe that there exists a TBA structure which is trivial and non empty.

2. Axiomatization of Ternary Boolean Algebras

Let T be a non empty TBA structure and a, b, c be elements of T. The functor $\mathrm{T}(a, b, c)$ yielding an element of T is defined by the term

(Def. 2) (the ternary operation of T)$(a,\ b,\ c)$.

We say that T is ternary-distributive if and only if

(Def. 3) for every elements a, b, c, d, e of T, $\mathrm{T}(\mathrm{T}(a, b, c), d, \mathrm{T}(a, b, e)) = \mathrm{T}(a, b, \mathrm{T}(c, d, e))$.

We say that T is ternary-left-idempotent if and only if

(Def. 4) for every elements a, b of T, $\mathrm{T}(b, b, a) = b$.

We say that T is ternary-right-idempotent if and only if

(Def. 5) for every elements a, b of T, $\mathrm{T}(a, b, b) = b$.

We say that T is ternary-left-absorbing if and only if

(Def. 6) for every elements a, b of T, $\mathrm{T}(b^{\mathrm{c}}, b, a) = a$.

We say that T is ternary-right-absorbing if and only if

(Def. 7) for every elements a, b of T, $\mathrm{T}(a, b, b^{\mathrm{c}}) = a$.

One can check that every non empty TBA structure which is trivial is also ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

A ternary Boolean algebra is a ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, ternary-right-absorbing, non empty TBA structure.

3. Converting TBAs into Ordinary Binary Boolean Algebras

Let T be a ternary Boolean algebra and x be an element of T. The functors: $\mathrm{JoinTBA}(T, x)$ and $\mathrm{MeetTBA}(T, x)$ yielding binary operations on the carrier of T are defined by conditions

(Def. 8) for every elements a, b of T, $\mathrm{JoinTBA}(T, x)(a, b) = \mathrm{T}(a, x, b)$,

(Def. 9) for every elements a, b of T, $\mathrm{MeetTBA}(T, x)(a, b) = \mathrm{T}(a, x^{\mathrm{c}}, b)$,

respectively. The functor $\mathrm{TBA2BA}(T, x)$ yielding a non empty lattice structure is defined by the term

(Def. 10) $\langle$the carrier of $T, \mathrm{JoinTBA}(T, x), \mathrm{MeetTBA}(T, x)\rangle$.

4. Basic Properties of Ternary Operation

From now on T denotes a ternary Boolean algebra, a, b, c, d, e denote elements of T, and x, y, z denote elements of T. Now we state the propositions:

(1) $\mathrm{T}(a, b, a) = a$.

(2) $\mathrm{T}(\mathrm{T}(a, b, c), b, a) = \mathrm{T}(a, b, c)$.

(3) $\mathrm{T}(a, b, \mathrm{T}(c, b, d)) = \mathrm{T}(\mathrm{T}(a, b, c), b, d)$. The theorem is a consequence of (2).

(4) $\mathrm{T}(b^{\mathrm{c}}, b, a) = \mathrm{T}(a, b, b^{\mathrm{c}})$.

(5) $\mathrm{T}(a, b^{\mathrm{c}}, b) = a$.

(6) $(a^{\mathrm{c}})^{\mathrm{c}} = a$. The theorem is a consequence of (5).

(7) $\mathrm{T}(a, b, a^{\mathrm{c}}) = b$. The theorem is a consequence of (6).

(8) $\mathrm{T}(a, b, c) = \mathrm{T}(a, c, b)$. The theorem is a consequence of (7) and (1).

(9) $\mathrm{T}(a, b, c) = \mathrm{T}(b, c, a)$. The theorem is a consequence of (7).

(10) $\mathrm{T}(a, b, c) = \mathrm{T}(c, b, a)$. The theorem is a consequence of (8) and (9).

(11) Let us consider an element x of T. Then $\mathrm{T}(a, b, c) = \mathrm{T}(\mathrm{T}(\mathrm{T}(a, x, b), x^{\mathrm{c}}, \mathrm{T}(b, x, c)), x^{\mathrm{c}}, \mathrm{T}(c, x, a))$. The theorem is a consequence of (8), (10), (7), (9), and (3).

5. The Rosetta Operation

Let L be a Boolean lattice and a, b, c be elements of L. The functor $\mathrm{Ros}(a, b, c)$ yielding an element of L is defined by the term

(Def. 11) $((a \sqcap b) \sqcup (b \sqcap c)) \sqcup (c \sqcap a)$.

Let B be a Boolean lattice. The functor $\mathrm{RosTrn}(B)$ yielding a ternary operation on the carrier of B is defined by

(Def. 12) for every elements a, b, c of B, $it(a, b, c) = \mathrm{Ros}(a, b, c)$.

Let B be a Boolean lattice. The functor BA2TBA(B) yielding a TBA structure is defined by the term

(Def. 13) $\langle$the carrier of B, comp B, RosTrn(B)$\rangle$.

The functor BA2TBAA(B) yielding a TBA lattice structure is defined by the term

(Def. 14) $\langle$the carrier of B, the join operation of B, the meet operation of B, comp B, RosTrn(B)$\rangle$.

Let us note that BA2TBA(B) is non empty and BA2TBAA(B) is non empty.

6. Proof that TBA2BA Satisfy Lattice Axioms

In the sequel T denotes a ternary Boolean algebra.

Let us consider T. Let x be an element of T. Let us observe that JoinTBA(T, x) is commutative and JoinTBA(T, x) is associative and MeetTBA(T, x) is commutative.

From now on x denotes an element of T.

Let us consider T. Let x be an element of T. Note that MeetTBA(T, x) is associative.

Let T be a ternary Boolean algebra and p be an element of T. One can verify that the lattice structure of TBA2BA(T, p) is lattice-like.

7. Proof that BA2TBAA Returns Standard Example of TBA

Let B be a Boolean lattice. One can verify that BA2TBAA(B) is lattice-like.

Now we state the propositions:

(12) Let us consider a Boolean lattice B, an element x of B, and an element xx of BA2TBA(B). If $xx = x$, then $x^{\mathrm{c}} = xx^{\mathrm{c}}$.

(13) Let us consider a Boolean lattice B, an element x of B, and an element xx of BA2TBAA(B). If $xx = x$, then $x^{\mathrm{c}} = xx^{\mathrm{c}}$.

Let B be a Boolean lattice. One can verify that BA2TBA(B) is ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing and BA2TBAA(B) is ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

In the sequel B denotes a Boolean lattice and v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_{103}, v_{100}, v_{102}, v_{104}, v_{105}, v_{101} denote elements of BA2TBAA(B).

Now we state the propositions:

(14) Suppose for every v_1 and v_0, $\mathrm{T}(v_0, v_0, v_1) = v_0$ and for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) = \mathrm{T}(v_2, v_0, v_1)$ and for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) =$

$\mathrm{T}(v_0, v_2, v_1)$ and for every v_3, v_2, v_1, and v_0, $\mathrm{T}(\mathrm{T}(v_0, v_1, v_2), v_1, v_3) = \mathrm{T}(v_0, v_1, \mathrm{T}(v_2, v_1, v_3))$. $\mathrm{T}(\mathrm{T}(v_1, v_2, v_3), v_4, \mathrm{T}(v_1, v_2, v_5)) = \mathrm{T}(v_1, v_2, \mathrm{T}(v_3, v_4, v_5))$.

(15) Suppose for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) = ((v_0 \sqcup v_1) \sqcap (v_1 \sqcup v_2)) \sqcap (v_0 \sqcup v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcup (v_1 \sqcap v_2) = (v_0 \sqcup v_1) \sqcap (v_0 \sqcup v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$. $\mathrm{T}(\mathrm{T}(v_1, v_2, v_3), v_2, v_4) = \mathrm{T}(v_1, v_2, \mathrm{T}(v_3, v_2, v_4))$.

(16) Let us consider a Boolean lattice B, elements v_0, v_1 of BA2TBAA(B), and elements a, b of B. If $a = v_0$ and $b = v_1$, then $v_0 \sqcup v_1 = a \sqcup b$.

Let B be a Boolean lattice. Observe that BA2TBAA(B) is ternary-distributive.

Let T be a ternary Boolean algebra and p be an element of T. Let us note that the lattice structure of TBA2BA(T, p) is distributive and the lattice structure of TBA2BA(T, p) is bounded.

Let us consider a ternary Boolean algebra T and an element p of T. Now we state the propositions:

(17) $\top_\alpha = p$, where α is the lattice structure of TBA2BA(T, p).

(18) $\perp_\alpha = p^{\mathrm{c}}$, where α is the lattice structure of TBA2BA(T, p).

Let T be a ternary Boolean algebra and p be an element of T. Note that the lattice structure of TBA2BA(T, p) is complemented.

Let us consider T. Observe that the lattice structure of TBA2BA(T, p) is Boolean.

8. Single Axiom for TBA

In the sequel T denotes a non empty TBA structure and v_0, v_1, v_2, v_3, v_4, v_5, v_6, u, w, v, v_{100}, v_{101}, v_{102}, v_{103}, v_{104} denote elements of T.

Let T be a non empty TBA structure. We say that T is satisfying TBA_1 if and only if

(Def. 15) for every elements x, y, z, u, v, v_6, w of T, $\mathrm{T}(\mathrm{T}(x, x^{\mathrm{c}}, y), \mathrm{T}(\mathrm{T}(z, u, v), w, \mathrm{T}(z, u, v_6))^{\mathrm{c}}, \mathrm{T}(u, \mathrm{T}(v_6, w, v), z)) = y$.

Now we state the proposition:

(19) Suppose for every v_4, v_3, v_2, v_1, and v_0, $\mathrm{T}(\mathrm{T}(v_0, v_1, v_2), v_3, \mathrm{T}(v_0, v_1, v_4)) = \mathrm{T}(v_0, v_1, \mathrm{T}(v_2, v_3, v_4))$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_1, v_1) = v_1$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_1, v_1{}^{\mathrm{c}}) = v_0$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_0, v_1) = v_0$. Let us consider elements x, y, z, u, v, v_6, w of T. Then $\mathrm{T}(\mathrm{T}(x, x^{\mathrm{c}}, y), \mathrm{T}(\mathrm{T}(z, u, v), w, \mathrm{T}(z, u, v_6))^{\mathrm{c}}, \mathrm{T}(u, \mathrm{T}(v_6, w, v), z)) = y$.

Let T be a non empty TBA structure. We say that T is TBA-like if and only if

(Def. 16) T is ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

Note that every non empty TBA structure which is ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing is also TBA-like and every non empty TBA structure which is TBA-like is also ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing and every non empty TBA structure which is TBA-like is also satisfying TBA_1.

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Duality Notions in Real Projective Plane[1]

Roland Coghetto
cafr-MSA2P asbl
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. In this article, we check with the Mizar system [1], [2], the converse of Desargues' theorem and the converse of Pappus' theorem of the real projective plane. It is well known that in the projective plane, the notions of points and lines are dual [11], [9], [15], [8]. Some results (analytical, synthetic, combinatorial) of projective geometry are already present in some libraries Lean/Hott [5], Isabelle/Hol [3], Coq [13], [14], [4], Agda [6], ...

Proofs of dual statements by proof assistants have already been proposed, using an axiomatic method (for example see in [13] - the section on duality: "[...] For every theorem we prove, we can easily derive its dual using our function `swap` [...][2]").

In our formalisation, we use an analytical rather than a synthetic approach using the definitions of Leończuk and Prażmowski of the projective plane [12]. Our motivation is to show that it is possible by developing dual definitions to find proofs of dual theorems in a few lines of code.

In the first part, rather technical, we introduce definitions that allow us to construct the duality between the points of the real projective plane and the lines associated to this projective plane. In the second part, we give a natural definition of line concurrency and prove that this definition is dual to the definition of alignment. Finally, we apply these results to find, in a few lines, the dual properties and theorems of those defined in the article [12] (`transitive`, `Vebleian`, `at_least_3rank`, `Fanoian`, `Desarguesian`, `2-dimensional`).

We hope that this methodology will allow us to continued more quickly the proof started in [7] that the Beltrami-Klein plane is a model satisfying the axioms of the hyperbolic plane (in the sense of Tarski geometry [10]).

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[2]https://github.com/coq-contribs/projective-geometry

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1. Preliminaries

Now we state the proposition:

(1) Let us consider real numbers a, b, c, d, e, f, g, h, i. Then $\langle|[a,b,c],[d,e,f],[g,h,i]|\rangle = a\cdot e\cdot i + b\cdot f\cdot g + c\cdot d\cdot h - g\cdot e\cdot c - h\cdot f\cdot a - i\cdot d\cdot b$.

Let us consider real numbers a, b, c, d, e. Now we state the propositions:

(2) $\langle|[a,1,0],[b,0,1],[c,d,e]|\rangle = c - a\cdot d - e\cdot b$.

(3) $\langle|[1,a,0],[0,b,1],[c,d,e]|\rangle = b\cdot e + a\cdot c - d$.

(4) $\langle|[1,0,a],[0,1,b],[c,d,e]|\rangle = e - c\cdot a - d\cdot b$.

(5) Let us consider an element u of $\mathcal{E}^3_{\mathrm{T}}$. Then u is zero if and only if $|(u,u)| = 0$.

Let us consider non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(6) If $\langle|u,v,w|\rangle = 0$, then there exists a non zero element p of $\mathcal{E}^3_{\mathrm{T}}$ such that $|(p,u)| = 0$ and $|(p,v)| = 0$ and $|(p,w)| = 0$.

(7) If $|(u,v)| = 0$ and w and v are proportional, then $|(u,w)| = 0$.

(8) Let us consider non zero elements a, u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose u and v are not proportional and $|(a,u)| = 0$ and $|(a,v)| = 0$. Then a and $u\times v$ are proportional.

(9) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and a real number r. If $r\neq 0$ and u and v are proportional, then $r\cdot u$ and v are proportional.

2. Dual of a Point - Dual of a Line

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P is π_1-zero if and only if

(Def. 1) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of u holds $u(1) = 0$.

Note that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is π_1-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non π_1-zero.

Now we state the proposition:

(10) Let us consider a non π_1-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. If $P =$ the direction of u, then $u(1)\neq 0$.

Let P be a non π_1-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\widetilde{\pi_1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 2) the direction of $it = P$ and $it(1) = 1$.

Now we state the propositions:

(11) Let us consider a non π_1-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u. Then $\widetilde{\pi_1}(P) = [1, \frac{u(2)}{u(1)}, \frac{u(3)}{u(1)}]$.

(12) Let us consider a non π_1-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $Q =$ the direction of $\widetilde{\pi_1}(P)$. Then Q is not π_1-zero.

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P is π_2-zero if and only if

(Def. 3) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of u holds $u(2) = 0$.

One can verify that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is π_2-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non π_2-zero.

Now we state the proposition:

(13) Let us consider a non π_2-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. If $P =$ the direction of u, then $u(2) \neq 0$.

Let P be a non π_2-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\widetilde{\pi_2}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 4) the direction of $it = P$ and $it(2) = 1$.

Now we state the propositions:

(14) Let us consider a non π_2-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u. Then $\widetilde{\pi_2}(P) = [\frac{u(1)}{u(2)}, 1, \frac{u(3)}{u(2)}]$.

(15) Let us consider a non π_2-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $Q =$ the direction of $\widetilde{\pi_2}(P)$. Then Q is not π_2-zero.

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P is π_3-zero if and only if

(Def. 5) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of u holds $u(3) = 0$.

Observe that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is π_3-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non π_3-zero.

Now we state the proposition:

(16) Let us consider a non π_3-zero point P of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element u of $\mathcal{E}^3_{\rm T}$. If $P =$ the direction of u, then $u(3) \neq 0$.

Let P be a non π_3-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\widetilde{\pi_3}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by

(Def. 6) the direction of $it = P$ and $it(3) = 1$.

Now we state the propositions:

(17) Let us consider a non π_3-zero point P of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element u of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of u. Then $\widetilde{\pi_3}(P) = [\frac{u(1)}{u(3)}, \frac{u(2)}{u(3)}, 1]$.

(18) Let us consider a non π_3-zero point P of the projective space over $\mathcal{E}^3_{\rm T}$, and a point Q of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $Q =$ the direction of $\widetilde{\pi_3}(P)$. Then Q is not π_3-zero.

Let us observe that there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non π_1-zero and non π_2-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non π_1-zero and non π_3-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non π_2-zero and non π_3-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non π_1-zero, non π_2-zero, and non π_3-zero.

Let P be a non π_1-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 7) $[-(\widetilde{\pi_1}(P))(2), 1, 0]$.

The functor $\mathrm{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 8) the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P)$.

The functor $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 9) $[-(\widetilde{\pi_1}(P))(3), 0, 1]$.

The functor $\mathrm{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 10) the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

Let us consider a non π_1-zero point P of the projective space over $\mathcal{E}^3_{\rm T}$. Now we state the propositions:

(19) $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P) \neq \mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

(20) The direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P) \neq$ the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

(21) Let us consider a non π_1-zero element P of the projective space over $\mathcal{E}^3_{\rm T}$, a non zero element u of $\mathcal{E}^3_{\rm T}$, and an element v of $\mathcal{E}^3_{\rm T}$. Suppose $u =$

$\widetilde{\pi_1}(P)$. Then $\langle|\operatorname{dir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{dir}_{(-\widetilde{\pi_1})_3,0,1}(P), v|\rangle = |(u,v)|$. The theorem is a consequence of (11) and (2).

(22) Let us consider a non π_1-zero element P of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element u of $\mathcal{E}^3_{\rm T}$. Suppose $u = \widetilde{\pi_1}(P)$. Then $\langle|\operatorname{dir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{dir}_{(-\widetilde{\pi_1})_3,0,1}(P), \widetilde{\pi_1}(P)|\rangle = 1 + u(2)\cdot u(2) + u(3)\cdot u(3)$. The theorem is a consequence of (21).

Let P be a non π_2-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 11) $[1, -(\widetilde{\pi_2}(P))(1), 0]$.

The functor $\operatorname{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 12) the direction of $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P)$.

The functor $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 13) $[0, -(\widetilde{\pi_2}(P))(3), 1]$.

The functor $\operatorname{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 14) the direction of $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

Let us consider a non π_2-zero point P of the projective space over $\mathcal{E}^3_{\rm T}$. Now we state the propositions:

(23) $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P) \neq \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

(24) The direction of $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P) \neq$ the direction of $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

(25) Let us consider a non π_2-zero element P of the projective space over $\mathcal{E}^3_{\rm T}$, a non zero element u of $\mathcal{E}^3_{\rm T}$, and an element v of $\mathcal{E}^3_{\rm T}$. Suppose $u = \widetilde{\pi_2}(P)$. Then $\langle|\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P), v|\rangle = -|(u,v)|$. The theorem is a consequence of (14) and (3).

(26) Let us consider a non π_2-zero element P of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element u of $\mathcal{E}^3_{\rm T}$. Suppose $u = \widetilde{\pi_2}(P)$. Then $\langle|\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P), \widetilde{\pi_2}(P)|\rangle = -(u(1)\cdot u(1) + 1 + u(3)\cdot u(3))$. The theorem is a consequence of (25).

Let P be a non π_3-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\operatorname{dir}_{1,0,(-\widetilde{\pi_3})_1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 15) $[1, 0, -(\widetilde{\pi_3}(P))(1)]$.

The functor $\operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\rm T}$ is defined by the term

(Def. 16) the direction of $\operatorname{dir}_{1,0,(-\widetilde{\pi_3})_1}(P)$.

The functor $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 17) $[0, 1, -(\widetilde{\pi_3}(P))(2)]$.

The functor $\mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 18) the direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

Let us consider a non π_3-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(27) $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P) \neq \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

(28) The direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P) \neq$ the direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

(29) Let us consider a non π_3-zero element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$, and an element v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_3}(P)$. Then $\langle| \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P), \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P), v|\rangle = |(u, v)|$. The theorem is a consequence of (17) and (4).

(30) Let us consider a non π_3-zero element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_3}(P)$. Then $\langle| \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P)$, $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P), \widetilde{\pi_3}(P)|\rangle = u(1) \cdot u(1) + u(2) \cdot u(2) + 1$. The theorem is a consequence of (29).

Let P be a non π_1-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_1(P)$ yielding an element of L(the real projective plane) is defined by the term

(Def. 19) $\mathrm{Line}(\mathrm{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P), \mathrm{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P))$.

Let P be a non π_2-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_2(P)$ yielding an element of L(the real projective plane) is defined by the term

(Def. 20) $\mathrm{Line}(\mathrm{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P), \mathrm{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P))$.

Let P be a non π_3-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_3(P)$ yielding an element of L(the real projective plane) is defined by the term

(Def. 21) $\mathrm{Line}(\mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P), \mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

Let us consider a non π_1-zero, non π_2-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(31) Suppose $P =$ the direction of u. Then

(i) $\widetilde{\pi_1}(P) = [1, \frac{u(2)}{u(1)}, \frac{u(3)}{u(1)}]$, and

(ii) $\widetilde{\pi_2}(P) = [\frac{u(1)}{u(2)}, 1, \frac{u(3)}{u(2)}]$.

(32) Suppose $P =$ the direction of u. Then

(i) $\widetilde{\pi_1}(P) = \frac{u(2)}{u(1)} \cdot (\widetilde{\pi_2}(P))$, and

(ii) $\widetilde{\pi_2}(P) = \frac{u(1)}{u(2)} \cdot (\widetilde{\pi_1}(P))$.

The theorem is a consequence of (10), (13), (11), and (14).

Let us consider a non π_1-zero, non π_2-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(33) $\mathrm{dual}_1(P) = \mathrm{dual}_2(P)$. The theorem is a consequence of (11), (14), (2), (10), (3), and (13).

(34) $\mathrm{dual}_2(P) = \mathrm{dual}_3(P)$. The theorem is a consequence of (17), (14), (3), (13), (16), and (4).

(35) $\mathrm{dual}_1(P) = \mathrm{dual}_3(P)$. The theorem is a consequence of (11), (17), (2), (10), (4), and (16).

(36) Let us consider a non π_1-zero, non π_2-zero, non π_3-zero point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Then

(i) $\mathrm{dual}_1(P) = \mathrm{dual}_2(P)$, and

(ii) $\mathrm{dual}_1(P) = \mathrm{dual}_3(P)$, and

(iii) $\mathrm{dual}_2(P) = \mathrm{dual}_3(P)$.

(37) Every element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is non π_1-zero or non π_2-zero or non π_3-zero non π_1-zero non π_2-zero or non π_3-zero.

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor dual P yielding an element of L(the real projective plane) is defined by

(Def. 22) (i) there exists a non π_1-zero point P' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and $it = \mathrm{dual}_1(P')$, **if** P is not π_1-zero,

(ii) there exists a non π_2-zero point P' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and $it = \mathrm{dual}_2(P')$, **if** P is π_1-zero and non π_2-zero,

(iii) there exists a non π_3-zero point P' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and $it = \mathrm{dual}_3(P')$, **if** P is π_1-zero, π_2-zero, and non π_3-zero.

Let P be a point of the real projective plane. The functor $\# P$ yielding an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 23) P.

The functor dual P yielding an element of L(the real projective plane) is defined by the term

(Def. 24) dual $\# P$.

Let us consider an element P of the real projective plane. Now we state the propositions:

(38) Suppose $\# P$ is not π_1-zero. Then there exists a non π_1-zero point P' of the projective space over $\mathcal{E}^3_{\rm T}$ such that

(i) $P = P'$, and

(ii) $\operatorname{dual} P = \operatorname{dual}_1(P')$.

(39) Suppose $\# P$ is not π_2-zero. Then there exists a non π_2-zero point P' of the projective space over $\mathcal{E}^3_{\rm T}$ such that

(i) $P = P'$, and

(ii) $\operatorname{dual} P = \operatorname{dual}_2(P')$.

The theorem is a consequence of (33).

(40) Suppose $\# P$ is not π_3-zero. Then there exists a non π_3-zero point P' of the projective space over $\mathcal{E}^3_{\rm T}$ such that

(i) $P = P'$, and

(ii) $\operatorname{dual} P = \operatorname{dual}_3(P')$.

The theorem is a consequence of (34) and (35).

Let us consider a non π_1-zero element P of the projective space over $\mathcal{E}^3_{\rm T}$. Now we state the propositions:

(41) $P \notin \operatorname{Line}(\operatorname{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P))$. The theorem is a consequence of (21) and (5).

(42) $P \notin \operatorname{Line}(\operatorname{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P))$. The theorem is a consequence of (25) and (5).

(43) $P \notin \operatorname{Line}(\operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P), \operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P)$. The theorem is a consequence of (29) and (5).

(44) Let us consider a point P of the real projective plane. Then $P \notin \operatorname{dual} P$. The theorem is a consequence of (37), (38), (41), (39), (42), (40), and (43).

Let l be an element of L(the real projective plane). The functor $\operatorname{dual} l$ yielding a point of the real projective plane is defined by

(Def. 25) there exist points P, Q of the real projective plane such that $P \neq Q$ and $l = \operatorname{Line}(P, Q)$ and $it = \operatorname{L2P}(P, Q)$.

Now we state the propositions:

(45) Let us consider a point P of the real projective plane. Then $\operatorname{dual} \operatorname{dual} P = P$. The theorem is a consequence of (37), (38), (11), (10), (8), (9), (39), (14), (13), (40), (17), and (16).

(46) Let us consider an element l of L(the real projective plane). Then $\operatorname{dual} \operatorname{dual} l = l$. The theorem is a consequence of (37), (38), (10), (11), (20), (2), (39), (13), (14), (24), (3), (40), (16), (17), (28), and (4).

(47) Let us consider points P, Q of the real projective plane. Then $P \neq Q$ if and only if $\operatorname{dual} P \neq \operatorname{dual} Q$. The theorem is a consequence of (45).

(48) Let us consider elements l, m of L(the real projective plane). Then $l \neq m$ if and only if $\operatorname{dual} l \neq \operatorname{dual} m$. The theorem is a consequence of (46).

3. Two Dual Notions: Concurrency and Collinearity

Let l_1, l_2, l_3 be elements of L(the real projective plane). We say that l_1, l_2, l_3 are concurrent if and only if

(Def. 26) there exists a point P of the real projective plane such that $P \in l_1$ and $P \in l_2$ and $P \in l_3$.

Let l be an element of L(the real projective plane). The functor $\# l$ yielding a line of Inc-ProjSp(the real projective plane) is defined by the term

(Def. 27) l.

Let l be a line of Inc-ProjSp(the real projective plane). The functor $\# l$ yielding an element of L(the real projective plane) is defined by the term

(Def. 28) l.

Now we state the propositions:

(49) Let us consider elements l_1, l_2, l_3 of L(the real projective plane). Then l_1, l_2, l_3 are concurrent if and only if $\# l_1$, $\# l_2$, $\# l_3$ are concurrent.

(50) Let us consider lines l_1, l_2, l_3 of Inc-ProjSp(the real projective plane). Then l_1, l_2, l_3 are concurrent if and only if $\# l_1$, $\# l_2$, $\# l_3$ are concurrent. The theorem is a consequence of (49).

(51) Let us consider elements P, Q, R of the real projective plane. Suppose P, Q and R are collinear. Then

- (i) Q, R and P are collinear, and
- (ii) R, P and Q are collinear, and
- (iii) P, R and Q are collinear, and
- (iv) R, Q and P are collinear, and
- (v) Q, P and R are collinear.

(52) Let us consider elements l_1, l_2, l_3 of L(the real projective plane). Suppose l_1, l_2, l_3 are concurrent. Then

- (i) l_2, l_1, l_3 are concurrent, and
- (ii) l_1, l_3, l_2 are concurrent, and
- (iii) l_3, l_2, l_1 are concurrent, and

(iv) l_3, l_2, l_1 are concurrent, and

(v) l_2, l_3, l_1 are concurrent.

(53) Let us consider points P, Q of the real projective plane, and elements P', Q' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P = P'$ and $Q = Q'$, then $\mathrm{Line}(P, Q) = \mathrm{Line}(P', Q')$.

Let us consider a point P of the real projective plane and an element l of L(the real projective plane). Now we state the propositions:

(54) If $P \in l$, then $\mathrm{dual}\, l \in \mathrm{dual}\, P$. The theorem is a consequence of (37), (38), (21), (7), (39), (25), (40), and (29).

(55) If $\mathrm{dual}\, l \in \mathrm{dual}\, P$, then $P \in l$. The theorem is a consequence of (54), (45), and (46).

(56) Let us consider points P, Q, R of the real projective plane. Suppose P, Q and R are collinear. Then $\mathrm{dual}\, P$, $\mathrm{dual}\, Q$, $\mathrm{dual}\, R$ are concurrent. The theorem is a consequence of (54).

(57) Let us consider an element l of L(the real projective plane), and points P, Q, R of the real projective plane. If $P, Q, R \in l$, then P, Q and R are collinear.

(58) Let us consider elements l_1, l_2, l_3 of L(the real projective plane). Suppose l_1, l_2, l_3 are concurrent. Then $\mathrm{dual}\, l_1$, $\mathrm{dual}\, l_2$ and $\mathrm{dual}\, l_3$ are collinear. The theorem is a consequence of (54) and (57).

(59) Let us consider points P, Q, R of the real projective plane. Then P, Q and R are collinear if and only if $\mathrm{dual}\, P$, $\mathrm{dual}\, Q$, $\mathrm{dual}\, R$ are concurrent. The theorem is a consequence of (56), (58), and (45).

(60) Let us consider elements l_1, l_2, l_3 of L(the real projective plane). Then l_1, l_2, l_3 are concurrent if and only if $\mathrm{dual}\, l_1$, $\mathrm{dual}\, l_2$ and $\mathrm{dual}\, l_3$ are collinear. The theorem is a consequence of (46) and (59).

4. Some Dual Properties of a Real Projective Plane

Now we state the propositions:

(61) The real projective plane is reflexive, transitive, Vebleian, at least 3 rank, Fanoian, Desarguesian, Pappian, and 2-dimensional.

(62) Converse reflexive:
Let us consider elements l, m, n of L(the real projective plane). Then

(i) l, m, l are concurrent, and

(ii) l, l, m are concurrent, and

(iii) l, m, m are concurrent.

The theorem is a consequence of (59) and (46).

(63) CONVERSE TRANSITIVE:

Let us consider elements l, m, n, n_1, n_2 of L(the real projective plane). Suppose $l \neq m$ and l, m, n are concurrent and l, m, n_1 are concurrent and l, m, n_2 are concurrent. Then n, n_1, n_2 are concurrent. The theorem is a consequence of (60), (48), (59), and (46).

(64) CONVERSE VEBLIEAN:

Let us consider elements l, l_1, l_2, n, n_1 of L(the real projective plane). Suppose l, l_1, n are concurrent and l_1, l_2, n_1 are concurrent. Then there exists an element n_2 of L(the real projective plane) such that

(i) l, l_2, n_2 are concurrent, and

(ii) n, n_1, n_2 are concurrent.

The theorem is a consequence of (60), (59), and (46).

(65) CONVERSE AT LEAST 3-RANK:

Let us consider elements l, m of L(the real projective plane). Then there exists an element n of L(the real projective plane) such that

(i) $l \neq n$, and

(ii) $m \neq n$, and

(iii) l, m, n are concurrent.

The theorem is a consequence of (45), (59), and (46).

(66) CONVERSE FANOIAN:

Let us consider elements l_1, n_2, m, n_1, m_1, l, n of L(the real projective plane). Suppose l_1, n_2, m are concurrent and n_1, m_1, m are concurrent and l_1, n_1, l are concurrent and n_2, m_1, l are concurrent and l_1, m_1, n are concurrent and n_2, n_1, n are concurrent and l, m, n are concurrent. Then

(i) l_1, n_2, m_1 are concurrent, or

(ii) l_1, n_2, n_1 are concurrent, or

(iii) l_1, n_1, m_1 are concurrent, or

(iv) n_2, n_1, m_1 are concurrent.

The theorem is a consequence of (60).

(67) CONVERSE DESARGUESIAN:

Let us consider elements k, l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2, n_3 of L(the real projective plane). Suppose $k \neq m_1$ and $l_1 \neq m_1$ and $k \neq m_2$ and $l_2 \neq m_2$ and $k \neq m_3$ and $l_3 \neq m_3$ and k, l_1, l_2 are not concurrent and k, l_1, l_3 are not concurrent and k, l_2, l_3 are not concurrent and l_1, l_2, n_3 are concurrent and m_1, m_2, n_3 are concurrent and l_2, l_3, n_1 are concurrent and m_2, m_3, n_1

are concurrent and l_1, l_3, n_2 are concurrent and m_1, m_3, n_2 are concurrent and k, l_1, m_1 are concurrent and k, l_2, m_2 are concurrent and k, l_3, m_3 are concurrent. Then n_1, n_2, n_3 are concurrent. The theorem is a consequence of (48) and (60).

(68) Converse Pappian:
Let us consider elements k, l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2, n_3 of L(the real projective plane). Suppose $k \neq l_2$ and $k \neq l_3$ and $l_2 \neq l_3$ and $l_1 \neq l_2$ and $l_1 \neq l_3$ and $k \neq m_2$ and $k \neq m_3$ and $m_2 \neq m_3$ and $m_1 \neq m_2$ and $m_1 \neq m_3$ and k, l_1, m_1 are not concurrent and k, l_1, l_2 are concurrent and k, l_1, l_3 are concurrent and k, m_1, m_2 are concurrent and k, m_1, m_3 are concurrent and l_1, m_2, n_3 are concurrent and m_1, l_2, n_3 are concurrent and l_1, m_3, n_2 are concurrent and l_3, m_1, n_2 are concurrent and l_2, m_3, n_1 are concurrent and l_3, m_2, n_1 are concurrent. Then n_1, n_2, n_3 are concurrent. The theorem is a consequence of (48) and (60).

(69) Converse 2-dimensional:
Let us consider elements l, l_1, m, m_1 of L(the real projective plane). Then there exists an element n of L(the real projective plane) such that

(i) l, l_1, n are concurrent, and

(ii) m, m_1, n are concurrent.

The theorem is a consequence of (59) and (46).

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Finite Dimensional Real Normed Spaces are Proper Metric Spaces[1]

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Karuizawa Hotch 244-1
Nagano, Japan

Summary. In this article, we formalize in Mizar [1], [2] the topological properties of finite-dimensional real normed spaces. In the first section, we formalize the Bolzano-Weierstrass theorem, which states that a bounded sequence of points in an n-dimensional Euclidean space has a certain subsequence that converges to a point. As a corollary, it is also shown the equivalence between a subset of an n-dimensional Euclidean space being compact and being closed and bounded.

In the next section, we formalize the definitions of L1-norm (Manhattan Norm) and maximum norm and show their topological equivalence in n-dimensional Euclidean spaces and finite-dimensional real linear spaces. In the last section, we formalize the linear isometries and their topological properties. Namely, it is shown that a linear isometry between real normed spaces preserves properties such as continuity, the convergence of a sequence, openness, closeness, and compactness of subsets. Finally, it is shown that finite-dimensional real normed spaces are proper metric spaces. We referred to [5], [9], and [7] in the formalization.

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1. Bolzano-Weierstrass Theorem and its Corollary

From now on X denotes a set, n, m, k denote natural numbers, K denotes a field, f denotes an n-element, real-valued finite sequence, and M denotes a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Now we state the propositions:

(1) Let us consider an element x of $\mathcal{R}^{n+1}$, and an element y of $\mathcal{R}^n$. If $y = x \restriction n$, then $|y| \leqslant |x|$.

(2) Let us consider an element x of $\mathcal{R}^{n+1}$, and an element w of $\mathbb{R}$. If $w = x(n+1)$, then $|w| \leqslant |x|$.

(3) Let us consider an element x of $\mathcal{R}^{n+1}$, an element y of $\mathcal{R}^n$, and an element w of $\mathbb{R}$. If $y = x \restriction n$ and $w = x(n+1)$, then $|x| \leqslant |y| + |w|$.

(4) Let us consider elements x, y of $\mathcal{R}^n$, and a natural number m. If $m \leqslant n$, then $(x - y) \restriction m = x \restriction m - y \restriction m$.

(5) Let us consider a natural number n, and a sequence x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose there exists a real number K such that for every natural number i, $\|x(i)\| < K$. Then there exists a subsequence x_0 of x such that x_0 is convergent.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every sequence x of $\langle \mathcal{E}^{\$_1}, \|\cdot\| \rangle$ such that there exists a real number K such that for every natural number i, $\|x(i)\| < K$ there exists a subsequence x_0 of x such that x_0 is convergent. $\mathcal{P}[0]$ by [4, (18)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(6) Let us consider a real normed space N, and a subset X of N. Suppose X is compact. Then

   (i) X is closed, and

   (ii) there exists a real number r such that for every point y of N such that $y \in X$ holds $\|y\| < r$.

(7) Let us consider a subset X of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then X is compact if and only if X is closed and there exists a real number r such that for every point y of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $y \in X$ holds $\|y\| < r$.

2. L1-norm and Maximum Norm

Now we state the propositions:

(8) Let us consider a non empty natural number n, and an element x of $\mathcal{R}^n$. Then there exists a real number x_4 such that

   (i) $x_4 \in \operatorname{rng}|x|$, and

   (ii) for every natural number i such that $i \in \operatorname{dom} x$ holds $|x|(i) \leqslant x_4$.

PROOF: Set $F = \mathrm{rng}|x|$. Set $x_4 = \sup F$. For every natural number i such that $i \in \mathrm{dom}\, x$ holds $|x|(i) \leqslant x_4$. □

(9) Let us consider a real-valued finite sequence x. Then $0 \leqslant \sum|x|$.

Let n be a natural number. Assume n is not empty. The functor max-norm(n) yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 1) for every element x of $\mathcal{R}^n$, $it(x) \in \mathrm{rng}|x|$ and for every natural number i such that $i \in \mathrm{dom}\, x$ holds $|x|(i) \leqslant it(x)$.

Assume n is not empty. The functor sum-norm(n) yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 2) for every element x of $\mathcal{R}^n$, $it(x) = \sum|x|$.

Now we state the proposition:

(10) Let us consider an element x of $\mathcal{R}^n$, and a real number x_4. Suppose $x_4 \in \mathrm{rng}|x|$ and for every natural number i such that $i \in \mathrm{dom}\, x$ holds $|x|(i) \leqslant x_4$. Then

(i) $\sum|x| \leqslant n \cdot x_4$, and

(ii) $x_4 \leqslant |x| \leqslant \sum|x|$.

PROOF: Set $F = n \mapsto x_4$. For every natural number j such that $j \in \mathrm{Seg}\, n$ holds $|x|(j) \leqslant F(j)$. Consider i being an object such that $i \in \mathrm{dom}|x|$ and $x_4 = |x|(i)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element x of $\mathcal{R}^{\$_1}$, $|x|^{\mathbf{2}} \leqslant (\sum|x|)^{\mathbf{2}}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

Let us consider a non empty natural number n, elements x, y of $\mathcal{R}^n$, and a real number a. Now we state the propositions:

(11) (i) $0 \leqslant (\text{max-norm}(n))(x)$, and

(ii) $(\text{max-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, and

(iii) $(\text{max-norm}(n))(a \cdot x) = |a| \cdot (\text{max-norm}(n))(x)$, and

(iv) $(\text{max-norm}(n))(x + y) \leqslant (\text{max-norm}(n))(x) + (\text{max-norm}(n))(y)$.

PROOF: Set $x_4 = (\text{max-norm}(n))(x)$. Set $y_2 = (\text{max-norm}(n))(y)$. Consider j_0 being an object such that $j_0 \in \mathrm{dom}|x|$ and $x_4 = |x|(j_0)$. Consider k_0 being an object such that $k_0 \in \mathrm{dom}|y|$ and $y_2 = |y|(k_0)$. $(\text{max-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. $(\text{max-norm}(n))(a \cdot x) = |a| \cdot (\text{max-norm}(n))(x)$. $(\text{max-norm}(n))(x + y) \leqslant (\text{max-norm}(n))(x) + (\text{max-norm}(n))(y)$. □

(12) (i) $0 \leqslant (\text{sum-norm}(n))(x)$, and

(ii) $(\text{sum-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, and

(iii) $(\text{sum-norm}(n))(a \cdot x) = |a| \cdot (\text{sum-norm}(n))(x)$, and

(iv) $(\text{sum-norm}(n))(x + y) \leqslant (\text{sum-norm}(n))(x) + (\text{sum-norm}(n))(y)$.

PROOF: $0 \leqslant \sum|x|$. $(\text{sum-norm}(n))(x) = 0$ iff $x = \langle\underbrace{0, \ldots, 0}_{n}\rangle$. For every natural number j such that $j \in \operatorname{Seg} n$ holds $|x+y|(j) \leqslant (|x| + |y|)(j)$. □

(13) Let us consider a non empty natural number n, and an element x of $\mathcal{R}^n$. Then

(i) $(\text{sum-norm}(n))(x) \leqslant n \cdot (\text{max-norm}(n))(x)$, and

(ii) $(\text{max-norm}(n))(x) \leqslant |x| \leqslant (\text{sum-norm}(n))(x)$.

The theorem is a consequence of (10).

(14) The RLS structure of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$.

(15) Let us consider a real number a, elements x, y of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and elements x_0, y_0 of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$. Suppose $x = x_0$ and $y = y_0$. Then

(i) the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle =$ the carrier of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$, and

(ii) $0_{\langle \mathcal{E}^n, \|\cdot\| \rangle} = 0_{\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}}$, and

(iii) $x + y = x_0 + y_0$, and

(iv) $a \cdot x = a \cdot x_0$, and

(v) $-x = -x_0$, and

(vi) $x - y = x_0 - y_0$.

The theorem is a consequence of (14).

Let X be a finite dimensional real linear space.

One can check that $\text{RLSp2RVSp}(X)$ is finite dimensional.

Now we state the proposition:

(16) Let us consider a finite dimensional real linear space X, an ordered basis b of $\text{RLSp2RVSp}(X)$, and an element y of $\text{RLSp2RVSp}(X)$. Then $y \rightarrow b$ is an element of $\mathcal{R}^{\dim(X)}$.

Let X be a finite dimensional real linear space and b be an ordered basis of $\text{RLSp2RVSp}(X)$. The functor $\text{max-norm}(X, b)$ yielding a function from X into $\mathbb{R}$ is defined by

(Def. 3) for every element x of X, there exists an element y of $\text{RLSp2RVSp}(X)$ and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \rightarrow b$ and $it(x) = (\text{max-norm}(\dim(X)))(z)$.

The functor $\text{sum-norm}(X, b)$ yielding a function from X into $\mathbb{R}$ is defined by

(Def. 4) for every element x of X, there exists an element y of $\text{RLSp2RVSp}(X)$ and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \rightarrow b$ and $it(x) = (\text{sum-norm}(\dim(X)))(z)$.

The functor Euclid-norm(X, b) yielding a function from X into $\mathbb{R}$ is defined by

(Def. 5) for every element x of X, there exists an element y of RLSp2RVSp(X) and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \to b$ and $it(x) = |z|$.

Now we state the proposition:

(17) Let us consider a natural number n, an element a of $\mathbb{R}$, an element a_1 of $\mathbb{R}_{\rm F}$, elements x, y of $\mathcal{R}^n$, and elements x_1, y_1 of (the carrier of $\mathbb{R}_{\rm F})^n$. Suppose $a = a_1$ and $x = x_1$ and $y = y_1$. Then

(i) $a \cdot x = a_1 \cdot x_1$, and

(ii) $x + y = x_1 + y_1$.

Let us consider a finite dimensional real linear space X, an ordered basis b of RLSp2RVSp(X), elements x, y of X, and a real number a. Now we state the propositions:

(18) Suppose $\dim(X) \neq 0$. Then

(i) $0 \leqslant (\text{max-norm}(X, b))(x)$, and

(ii) $(\text{max-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{max-norm}(X, b))(a \cdot x) = |a| \cdot (\text{max-norm}(X, b))(x)$, and

(iv) $(\text{max-norm}(X, b))(x+y) \leqslant (\text{max-norm}(X, b))(x) + (\text{max-norm}(X, b))(y)$.

The theorem is a consequence of (11).

(19) Suppose $\dim(X) \neq 0$. Then

(i) $0 \leqslant (\text{sum-norm}(X, b))(x)$, and

(ii) $(\text{sum-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{sum-norm}(X, b))(a \cdot x) = |a| \cdot (\text{sum-norm}(X, b))(x)$, and

(iv) $(\text{sum-norm}(X, b))(x + y) \leqslant (\text{sum-norm}(X, b))(x) + (\text{sum-norm}(X, b))(y)$.

The theorem is a consequence of (12).

(20) (i) $0 \leqslant (\text{Euclid-norm}(X, b))(x)$, and

(ii) $(\text{Euclid-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{Euclid-norm}(X, b))(a \cdot x) = |a| \cdot (\text{Euclid-norm}(X, b))(x)$, and

(iv) $(\text{Euclid-norm}(X, b))(x+y) \leqslant (\text{Euclid-norm}(X, b))(x) + (\text{Euclid-norm}(X, b))(y)$.

(21) Let us consider a finite dimensional real linear space X, an ordered basis b of RLSp2RVSp(X), and an element x of X. Suppose $\dim(X) \neq 0$. Then

(i) $(\text{sum-norm}(X, b))(x) \leqslant (\dim(X)) \cdot (\text{max-norm}(X, b))(x)$, and

(ii) $(\text{max-norm}(X, b))(x) \leqslant (\text{Euclid-norm}(X, b))(x) \leqslant (\text{sum-norm}(X, b))$ (x).

The theorem is a consequence of (13).

(22) Let us consider a finite dimensional real linear space V, and an ordered basis b of RLSp2RVSp(V). Suppose $\dim(V) \neq 0$. Then there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that

(i) S is bijective, and

(ii) for every element x of RLSp2RVSp(V), $S(x) = x \rightarrow b$.

The theorem is a consequence of (15).

(23) Let us consider a finite dimensional real normed space V. Suppose $\dim(V)$ $\neq 0$. Then there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ and there exists a finite dimensional vector space W over $\mathbb{R}_{\mathrm{F}}$ and there exists an ordered basis b of W such that $W =$ RLSp2RVSp(V) and S is bijective and for every element x of W, $S(x) = x \rightarrow b$. The theorem is a consequence of (15).

(24) Let us consider a real normed space V, a finite dimensional real linear space W, and an ordered basis b of RLSp2RVSp(W). Suppose V is finite dimensional and $\dim(V) \neq 0$ and the RLS structure of $V =$ the RLS structure of W. Then there exist real numbers k_1, k_2 such that

(i) $0 < k_1$, and

(ii) $0 < k_2$, and

(iii) for every point x of V, $\|x\| \leqslant k_1 \cdot (\text{max-norm}(W, b))(x)$ and $(\text{max-norm}$ $(W, b))(x) \leqslant k_2 \cdot \|x\|$.

PROOF: Reconsider $e = b$ as a finite sequence of elements of W. Reconsider $e_1 = e$ as a finite sequence of elements of V. Define $\mathcal{F}(\text{natural number}) =$ $\|e_{1/\$_1}\| (\in \mathbb{R})$. Consider k being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} k = \operatorname{len} b$ and for every natural number i such that $i \in \operatorname{dom} k$ holds $k(i) = \mathcal{F}(i)$. Set $k_1 = \sum k$. For every natural number i such that $i \in \operatorname{dom} k$ holds $0 \leqslant k(i)$. For every point x of V, $\|x\| \leqslant (k_1 + 1) \cdot (\text{max-norm}(W, b))(x)$ by [6, (12), (15)], [8, (7)].

Consider S_0 being a linear operator from W into $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that S_0 is bijective and for every element x of RLSp2RVSp(W), $S_0(x) =$ $x \rightarrow b$. Reconsider $S = S_0$ as a function from the carrier of V into the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. For every elements x, y of V, $S(x+y) = S(x)+S(y)$. For every real number a and for every vector x of V, $S(a \cdot x) = a \cdot S(x)$.

Consider T being a linear operator from $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ into V such that $T = S^{-1}$ and T is one-to-one and onto. For every element x of V, $\|x\| \leqslant (k_1+1) \cdot \|S(x)\|$. For every element y of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$, $\|T(y)\| \leqslant (k_1+1) \cdot \|y\|$. Set $C_2 = \{y$, where y is an element of $V : (\text{max-norm}(W, b))(y) = 1\}$.

Set $C_1 = \{x$, where x is an element of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle : (\text{max-norm}(\dim (W)))(x) = 1\}$. For every object z such that $z \in C_2$ holds $z \in$ the carrier of V. For every object z such that $z \in C_1$ holds $z \in$ the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. Consider z_5 being a point of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $z_5 \neq 0_{\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle}$. Reconsider $z_6 = z_5$ as an element of $\mathcal{R}^{\dim(W)}$. $(\text{max-norm}(\dim (W)))(z_6) \neq 0$. $0 < (\text{max-norm}(\dim(W)))(z_5)$. For every object y, $y \in T^{\circ}C_1$ iff $y \in C_2$. Reconsider $g = \text{max-norm}(\dim(W))$ as a function from the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ into $\mathbb{R}$. Set $D =$ the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. For every point x_0 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ and for every real number r such that $x_0 \in D$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $x_1 \in D$ and $\|x_1 - x_0\| < s$ holds $|g_{/x_1} - g_{/x_0}| < r$.

For every sequence s_1 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $\operatorname{rng} s_1 \subseteq C_1$ and s_1 is convergent holds $\lim s_1 \in C_1$. There exists a real number r such that for every point y of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $y \in C_1$ holds $\|y\| < r$ by (13), [3, (1)]. Reconsider $f = \mathrm{id}_{C_2}$ as a partial function from V to V. Consider y_0 being an element of V such that $y_0 \in \operatorname{dom}\|f\|$ and $\inf \operatorname{rng}\|f\| = \|f\|(y_0)$. Set $k_2 = \|f_{/y_0}\|$. For every element x of V such that $x \in C_2$ holds $k_2 \leqslant \|x\|$. $k_2 \neq 0$. For every point x of V, $(\text{max-norm}(W, b))(x) \leqslant \frac{1}{k_2} \cdot \|x\|$. $\square$

(25) Let us consider real normed spaces X, Y. Suppose the RLS structure of $X =$ the RLS structure of Y and X is finite dimensional and $\dim(X) \neq 0$. Then there exist real numbers k_1, k_2 such that

- (i) $0 < k_1$, and
- (ii) $0 < k_2$, and
- (iii) for every element x of X and for every element y of Y such that $x = y$ holds $\|x\| \leqslant k_1 \cdot \|y\|$ and $\|y\| \leqslant k_2 \cdot \|x\|$.

The theorem is a consequence of (24).

(26) Let us consider a real normed space V. Suppose V is finite dimensional and $\dim(V) \neq 0$. Then there exist real numbers k_1, k_2 and there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that S is bijective and $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|S(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|S(x)\|$. The theorem is a consequence of (23), (24), and (21).

3. Linear Isometry and its Topological Properties

Let V, W be real normed spaces and L be a linear operator from V into W. We say that L is isometric-like if and only if

(Def. 6) there exist real numbers k_1, k_2 such that $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|L(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|L(x)\|$.

Now we state the proposition:

(27) Let us consider a finite dimensional real normed space V. Suppose $\dim(V) \neq 0$. Then there exists a linear operator L from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that L is one-to-one, onto, and isometric-like.
The theorem is a consequence of (26).

Let us consider real normed spaces V, W and a linear operator L from V into W. Now we state the propositions:

(28) Suppose L is one-to-one, onto, and isometric-like. Then there exists a linear operator K from W into V such that

(i) $K = L^{-1}$, and

(ii) K is one-to-one, onto, and isometric-like.

Proof: Consider K being a linear operator from W into V such that $K = L^{-1}$ and K is one-to-one and onto. Consider k_1, k_2 being real numbers such that $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|L(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|L(x)\|$. For every element y of W, $\|K(y)\| \leqslant k_2 \cdot \|y\|$ and $\|y\| \leqslant k_1 \cdot \|K(y)\|$. □

(29) If L is one-to-one, onto, and isometric-like, then L is Lipschitzian.

(30) If L is one-to-one, onto, and isometric-like, then L is continuous on the carrier of V.

(31) Let us consider real normed spaces S, T, a linear operator I from S into T, and a point x of S. If I is one-to-one, onto, and isometric-like, then I is continuous in x.
The theorem is a consequence of (29).

(32) Let us consider real normed spaces S, T, a linear operator I from S into T, and a subset Z of S. If I is one-to-one, onto, and isometric-like, then I is continuous on Z.
The theorem is a consequence of (31).

Let us consider real normed spaces S, T, a linear operator I from S into T, and a sequence s_1 of S. Now we state the propositions:

(33) Suppose I is one-to-one, onto, and isometric-like and s_1 is convergent. Then

(i) $I \cdot s_1$ is convergent, and

(ii) $\lim I \cdot s_1 = I(\lim s_1)$.

The theorem is a consequence of (31).

(34) If I is one-to-one, onto, and isometric-like, then s_1 is convergent iff $I \cdot s_1$ is convergent. The theorem is a consequence of (28) and (33).

Let us consider real normed spaces S, T, a linear operator I from S into T, and a subset Z of S. Now we state the propositions:

(35) If I is one-to-one, onto, and isometric-like, then Z is closed iff $I^\circ Z$ is closed.

Proof: Consider J being a linear operator from T into S such that $J = I^{-1}$ and J is one-to-one, onto, and isometric-like. Z is closed iff $I^\circ Z$ is closed. □

(36) If I is one-to-one, onto, and isometric-like, then Z is open iff $I^\circ Z$ is open. The theorem is a consequence of (28) and (35).

(37) If I is one-to-one, onto, and isometric-like, then Z is compact iff $I^\circ Z$ is compact.

Proof: Consider J being a linear operator from T into S such that $J = I^{-1}$ and J is one-to-one, onto, and isometric-like. If $I^\circ Z$ is compact, then Z is compact. □

(38) Let us consider a finite dimensional real normed space V, and a subset X of V. Suppose $\dim(V) \neq 0$. Then X is compact if and only if X is closed and there exists a real number r such that for every point y of V such that $y \in X$ holds $\|y\| < r$. The theorem is a consequence of (6), (27), (35), and (37).

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Relationship between the Riemann and Lebesgue Integrals

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. The goal of this article is to clarify the relationship between Riemann and Lebesgue integrals. In previous article [5], we constructed a one-dimensional Lebesgue measure. The one-dimensional Lebesgue measure provides a measure of any intervals, which can be used to prove the well-known relationship [6] between the Riemann and Lebesgue integrals [1]. We also proved the relationship between the integral of a given measure and that of its complete measure. As the result of this work, the Lebesgue integral of a bounded real valued function in the Mizar system [2], [3] can be calculated by the Riemann integral.

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1. Preliminaries

Let us consider a non empty set X and a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(1) (i) $\operatorname{rng}\max_{+}(f) \subseteq \operatorname{rng} f \cup \{0\}$, and

(ii) $\operatorname{rng}\max_{-}(f) \subseteq \operatorname{rng}(-f) \cup \{0\}$.

(2) If f is real-valued, then $-f$ is real-valued and $\max_{+}(f)$ is real-valued and $\max_{-}(f)$ is real-valued. The theorem is a consequence of (1).

(3) If f is without $-\infty$ and without $+\infty$, then f is a partial function from X to $\mathbb{R}$.

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(4) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then

(i) $\max_+(f)$ is simple function in S, and

(ii) $\max_-(f)$ is simple function in S.

PROOF: Consider F being a finite sequence of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $(\max_+(f))(x) = (\max_+(f))(y)$. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $(\max_-(f))(x) = (\max_-(f))(y)$. □

Let us consider real numbers a, b. Now we state the propositions:

(5) Suppose $a \leqslant b$. Then

(i) (B-Meas)($[a, b]$) $= b - a$, and

(ii) (B-Meas)($[a, b[$) $= b - a$, and

(iii) (B-Meas)($]a, b]$) $= b - a$, and

(iv) (B-Meas)($]a, b[$) $= b - a$, and

(v) (L-Meas)($[a, b]$) $= b - a$, and

(vi) (L-Meas)($[a, b[$) $= b - a$, and

(vii) (L-Meas)($]a, b]$) $= b - a$, and

(viii) (L-Meas)($]a, b[$) $= b - a$.

(6) Suppose $a > b$. Then

(i) (B-Meas)($[a, b]$) $= 0$, and

(ii) (B-Meas)($[a, b[$) $= 0$, and

(iii) (B-Meas)($]a, b]$) $= 0$, and

(iv) (B-Meas)($]a, b[$) $= 0$, and

(v) (L-Meas)($[a, b]$) $= 0$, and

(vi) (L-Meas)($[a, b[$) $= 0$, and

(vii) (L-Meas)($]a, b]$) $= 0$, and

(viii) (L-Meas)($]a, b[$) $= 0$.

(7) Let us consider an element A_1 of the Borel sets, an element A_2 of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $A_1 = A_2$ and f is A_1-measurable, then f is A_2-measurable.

(8) Let us consider real numbers a, b, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $a < b$ and $A = [a, b]$. Let us consider a natural number n. If $n > 0$, then there exists a partition D of A such that D divides into equal n.

Let F be a finite sequence of elements of the Borel sets and n be a natural number. One can check that the functor $F(n)$ yields an extended real-membered set. Now we state the proposition:

(9) Let us consider real numbers a, b, a non empty, closed interval subset A of $\mathbb{R}$, and a partition D of A. Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets such that

(i) $\operatorname{dom} D = \operatorname{dom} F$, and

(ii) $\bigcup \operatorname{rng} F = A$, and

(iii) for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ if $\operatorname{len} D = 1$, then $\$_2 = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $\$_1 = 1$, then $\$_2 = [a, D(\$_1)[$ and if $1 < \$_1 < \operatorname{len} D$, then $\$_2 = [D(\$_1 -' 1), D(\$_1)[$ and if $\$_1 = \operatorname{len} D$, then $\$_2 = [D(\$_1 -' 1), D(\$_1)]$. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} D$ there exists an element x of the Borel sets such that $\mathcal{P}[k, x]$ by [4, (5)]. Consider F being a finite sequence of elements of the Borel sets such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} D$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} D$ holds $\mathcal{P}[k, F(k)]$. For every objects x, y such that $x \neq y$ holds $F(x)$ misses $F(y)$. For every natural number k such that $k \in \operatorname{dom} F$ and $k \neq \operatorname{len} D$ holds $\bigcup \operatorname{rng}(F \restriction k) = [a, D(k)[$. $\bigcup \operatorname{rng} F = A$. $\square$

Let us consider real numbers a, b, a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, and a partial function f from A to $\mathbb{R}$. Now we state the propositions:

(10) Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$.

Proof: Consider F being a finite sequence of separated subsets of the Borel sets such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $\$_1 \in F(k)$ and $\$_2 = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Consider g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that for every object x, $x \in \operatorname{dom} g$ iff $x \in \mathbb{R}$ and there exists an object y such that $\mathcal{P}[x, y]$ and for every object x such that $x \in \operatorname{dom} g$ holds $\mathcal{P}[x, g(x)]$. For every natural number k and for every elements x, y of $\mathbb{R}$ such that $k \in \operatorname{dom} F$ and $x, y \in F(k)$ holds $g(x) = g(y)$. $\square$

(11) Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$.

Proof: Consider F being a finite sequence of separated subsets of the Borel sets such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $\$_1 \in F(k)$ and $\$_2 = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Consider g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that for every object x, $x \in \operatorname{dom} g$ iff $x \in \mathbb{R}$ and there exists an object y such that $\mathcal{P}[x, y]$ and for every object x such that $x \in \operatorname{dom} g$ holds $\mathcal{P}[x, g(x)]$. For every natural number k and for every elements x, y of $\mathbb{R}$ such that $k \in \operatorname{dom} F$ and $x, y \in F(k)$ holds $g(x) = g(y)$. $\square$

(12) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, a finite sequence F of separated subsets of S, a finite sequence a of elements of $\overline{\mathbb{R}}$, and a natural number n. Suppose f is simple function in S and F and a are representation of f and $n \in \operatorname{dom} F$. Then

(i) $F(n) = \emptyset$, or

(ii) $a(n)$ is a real number.

Let A be a non empty, closed interval subset of $\mathbb{R}$ and n be a natural number. Assume $n > 0$ and $\mathrm{vol}(A) > 0$. The functor $\mathrm{EqDiv}(A, n)$ yielding a partition of A is defined by

(Def. 1) *it* divides into equal n.

Now we state the propositions:

(13) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a natural number n. If $\mathrm{vol}(A) > 0$ and $\mathrm{len}\,\mathrm{EqDiv}(A, 2^n) = 1$, then $n = 0$.

(14) Let us consider real numbers a, b, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $a < b$ and $A = [a, b]$. Then there exists a division sequence D of A such that for every natural number n, $D(n)$ divides into equal 2^n.
PROOF: Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ there exists a partition D of A such that $D = \$_2$ and D divides into equal $2^{\$_1}$. For every element n of $\mathbb{N}$, there exists an element D of $\mathrm{divs}\,A$ such that $\mathcal{P}[n, D]$. Consider D being a function from $\mathbb{N}$ into $\mathrm{divs}\,A$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, D(n)]$. For every natural number n, $D(n)$ divides into equal 2^n. $\square$

(15) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, and natural numbers n, k. Suppose D divides into equal n and $k \in \mathrm{dom}\, D$. Then $\mathrm{vol}(\mathrm{divset}(D, k)) = \frac{\mathrm{vol}(A)}{n}$.

(16) Let us consider a complex number x, and a natural number r. If $x \neq 0$, then $(x^r)^{-1} = (x^{-1})^r$.

(17) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a sequence T of $\mathrm{divs}\,A$. Suppose $\mathrm{vol}(A) > 0$ and for every natural number n, $T(n) = \mathrm{EqDiv}(A, 2^n)$. Then δ_T is 0-convergent and non-zero.
PROOF: For every natural number n, $(\delta_T)(n) = 2 \cdot (\mathrm{vol}(A)) \cdot ((2^{-1})^{n+1})$. Define $\mathcal{S}(\text{natural number}) = (2^{-1})^{\$_1+1}$. Consider s being a sequence of real numbers such that for every natural number n, $s(n) = \mathcal{S}(n)$. For every natural number n, $(\delta_T)(n) = 2 \cdot (\mathrm{vol}(A)) \cdot s(n)$. $\square$

(18) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, a partial function f from X to $\overline{\mathbb{R}}$, a finite sequence F of separated subsets of S, and finite sequences a, x of elements of $\overline{\mathbb{R}}$. Suppose f is simple function in S and $E = \mathrm{dom}\, f$ and $M(E) < +\infty$ and F and a are representation of f and $\mathrm{dom}\, x = \mathrm{dom}\, F$ and for every natural number i such that $i \in \mathrm{dom}\, x$ holds $x(i) = a(i) \cdot (M \cdot F)(i)$. Then $\int f \,\mathrm{d}M = \sum x$.
PROOF: $\max_+(f)$ is simple function in S and $\max_-(f)$ is simple function in S. Define $\mathcal{P}[\text{natural number}, \text{extended real}] \equiv$ for every object x such that $x \in F(\$_1)$ holds $\$_2 = \max(f(x), 0)$. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, a$ there exists an element y of $\overline{\mathbb{R}}$ such that $\mathcal{P}[k, y]$. Consider

a_1 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_1 = \operatorname{Seg} \operatorname{len} a$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} a$ holds $\mathcal{P}[k, a_1(k)]$. For every natural number k such that $k \in \operatorname{dom} F$ for every object x such that $x \in F(k)$ holds $(\max_+(f))(x) = a_1(k)$. Define $\mathcal{Q}[\text{natural number, extended real}] \equiv \$_2 = a_1(\$_1) \cdot (M \cdot F)(\$_1)$. Consider x_1 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} x_1 = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{Q}[k, x_1(k)]$. Reconsider $r_1 = x_1$ as a finite sequence of elements of $\mathbb{R}$. $\int' \max_+(f) \, \mathrm{d}M = \sum x_1$.

Define $\mathcal{P}[\text{natural number, extended real}] \equiv$ for every object x such that $x \in F(\$_1)$ holds $\$_2 = \max(-f(x), 0)$. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} a$ there exists an element y of $\overline{\mathbb{R}}$ such that $\mathcal{P}[k, y]$. Consider a_2 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_2 = \operatorname{Seg} \operatorname{len} a$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} a$ holds $\mathcal{P}[k, a_2(k)]$. For every natural number k such that $k \in \operatorname{dom} F$ for every object x such that $x \in F(k)$ holds $(\max_-(f))(x) = a_2(k)$. Define $\mathcal{Q}[\text{natural number, extended real}] \equiv \$_2 = a_2(\$_1) \cdot (M \cdot F)(\$_1)$. Consider x_2 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} x_2 = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{Q}[k, x_2(k)]$. Reconsider $r_2 = x_2$ as a finite sequence of elements of $\mathbb{R}$. $\int' \max_-(f) \, \mathrm{d}M = \sum x_2$. For every object k such that $k \in \operatorname{dom} x$ holds $x(k) = (r_1 - r_2)(k)$. $\square$

Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partial function f from A to $\mathbb{R}$, and a partition D of A. Now we state the propositions:

(19) Suppose f is bounded and $A \subseteq \operatorname{dom} f$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [\inf A, \sup A]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [\inf A, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$ and $\operatorname{dom} g = A$ and $\int g \, \mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, D)$ and for every real number x such that $x \in A$ holds $\inf \operatorname{rng} f \leqslant g(x) \leqslant f(x)$.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Consider F being a finite sequence of separated subsets of the Borel sets, g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Define $\mathcal{H}[$natural number, extended real$] \equiv \$_2 = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, \$_1))$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{H}[k, r]$.

Consider h being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} h = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{H}[k, h(k)]$. For every natural number k such that $k \in \operatorname{dom} F$ for every object x such that $x \in F(k)$ holds $g(x) = h(k)$. Define $\mathcal{Z}[$natural number, extended real$] \equiv \$_2 = h(\$_1) \cdot ((\text{B-Meas}) \cdot F)(\$_1)$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{Z}[k, r]$. Consider z being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} z = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{Z}[k, z(k)]$. $\int g \, \mathrm{d}\, \text{B-Meas} = \sum z$. For every object p such that $p \in \operatorname{dom} z$ holds $z(p) = (\text{lower\_volume}(f, D))(p)$. For every real number x such that $x \in A$ holds $\inf \operatorname{rng} f \leqslant g(x) \leqslant f(x)$. $\square$

(20) Suppose f is bounded and $A \subseteq \operatorname{dom} f$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [\inf A, \sup A]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [\inf A, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$ and $\operatorname{dom} g = A$ and $\int g \, \mathrm{d}\, \text{B-Meas} = \text{upper\_sum}(f, D)$ and for every real number x such that $x \in A$ holds $\sup \operatorname{rng} f \geqslant g(x) \geqslant f(x)$.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Consider F being a finite sequence of separated subsets of the Borel sets, g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Define $\mathcal{H}[$natural number, extended real$] \equiv \$_2 = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, \$_1))$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{H}[k, r]$.

Consider h being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} h = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{H}[k, h(k)]$. For every natural number k such that $k \in \operatorname{dom} F$ for every object x such that $x \in F(k)$ holds $g(x) = h(k)$. Define $\mathcal{Z}[\text{natural number}, \text{extended real}] \equiv \$_2 = h(\$_1) \cdot (\text{B-Meas} \cdot F)(\$_1)$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{Z}[k, r]$. Consider z being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} z = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{Z}[k, z(k)]$. $\int g \, \mathrm{d}\,\text{B-Meas} = \sum z$. For every object p such that $p \in \operatorname{dom} z$ holds $z(p) = \text{upper\_volume}(f, D)(p)$. For every real number x such that $x \in A$ holds $\sup \operatorname{rng} f \geqslant g(x) \geqslant f(x)$. $\square$

Let us consider a non empty, closed interval subset A of $\mathbb{R}$ and a partial function f from A to $\mathbb{R}$. Now we state the propositions:

(21) Suppose f is bounded and $A \subseteq \operatorname{dom} f$ and $\operatorname{vol}(A) > 0$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$ with the same dom and there exists a sequence I of extended reals such that $A = \operatorname{dom}(F(0))$ and for every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n) \, \mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \text{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\inf \operatorname{rng} f \leqslant F(n)(x) \leqslant f(x)$ and for every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \leqslant F(m)(x)$ and for every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \sup(F\#x)$ and $\sup(F\#x) \leqslant f(x)$ and $\lim F$ is integrable on B-Meas and for every natural number n, $I(n) = \int F(n) \, \mathrm{d}\,\text{B-Meas}$ and I is convergent and $\lim I = \int \lim F \, \mathrm{d}\,\text{B-Meas}$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{partial function from } \mathbb{R} \text{ to } \overline{\mathbb{R}}] \equiv A = \operatorname{dom} \$_2$ and $\$_2$ is simple function in the Borel sets and $\int \$_2 \, \mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \text{EqDiv}(A, 2^{\$_1}))$ and for every real number x such that $x \in A$ holds $\inf \operatorname{rng} f \leqslant \$_2(x) \leqslant f(x)$ and there exists a finite sequence K of separated subsets of the Borel sets such that $\operatorname{dom} K = \operatorname{dom}(\text{EqDiv}(A, 2^{\$_1}))$ and $\bigcup \operatorname{rng} K = A$.

For every natural number k such that $k \in \operatorname{dom} K$ holds if $\operatorname{len} \text{EqDiv}(A, 2^{\$_1}) = 1$, then $K(k) = [\inf A, \sup A]$ and if $\operatorname{len} \text{EqDiv}(A, 2^{\$_1}) \neq 1$, then if $k = 1$, then $K(k) = [\inf A, (\text{EqDiv}(A, 2^{\$_1}))(k)[$ and if $1 < k < \operatorname{len} \text{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\text{EqDiv}(A, 2^{\$_1}))(k -' 1), (\text{EqDiv}(A, 2^{\$_1}))(k)[$ and if $k = \operatorname{len} \text{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\text{EqDiv}(A, 2^{\$_1}))(k -' 1), (\text{EqDiv}(A, 2^{\$_1}))(k)]$ and for every real number x such that $x \in \operatorname{dom} \$_2$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} K$ and $x \in K(k)$ and $\$_2(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(\text{EqDiv}(A, 2^{\$_1}), k))$. For every element n of $\mathbb{N}$, there exists an element g of $\mathbb{R} \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, g]$.

Consider F being a function from $\mathbb{N}$ into $\mathbb{R} \dot{\to} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every natural numbers n, m, $\mathrm{dom}(F(n)) = \mathrm{dom}(F(m))$. For every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\inf \mathrm{rng}\, f \leqslant F(n)(x) \leqslant f(x)$. For every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \leqslant F(m)(x)$. For every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \sup(F\#x)$ and $\sup(F\#x) \leqslant f(x)$. Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Reconsider $K = \max(|\inf \mathrm{rng}\, f|, |\sup \mathrm{rng}\, f|)$ as a real number. For every natural number n and for every set x such that $x \in \mathrm{dom}(F(0))$ holds $|F(n)(x)| \leqslant K$. $\square$

(22) Suppose f is bounded and $A \subseteq \mathrm{dom}\, f$ and $\mathrm{vol}(A) > 0$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$ with the same dom and there exists a sequence I of extended reals such that $A = \mathrm{dom}(F(0))$ and for every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant F(n)(x) \geqslant f(x)$ and for every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \geqslant F(m)(x)$ and for every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \inf(F\#x)$ and $\inf(F\#x) \geqslant f(x)$ and $\lim F$ is integrable on B-Meas and for every natural number n, $I(n) = \int F(n)\,\mathrm{d}\,\text{B-Meas}$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}\,\text{B-Meas}$.

Proof: Define $\mathcal{P}[\text{natural number}, \text{partial function from } \mathbb{R} \text{ to } \overline{\mathbb{R}}] \equiv A = \mathrm{dom}\, \$_2$ and $\$_2$ is simple function in the Borel sets and $\int \$_2\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \mathrm{EqDiv}(A, 2^{\$_1}))$ and for every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant \$_2(x) \geqslant f(x)$ and there exists a finite sequence K of separated subsets of the Borel sets such that $\mathrm{dom}\, K = \mathrm{dom}(\mathrm{EqDiv}(A, 2^{\$_1}))$ and $\bigcup \mathrm{rng}\, K = A$.

For every natural number k such that $k \in \mathrm{dom}\, K$ holds if $\mathrm{len}\, \mathrm{EqDiv}(A, 2^{\$_1}) = 1$, then $K(k) = [\inf A, \sup A]$ and if $\mathrm{len}\, \mathrm{EqDiv}(A, 2^{\$_1}) \neq 1$, then if $k = 1$, then $K(k) = [\inf A, (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $1 < k < \mathrm{len}\, \mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $k = \mathrm{len}\, \mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)]$ and for every real number x such that $x \in \mathrm{dom}\, \$_2$ there exists a natural number k such that $1 \leqslant k \leqslant \mathrm{len}\, K$ and $x \in K(k)$ and $\$_2(x) = \sup \mathrm{rng}(f \restriction \mathrm{divset}(\mathrm{EqDiv}(A, 2^{\$_1}), k))$.

For every element n of $\mathbb{N}$, there exists an element g of $\mathbb{R} \dot{\to} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, g]$. Consider F being a function from $\mathbb{N}$ into $\mathbb{R} \dot{\to} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every natural numbers n, m,

$\mathrm{dom}(F(n)) = \mathrm{dom}(F(m))$. For every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant F(n)(x) \geqslant f(x)$. For every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \geqslant F(m)(x)$. For every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \inf(F\#x)$ and $\inf(F\#x) \geqslant f(x)$ by [7, (7),(36)]. Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Set $K = \max(|\inf \mathrm{rng}\, f|, |\sup \mathrm{rng}\, f|)$. For every natural number n and for every set x such that $x \in \mathrm{dom}(F(0))$ holds $|F(n)(x)| \leqslant K$. □

2. Properties of Complete Measure Space

Now we state the propositions:

(23) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element E of S, and a natural number n. Suppose $E = \mathrm{dom}\, f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $M(E \cap \text{GTE-dom}(f, \frac{1}{n+1})) = 0$.

(24) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $E = \mathrm{dom}\, f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $M(E \cap \text{GT-dom}(f, 0)) = 0$.
PROOF: Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = E \cap \text{GTE-dom}(f, \frac{1}{\$_1+1})$. For every element n of $\mathbb{N}$, there exists an element y of S such that $\mathcal{P}[n, y]$. Consider F being a function from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every element n of $\mathbb{N}$, $(M \cdot F)(n) = 0$. □

(25) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and an element E of S. Suppose $E = \mathrm{dom}\, f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $f =^{M}_{\text{a.e.}} (X \longmapsto 0){\restriction}E$. The theorem is a consequence of (24).

(26) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\mathbb{R}$, and an element E_1 of S. Suppose M is complete and f is E_1-measurable and $f =^{M}_{\text{a.e.}} g$ and $E_1 = \mathrm{dom}\, f$. Then g is E_1-measurable.
PROOF: Consider E being an element of S such that $M(E) = 0$ and $f{\restriction}E^{\mathrm{c}} = g{\restriction}E^{\mathrm{c}}$. For every real number r, $E_1 \cap \text{LE-dom}(\overline{\mathbb{R}}(g), r) \in S$. □

(27) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then every element of S is an element of $\mathrm{COM}(S, M)$.

(28) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and partial functions f, g from X to $\mathbb{R}$. If $f =_{\text{a.e.}}^{M} g$, then $f =_{\text{a.e.}}^{\text{COM}(M)} g$. The theorem is a consequence of (27).

(29) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f =_{\text{a.e.}}^{\text{B-Meas}} g$. Then $f =_{\text{a.e.}}^{\text{L-Meas}} g$.

(30) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E_1 of S, an element E_2 of $\text{COM}(S, M)$, and a partial function f from X to $\overline{\mathbb{R}}$. If $E_1 = E_2$ and f is E_1-measurable, then f is E_2-measurable. The theorem is a consequence of (27).

(31) Let us consider an element E_1 of the Borel sets, an element E_2 of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $E_1 = E_2$ and f is E_1-measurable, then f is E_2-measurable.

(32) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then every finite sequence of separated subsets of S is a finite sequence of separated subsets of $\text{COM}(S, M)$. The theorem is a consequence of (27).

(33) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. If f is simple function in S, then f is simple function in $\text{COM}(S, M)$. The theorem is a consequence of (32).

(34) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then $\emptyset$ is a set with measure zero w.r.t. M.

(35) Let us consider a set X, a σ-field S of subsets of X, a σ-measure M on S, and an element E of S. Then $M(E) = \text{COM}(M)(E)$. The theorem is a consequence of (34).

(36) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int\limits_f M(x)dx = \int\limits_f \text{COM}(M)(x)dx$.

Proof: Consider F being a finite sequence of separated subsets of S, a, x being finite sequences of elements of $\overline{\mathbb{R}}$ such that F and a are representation of f and $a(1) = 0_{\overline{\mathbb{R}}}$ and for every natural number n such that $2 \leqslant n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n) < +\infty$ and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$ and $\int\limits_f M(x)dx = \sum x$. f is simple function in $\text{COM}(S, M)$. Reconsider $F_1 = F$ as a finite sequence of separated subsets of $\text{COM}(S, M)$. For every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (\text{COM}(M) \cdot$

$F_1)(n)$. □

(37) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $E = \operatorname{dom} f$ and f is E-measurable and non-negative. Then $\int^{+} f \, \mathrm{d}M = \int^{+} f \, \mathrm{dCOM}(M)$.

Proof: Consider F being a sequence of partial functions from X into $\overline{\mathbb{R}}$ such that for every natural number n, $F(n)$ is simple function in S and $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every natural number n, $F(n)$ is non-negative and for every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leqslant F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$. Reconsider $g = F(0)$ as a partial function from X to $\overline{\mathbb{R}}$. For every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leqslant \lim(F\#x)$.

Consider K being a sequence of extended reals such that for every natural number n, $K(n) = \int' F(n) \, \mathrm{d}M$ and K is convergent and $\sup \operatorname{rng} K = \lim K$ and $\int' g \, \mathrm{d}M \leqslant \lim K$. Reconsider $E_1 = E$ as an element of COM(S, M). f is E_1-measurable. For every natural number n, $F(n)$ is simple function in $\mathrm{COM}(S, M)$ and $\operatorname{dom}(F(n)) = \operatorname{dom} f$. For every natural number n, $K(n) = \int' F(n) \, \mathrm{dCOM}(M)$. □

(38) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then

(i) f is integrable on COM(M), and

(ii) $\int f \, \mathrm{d}M = \int f \, \mathrm{dCOM}(M)$.

The theorem is a consequence of (27), (37), and (30).

3. Relation Between Riemann and Lebesgue Integrals

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and partial functions f, g from X to $\mathbb{R}$. Now we state the propositions:

(39) If ($E = \operatorname{dom} f$ or $E = \operatorname{dom} g$) and $f =^{M}_{\text{a.e.}} g$, then $f - g =^{M}_{\text{a.e.}} (X \longmapsto 0) \restriction E$.

Proof: Consider A being an element of S such that $M(A) = 0$ and $f \restriction A^{\mathrm{c}} = g \restriction A^{\mathrm{c}}$. For every element x of X such that $x \in \operatorname{dom}((f - g) \restriction A^{\mathrm{c}})$ holds $((f - g) \restriction A^{\mathrm{c}})(x) = (((X \longmapsto 0) \restriction E) \restriction A^{\mathrm{c}})(x)$. □

(40) If $E = \operatorname{dom}(f - g)$ and $f - g =^{M}_{\text{a.e.}} (X \longmapsto 0) \restriction E$, then $f \restriction E =^{M}_{\text{a.e.}} g \restriction E$.

PROOF: Consider A being an element of S such that $M(A) = 0$ and $(f - g){\restriction}A^{\mathrm{c}} = ((X \longmapsto 0){\restriction}E){\restriction}A^{\mathrm{c}}$. For every element x of X such that $x \in \mathrm{dom}((f{\restriction}E){\restriction}A^{\mathrm{c}})$ holds $((f{\restriction}E){\restriction}A^{\mathrm{c}})(x) = ((g{\restriction}E){\restriction}A^{\mathrm{c}})(x)$. $\square$

(41) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a partial function f from X to $\mathbb{R}$. Suppose $E = \mathrm{dom}\, f$ and $M(E) < +\infty$ and f is bounded and E-measurable. Then f is integrable on M.

(42) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and partial functions f, g from X to $\mathbb{R}$. Then $f =^{M}_{\text{a.e.}} g$ if and only if $\max_+(f) =^{M}_{\text{a.e.}} \max_+(g)$ and $\max_-(f) =^{M}_{\text{a.e.}} \max_-(g)$.
PROOF: Consider E_1 being an element of S such that $M(E_1) = 0$ and $\max_+(f){\restriction}{E_1}^{\mathrm{c}} = \max_+(g){\restriction}{E_1}^{\mathrm{c}}$. Consider E_2 being an element of S such that $M(E_2) = 0$ and $\max_-(f){\restriction}{E_2}^{\mathrm{c}} = \max_-(g){\restriction}{E_2}^{\mathrm{c}}$. Set $E = E_1 \cup E_2$. For every element x of X such that $x \in \mathrm{dom}(f{\restriction}E^{\mathrm{c}})$ holds $(f{\restriction}E^{\mathrm{c}})(x) = (g{\restriction}E^{\mathrm{c}})(x)$. $\square$

(43) Let us consider a non empty set X, and a partial function f from X to $\mathbb{R}$. Then

(i) $\max_+(\overline{\mathbb{R}}(f)) = \overline{\mathbb{R}}(\max_+(f))$, and

(ii) $\max_-(\overline{\mathbb{R}}(f)) = \overline{\mathbb{R}}(\max_-(f))$.

(44) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\mathbb{R}$, and an element E of S. Suppose M is complete and f is integrable on M and $f =^{M}_{\text{a.e.}} g$ and $E = \mathrm{dom}\, f$ and $E = \mathrm{dom}\, g$. Then

(i) g is integrable on M, and

(ii) $\int f \,\mathrm{d}M = \int g \,\mathrm{d}M$.

The theorem is a consequence of (26), (43), and (42).

(45) Let us consider a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$, and a real number a. Suppose $a \in \mathrm{dom}\, f$. Then there exists an element A of the Borel sets such that

(i) $A = \{a\}$, and

(ii) f is A-measurable, and

(iii) $f{\restriction}A$ is integrable on B-Meas, and

(iv) $\int f{\restriction}A \,\mathrm{d}\,\text{B-Meas} = 0$.

(46) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $a \in \mathrm{dom}\, f$. Then there exists an element A of the Borel sets such that

(i) $A = \{a\}$, and

(ii) f is A-measurable, and

(iii) $f \restriction A$ is integrable on B-Meas, and

(iv) $\int f \restriction A \, \mathrm{d}\, \text{B-Meas} = 0$.

The theorem is a consequence of (45).

(47) Let us consider a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. Suppose f is integrable on B-Meas. Then

(i) f is integrable on L-Meas, and

(ii) $\int f \, \mathrm{d}\, \text{B-Meas} = \int f \, \mathrm{d}\, \text{L-Meas}$.

(48) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is integrable on B-Meas. Then

(i) f is integrable on L-Meas, and

(ii) $\int f \, \mathrm{d}\, \text{B-Meas} = \int f \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (38).

(49) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, an element A_1 of L-Field, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A = A_1$ and $A \subseteq \operatorname{dom} f$ and $f \restriction A$ is bounded and f is integrable on A. Then

(i) f is A_1-measurable, and

(ii) $f \restriction A_1$ is integrable on L-Meas, and

(iii) $\operatorname{integral} f \restriction A = \int f \restriction A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (46), (30), (48), (21), (22), (17), (3), (25), (29), (40), (26), (41), (38), and (44).

(50) Let us consider real numbers a, b, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$. Then $\int\limits_a^b f(x)dx = \int f \restriction [a, b] \, \mathrm{d}\, \text{L-Meas}$. The theorem is a consequence of (49).

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Improper Integral. Part I

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. In this article, we deal with Riemann's improper integral [1], using the Mizar system [2], [3]. Improper integrals with finite values are discussed in [5] by Yamazaki et al., but in general, improper integrals do not assume that they are finite. Therefore, we have formalized general improper integrals that does not limit the integral value to a finite value. In addition, each theorem in [5] assumes that the domain of integrand includes a closed interval, but since the improper integral should be discusses based on the half-open interval, we also corrected it.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $a \leqslant b \leqslant c$ and $[a, c] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is bounded and $f{\restriction}[b, c]$ is bounded and f is integrable on $[a, b]$ and f is integrable on $[b, c]$. Then

(i) f is integrable on $[a, c]$, and

(ii) $\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx$.

Let us consider a sequence s of real numbers. Now we state the propositions:

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(2) If s is divergent to $+\infty$, then s is not divergent to $-\infty$ and s is not convergent.

(3) If s is divergent to $-\infty$, then s is not divergent to $+\infty$ and s is not convergent.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0. Now we state the propositions:

(4) Suppose f is left convergent in x_0 or left divergent to $+\infty$ in x_0 or left divergent to $-\infty$ in x_0. Then there exists a sequence s of real numbers such that

 (i) s is convergent, and

 (ii) $\lim s = x_0$, and

 (iii) $\operatorname{rng} s \subseteq \operatorname{dom} f \cap]-\infty, x_0[$.

 PROOF: Define $\mathcal{F}[\text{natural number}, \text{real number}] \equiv x_0 - \frac{1}{\$_1+1} < \$_2 < x_0$ and $\$_2 \in \operatorname{dom} f$. For every element n of $\mathbb{N}$, there exists an element r of $\mathbb{R}$ such that $\mathcal{F}[n, r]$. Consider s being a sequence of real numbers such that for every element n of $\mathbb{N}$, $\mathcal{F}[n, s(n)]$. For every natural number n, $x_0 - \frac{1}{n+1} < s(n) < x_0$ and $s(n) \in \operatorname{dom} f$. $\square$

(5) Suppose f is right convergent in x_0 or right divergent to $+\infty$ in x_0 or right divergent to $-\infty$ in x_0. Then there exists a sequence s of real numbers such that

 (i) s is convergent, and

 (ii) $\lim s = x_0$, and

 (iii) $\operatorname{rng} s \subseteq \operatorname{dom} f \cap]x_0, +\infty[$.

 PROOF: Define $\mathcal{F}[\text{natural number}, \text{real number}] \equiv x_0 < \$_2 < x_0 + \frac{1}{\$_1+1}$ and $\$_2 \in \operatorname{dom} f$. For every element n of $\mathbb{N}$, there exists an element r of $\mathbb{R}$ such that $\mathcal{F}[n, r]$. Consider s being a sequence of real numbers such that for every element n of $\mathbb{N}$, $\mathcal{F}[n, s(n)]$. For every natural number n, $x_0 < s(n) < x_0 + \frac{1}{n+1}$ and $s(n) \in \operatorname{dom} f$. $\square$

(6) If f is left divergent to $+\infty$ in x_0, then f is not left divergent to $-\infty$ in x_0 and f is not left convergent in x_0. The theorem is a consequence of (4) and (2).

(7) If f is left divergent to $-\infty$ in x_0, then f is not left divergent to $+\infty$ in x_0 and f is not left convergent in x_0. The theorem is a consequence of (4) and (3).

(8) If f is right divergent to $+\infty$ in x_0, then f is not right divergent to $-\infty$ in x_0 and f is not right convergent in x_0. The theorem is a consequence of (5) and (2).

(9) If f is right divergent to $-\infty$ in x_0, then f is not right divergent to $+\infty$ in x_0 and f is not right convergent in x_0. The theorem is a consequence of (5) and (3).

(10) Suppose f is right convergent in x_0. Then

(i) there exists a real number r such that $0 < r$ and $f{\restriction}]x_0, x_0 + r[$ is lower bounded, and

(ii) there exists a real number r such that $0 < r$ and $f{\restriction}]x_0, x_0 + r[$ is upper bounded.

Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f{\restriction}]x_0, x_0 + R[)$ holds $-1 + g < (f{\restriction}]x_0, x_0 + R[)(r_1)$. Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f{\restriction}]x_0, x_0 + R[)$ holds $(f{\restriction}]x_0, x_0 + R[)(r_1) < g + 1$. $\square$

(11) Suppose f is left convergent in x_0. Then

(i) there exists a real number r such that $0 < r$ and $f{\restriction}]x_0 - r, x_0[$ is lower bounded, and

(ii) there exists a real number r such that $0 < r$ and $f{\restriction}]x_0 - r, x_0[$ is upper bounded.

Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f{\restriction}]x_0 - R, x_0[)$ holds $-1 + g < (f{\restriction}]x_0 - R, x_0[)(r_1)$. Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f{\restriction}]x_0 - R, x_0[)$ holds $(f{\restriction}]x_0 - R, x_0[)(r_1) < g + 1$. $\square$

(12) Suppose f is right divergent to $+\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f{\restriction}]x_0, x_0 + r[$ is lower bounded.

PROOF: Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $1 < f(r_1)$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0, x_0 + R[)$ holds $1 < (f\restriction]x_0, x_0 + R[)(r_1)$. $\square$

(13) Suppose f is right divergent to $-\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f\restriction]x_0, x_0 + r[$ is upper bounded.

PROOF: Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0, x_0 + R[)$ holds $(f\restriction]x_0, x_0 + R[)(r_1) < 1$. $\square$

(14) Suppose f is left divergent to $+\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f\restriction]x_0 - r, x_0[$ is lower bounded.

PROOF: Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $1 < f(r_1)$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0 - R, x_0[)$ holds $1 < (f\restriction]x_0 - R, x_0[)(r_1)$. $\square$

(15) Suppose f is left divergent to $-\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f\restriction]x_0 - r, x_0[$ is upper bounded.

PROOF: Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0 - R, x_0[)$ holds $(f\restriction]x_0 - R, x_0[)(r_1) < 1$. $\square$

Let us consider partial functions f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0.

(16) Suppose f_1 is right divergent to $-\infty$ in x_0 and for every real number r such that $x_0 < r$ there exists a real number g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $0 < r$ and $f_2\restriction]x_0, x_0 + r[$ is upper bounded. Then $f_1 + f_2$ is right divergent to $-\infty$ in x_0.

(17) Suppose f_1 is left divergent to $-\infty$ in x_0 and for every real number r such that $r < x_0$ there exists a real number g such that $r < g < x_0$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $0 < r$ and

$f_2\restriction]x_0 - r, x_0[$ is upper bounded. Then $f_1 + f_2$ is left divergent to $-\infty$ in x_0.

2. Properties of Extended Riemann Integral

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(18) Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f\restriction[a, b]$ is bounded. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

Proof: Reconsider $A =]a, b]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}$(element of A) $= (\int\limits_{\$_1}^{b} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. Consider M_0 being a real number such that for every object x such that $x \in [a, b] \cap \operatorname{dom} f$ holds $|f(x)| \leqslant M_0$. Reconsider $M = M_0 + 1$ as a real number. For every real number x such that $x \in [a, b]$ holds $|f(x)| < M$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - \int\limits_{a}^{b} f(x)dx| < g_1$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom} I_1$. $\square$

(19) Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f\restriction[a, b]$ is bounded. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

Proof: Reconsider $A = [a, b[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}$(element of A) $= (\int\limits_{a}^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. Consider M_0 being a real number such that for every object x such that $x \in [a,b] \cap \operatorname{dom} f$ holds $|f(x)| \leqslant M_0$. Reconsider $M = M_0 + 1$ as a real number. For every real number x such that $x \in [a,b]$ holds $|f(x)| < M$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < b$ and for every real number r_1 such that $r < r_1 < b$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - \int\limits_a^b f(x)dx| < g_1$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. For every real number r such that $r < b$ there exists a real number g such that $r < g < b$ and $g \in \operatorname{dom} I_1$. $\square$

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(20) Suppose $a < b \leqslant c$ and $]a,c] \subseteq \operatorname{dom} f$ and $f{\restriction}[b,c]$ is bounded and f is integrable on $[b,c]$ and f is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, c, and

(ii) $(R^<)\int\limits_a^c f(x)dx = (R^<)\int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx.$

Proof: For every real number e such that $a < e \leqslant c$ holds f is integrable on $[e,c]$ and $f{\restriction}[e,c]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a,b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is right convergent in a. Reconsider $A =]a,c]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_{\$_1}^c f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^c f(x)dx$.

For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_b^c f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r

such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - ((R^{<})\int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx)| < g_1$. $\square$

(21) Suppose $a \leqslant b < c$ and $[a, c[\subseteq \operatorname{dom} f$ and $f\restriction[a, b]$ is bounded and f is integrable on $[a, b]$ and f is right extended Riemann integrable on b, c. Then

(i) f is right extended Riemann integrable on a, c, and

(ii) $(R^{>})\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + (R^{>})\int\limits_b^c f(x)dx.$

Proof: For every real number e such that $a \leqslant e < c$ holds f is integrable on $[a, e]$ and $f\restriction[a, e]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [b, c[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_b^x f(x)dx$ and I is left convergent in c. Reconsider $A = [a, c[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_a^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. For every real number r such that $r < c$ there exists a real number g such that $r < g < c$ and $g \in \operatorname{dom} I_1$.

Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_a^b f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - (\int\limits_a^b f(x)dx + (R^{>})\int\limits_b^c f(x)dx)| < g_1$. $\square$

(22) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Let us consider a real number d. Suppose $a < d \leqslant b$. Then

(i) f is left extended Riemann integrable on a, d, and

(ii) $(R^{<})\int\limits_{a}^{b} f(x)dx = (R^{<})\int\limits_{a}^{d} f(x)dx + \int\limits_{d}^{b} f(x)dx$.

The theorem is a consequence of (20).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and real numbers c, d. Now we state the propositions:

(23) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then suppose $a \leqslant c < d \leqslant b$. Then

(i) f is left extended Riemann integrable on c, d, and

(ii) if $a < c$, then $(R^{<})\int\limits_{c}^{d} f(x)dx = \int\limits_{c}^{d} f(x)dx$.

The theorem is a consequence of (22).

(24) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then if $a < c < d \leqslant b$, then f is right extended Riemann integrable on c, d and $(R^{>})\int\limits_{c}^{d} f(x)dx = \int\limits_{c}^{d} f(x)dx$. The theorem is a consequence of (19).

(25) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Let us consider a real number c. Suppose $a \leqslant c < b$. Then

(i) f is right extended Riemann integrable on c, b, and

(ii) $(R^{>})\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{c} f(x)dx + (R^{>})\int\limits_{c}^{b} f(x)dx$.

The theorem is a consequence of (21).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and real numbers c, d. Now we state the propositions:

(26) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then suppose $a \leqslant c < d \leqslant b$. Then

(i) f is right extended Riemann integrable on c, d, and

(ii) if $d < b$, then $(R^{>})\int\limits_{c}^{d} f(x)dx = \int\limits_{c}^{d} f(x)dx$.

The theorem is a consequence of (25).

(27) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then if $a \leqslant c < d < b$, then f is left extended Riemann integrable on c, d and $(R^{<})\int\limits_{c}^{d} f(x)dx = \int\limits_{c}^{d} f(x)dx$. The theorem is a consequence of (18).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(28) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and $]a, b] \subseteq \operatorname{dom} g$ and f is left extended Riemann integrable on a, b and g is left extended Riemann integrable on a, b. Then

(i) $f + g$ is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int\limits_{a}^{b}(f+g)(x)dx = (R^{<})\int\limits_{a}^{b} f(x)dx + (R^{<})\int\limits_{a}^{b} g(x)dx$.

Proof: Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_{x}^{b} g(x)dx$ and I_2 is right convergent in a and $(R^{<})\int\limits_{a}^{b} g(x)dx = \lim\limits_{a^+} I_2$. Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and I_1 is right convergent in a and $(R^{<})\int\limits_{a}^{b} f(x)dx = \lim\limits_{a^+} I_1$. Set $I_3 = I_1 + I_2$. $\operatorname{dom} I_3 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_{x}^{b}(f+g)(x)dx$. For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom}(I_1 + I_2)$. For every real number d such that $a < d \leqslant b$ holds $f + g$ is integrable on $[d, b]$ and $(f + g)\restriction[d, b]$ is bounded. □

(29) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and $[a, b[\subseteq \operatorname{dom} g$ and f is right extended Riemann integrable on a, b and g is right extended Riemann integrable on a, b. Then

(i) $f + g$ is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int\limits_{a}^{b}(f+g)(x)dx = (R^{>})\int\limits_{a}^{b} f(x)dx + (R^{>})\int\limits_{a}^{b} g(x)dx$.

Proof: Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that

$\operatorname{dom} I_2 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_a^x g(x)dx$ and I_2 is left convergent in b and $(R^{>})\int\limits_a^b g(x)dx = \lim\limits_{b^-} I_2$. Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and I_1 is left convergent in b and $(R^{>})\int\limits_a^b f(x)dx = \lim\limits_{b^-} I_1$. Set $I_3 = I_1 + I_2$. $\operatorname{dom} I_3 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_a^x (f+g)(x)dx$. For every real number r such that $r < b$ there exists a real number g such that $r < g < b$ and $g \in \operatorname{dom}(I_1 + I_2)$. For every real number d such that $a \leqslant d < b$ holds $f + g$ is integrable on $[a, d]$ and $(f + g)\restriction[a, d]$ is bounded. $\square$

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a real number r. Now we state the propositions:

(30) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then

(i) $r \cdot f$ is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int\limits_a^b (r \cdot f)(x)dx = r \cdot ((R^{<})\int\limits_a^b f(x)dx)$.

PROOF: For every real number r, $r \cdot f$ is left extended Riemann integrable on a, b and $(R^{<})\int\limits_a^b (r \cdot f)(x)dx = r \cdot ((R^{<})\int\limits_a^b f(x)dx)$. $\square$

(31) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then

(i) $r \cdot f$ is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int\limits_a^b (r \cdot f)(x)dx = r \cdot ((R^{>})\int\limits_a^b f(x)dx)$.

PROOF: For every real number r, $r \cdot f$ is right extended Riemann integrable on a, b and $(R^{>})\int\limits_a^b (r \cdot f)(x)dx = r \cdot ((R^{>})\int\limits_a^b f(x)dx)$. $\square$

3. Definition of Improper Integral

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. We say that f is left improper integrable on a and b if and only if

(Def. 1) for every real number d such that $a < d \leqslant b$ holds f is integrable on $[d, b]$ and $f \restriction [d, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and (I_1 is right convergent in a or right divergent to $+\infty$ in a or I_1 is right divergent to $-\infty$ in a).

We say that f is right improper integrable on a and b if and only if

(Def. 2) for every real number d such that $a \leqslant d < b$ holds f is integrable on $[a, d]$ and $f \restriction [a, d]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$ and (I_1 is left convergent in b or left divergent to $+\infty$ in b or I_1 is left divergent to $-\infty$ in b).

Assume f is left improper integrable on a and b. The functor left-improper-integral(f, a, b) yielding an extended real is defined by

(Def. 3) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and (I_1 is right convergent in a and $it = \lim_{a^+} I_1$ or I_1 is right divergent to $+\infty$ in a and $it = +\infty$ or I_1 is right divergent to $-\infty$ in a and $it = -\infty$).

Assume f is right improper integrable on a and b. The functor right-improper-integral(f, a, b) yielding an extended real is defined by

(Def. 4) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$ and (I_1 is left convergent in b and $it = \lim_{b^-} I_1$ or I_1 is left divergent to $+\infty$ in b and $it = +\infty$ or I_1 is left divergent to $-\infty$ in b and $it = -\infty$).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(32) If f is left extended Riemann integrable on a, b, then f is left improper integrable on a and b.

(33) If f is right extended Riemann integrable on a, b, then f is right improper integrable on a and b.

(34) Suppose f is left improper integrable on a and b. Then

(i) f is left extended Riemann integrable on a, b and left-improper-integral $(f, a, b) = (R^<)\int\limits_a^b f(x)dx$, or

(ii) f is not left extended Riemann integrable on a, b and left-improper-integral$(f, a, b) = +\infty$, or

(iii) f is not left extended Riemann integrable on a, b and left-improper-integral$(f, a, b) = -\infty$.

The theorem is a consequence of (8) and (9).

(35) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and I_1 is right divergent to $+\infty$ in a or right divergent to $-\infty$ in a. Then f is not left extended Riemann integrable on a, b. The theorem is a consequence of (8) and (9).

(36) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is left improper integrable on a and b and $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and I_1 is right convergent in a. Then left-improper-integral$(f, a, b) = \lim_{a^+} I_1$. The theorem is a consequence of (34).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(37) Suppose $a < b \leqslant c$ and $]a, c] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and c. Then

(i) f is left improper integrable on a and b, and

(ii) if left-improper-integral$(f, a, c) = (R^<)\int\limits_a^c f(x)dx$, then left-improper-integral$(f, a, b) = (R^<)\int\limits_a^b f(x)dx$, and

(iii) if left-improper-integral$(f, a, c) = +\infty$, then left-improper-integral $(f, a, b) = +\infty$, and

(iv) if left-improper-integral$(f, a, c) = -\infty$, then left-improper-integral $(f, a, b) = -\infty$.

The theorem is a consequence of (34).

(38) Suppose $a < b \leqslant c$ and $]a, c] \subseteq \operatorname{dom} f$ and $f \restriction [b, c]$ is bounded and f is left improper integrable on a and b and f is integrable on $[b, c]$. Then

  (i) f is left improper integrable on a and c, and

  (ii) if left-improper-integral$(f, a, b) = (R^{<})\int\limits_a^b f(x)dx$, then left-improper-integral$(f, a, c) =$ left-improper-integral$(f, a, b) + \int\limits_b^c f(x)dx$, and

  (iii) if left-improper-integral$(f, a, b) = +\infty$, then left-improper-integral $(f, a, c) = +\infty$, and

  (iv) if left-improper-integral$(f, a, b) = -\infty$, then left-improper-integral $(f, a, c) = -\infty$.

The theorem is a consequence of (34).

(39) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is right improper integrable on a and b. Then

  (i) f is right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = (R^{>})\int\limits_a^b f(x)dx$, or

  (ii) f is not right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = +\infty$, or

  (iii) f is not right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = -\infty$.

The theorem is a consequence of (6) and (7).

(40) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and I_1 is left divergent to $+\infty$ in b or left divergent to $-\infty$ in b. Then f is not right extended Riemann integrable on a, b. The theorem is a consequence of (6) and (7).

(41) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is right improper integrable on a and b and $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and

I_1 is left convergent in b. Then right-improper-integral$(f, a, b) = \lim_{b^-} I_1$. The theorem is a consequence of (39).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(42) Suppose $a \leqslant b < c$ and $[a, c[\, \subseteq \operatorname{dom} f$ and f is right improper integrable on a and c. Then

(i) f is right improper integrable on b and c, and

(ii) if right-improper-integral$(f, a, c) = (R^{>})\int\limits_{a}^{c} f(x)dx$, then right-improper-integral$(f, b, c) = (R^{>})\int\limits_{b}^{c} f(x)dx$, and

(iii) if right-improper-integral$(f, a, c) = +\infty$, then right-improper-integral$(f, b, c) = +\infty$, and

(iv) if right-improper-integral$(f, a, c) = -\infty$, then right-improper-integral$(f, b, c) = -\infty$.

The theorem is a consequence of (39).

(43) Suppose $a \leqslant b < c$ and $[a, c[\, \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is bounded and f is right improper integrable on b and c and f is integrable on $[a, b]$. Then

(i) f is right improper integrable on a and c, and

(ii) if right-improper-integral$(f, b, c) = (R^{>})\int\limits_{b}^{c} f(x)dx$, then right-improper-integral$(f, a, c) =$ right-improper-integral$(f, b, c) + \int\limits_{a}^{b} f(x)dx$, and

(iii) if right-improper-integral$(f, b, c) = +\infty$, then right-improper-integral$(f, a, c) = +\infty$, and

(iv) if right-improper-integral$(f, b, c) = -\infty$, then right-improper-integral$(f, a, c) = -\infty$.

The theorem is a consequence of (39).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, c be real numbers. We say that f is improper integrable on a and c if and only if

(Def. 5) there exists a real number b such that $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$.

Now we state the propositions:

(44) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose f is improper integrable on a and c. Then there exists a real number b such that

(i) $a < b < c$, and

(ii) left-improper-integral$(f, a, b) = (R^{<})\int\limits_{a}^{b} f(x)dx$ and right-improper-integral$(f, b, c) = (R^{>})\int\limits_{b}^{c} f(x)dx$ or left-improper-integral(f, a, b) + right-improper-integral$(f, b, c) = +\infty$ or left-improper-integral (f, a, b) + right-improper-integral$(f, b, c) = -\infty$.

The theorem is a consequence of (34) and (39).

(45) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $]a, c[\subseteq \operatorname{dom} f$ and $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$. Let us consider a real number b_1. Suppose $a < b_1 \leqslant b$. Then left-improper-integral(f, a, b) + right-improper-integral(f, b, c) = left-improper-integral(f, a, b_1) + right-improper-integral (f, b_1, c). The theorem is a consequence of (34) and (39).

(46) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $]a, c[\subseteq \operatorname{dom} f$ and $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$. Let us consider a real number b_2. Suppose $b \leqslant b_2 < c$. Then left-improper-integral(f, a, b) + right-improper-integral(f, b, c) = left-improper-integral(f, a, b_2) + right-improper-integral (f, b_2, c). The theorem is a consequence of (39) and (34).

(47) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose $]a, c[\subseteq \operatorname{dom} f$ and f is improper integrable on a and c. Let us consider real numbers b_1, b_2. Suppose $a < b_1 < c$ and $a < b_2 < c$. Then left-improper-integral(f, a, b_1) + right-improper-integral(f, b_1, c) = left-improper-integral(f, a, b_2) + right-improper-integral(f, b_2, c). The theorem is a consequence of (45) and (46).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Assume $]a, b[\subseteq \operatorname{dom} f$ and f is improper integrable on a and b. The functor improper-integral(f, a, b) yielding an extended real is defined by

(Def. 6) there exists a real number c such that $a < c < b$ and f is left improper integrable on a and c and f is right improper integrable on c and b and $it =$ left-improper-integral(f, a, c) + right-improper-integral(f, c, b).

Now we state the proposition:

(48) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose $]a, c[\subseteq \operatorname{dom} f$ and f is improper integrable on a and c. Let us consider a real number b. Suppose $a < b < c$. Then

(i) f is left improper integrable on a and b, and

(ii) f is right improper integrable on b and c, and

(iii) improper-integral(f, a, c) = left-improper-integral(f, a, b) + right-improper-integral(f, b, c).

The theorem is a consequence of (37), (43), (47), (38), and (42).

4. Linearity of Improper Integral

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(49) Suppose f is left improper integrable on a and b and left-improper-integral $(f, a, b) = +\infty$. Then suppose $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$. Then I_1 is right divergent to $+\infty$ in a.

(50) Suppose f is left improper integrable on a and b and left-improper-integral $(f, a, b) = -\infty$. Then suppose $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$. Then I_1 is right divergent to $-\infty$ in a.

(51) Suppose f is right improper integrable on a and b and right-improper-integral$(f, a, b) = +\infty$. Then suppose $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. Then I_1 is left divergent to $+\infty$ in b.

(52) Suppose f is right improper integrable on a and b and right-improper-integral$(f,a,b) = -\infty$. Then suppose $\operatorname{dom} I_1 = [a,b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. Then I_1 is left divergent to $-\infty$ in b.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, r.

(53) Suppose $a < b$ and $]a,b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b. Then

(i) $r \cdot f$ is left improper integrable on a and b, and

(ii) left-improper-integral$(r \cdot f, a, b) = r \cdot$ left-improper-integral(f,a,b).

Proof: For every real number d such that $a < d \leqslant b$ holds $r \cdot f$ is integrable on $[d,b]$ and $(r \cdot f){\restriction}[d,b]$ is bounded. $\square$

(54) Suppose $a < b$ and $[a,b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b. Then

(i) $r \cdot f$ is right improper integrable on a and b, and

(ii) right-improper-integral$(r \cdot f, a, b) = r \cdot$right-improper-integral$(f,a,b)$.

Proof: For every real number d such that $a \leqslant d < b$ holds $r \cdot f$ is integrable on $[a,d]$ and $(r \cdot f){\restriction}[a,d]$ is bounded. $\square$

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(55) Suppose $a < b$ and $]a,b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b. Then

(i) $-f$ is left improper integrable on a and b, and

(ii) left-improper-integral$(-f, a, b) = -$left-improper-integral(f,a,b).

The theorem is a consequence of (53).

(56) Suppose $a < b$ and $[a,b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b. Then

(i) $-f$ is right improper integrable on a and b, and

(ii) right-improper-integral$(-f, a, b) = -$right-improper-integral(f,a,b).

The theorem is a consequence of (54).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(57) Suppose $a < b$ and $]a,b] \subseteq \operatorname{dom} f$ and $]a,b] \subseteq \operatorname{dom} g$ and f is left improper integrable on a and b and g is left improper integrable on a and b and it is not true that left-improper-integral$(f,a,b) = +\infty$ and left-improper-integral$(g,a,b) = -\infty$ and it is not true that left-improper-integral$(f,a,b) = -\infty$ and left-improper-integral$(g,a,b) = +\infty$. Then

(i) $f+g$ is left improper integrable on a and b, and

(ii) left-improper-integral$(f+g,a,b)$ = left-improper-integral(f,a,b) + left-improper-integral(g,a,b).

Proof: For every real number d such that $a < d \leqslant b$ holds $f+g$ is integrable on $[d,b]$ and $(f+g)\restriction[d,b]$ is bounded. □

(58) Suppose $a < b$ and $[a,b[\ \subseteq \operatorname{dom} f$ and $[a,b[\ \subseteq \operatorname{dom} g$ and f is right improper integrable on a and b and g is right improper integrable on a and b and it is not true that right-improper-integral$(f,a,b) = +\infty$ and right-improper-integral$(g,a,b) = -\infty$ and it is not true that right-improper-integral$(f,a,b) = -\infty$ and right-improper-integral$(g,a,b) = +\infty$. Then

(i) $f+g$ is right improper integrable on a and b, and

(ii) right-improper-integral$(f+g,a,b)$ = right-improper-integral(f,a,b)+ right-improper-integral(g,a,b).

Proof: For every real number d such that $a \leqslant d < b$ holds $f+g$ is integrable on $[a,d]$ and $(f+g)\restriction[a,d]$ is bounded by [4, (11)]. □

(59) Suppose $a < b$ and $]a,b] \subseteq \operatorname{dom} f$ and $]a,b] \subseteq \operatorname{dom} g$ and f is left improper integrable on a and b and g is left improper integrable on a and b and it is not true that left-improper-integral$(f,a,b) = +\infty$ and left-improper-integral$(g,a,b) = +\infty$ and it is not true that left-improper-integral$(f,a,b) = -\infty$ and left-improper-integral$(g,a,b) = -\infty$. Then

(i) $f-g$ is left improper integrable on a and b, and

(ii) left-improper-integral$(f-g,a,b)$ = left-improper-integral(f,a,b) − left-improper-integral(g,a,b).

The theorem is a consequence of (55) and (57).

(60) Suppose $a < b$ and $[a,b[\ \subseteq \operatorname{dom} f$ and $[a,b[\ \subseteq \operatorname{dom} g$ and f is right improper integrable on a and b and g is right improper integrable on a and b and it is not true that right-improper-integral$(f,a,b) = +\infty$ and right-improper-integral$(g,a,b) = +\infty$ and it is not true that right-improper-integral$(f,a,b) = -\infty$ and right-improper-integral$(g,a,b) = -\infty$. Then

(i) $f-g$ is right improper integrable on a and b, and

(ii) right-improper-integral$(f-g,a,b)$ = right-improper-integral(f,a,b)− right-improper-integral(g,a,b).

The theorem is a consequence of (56) and (58).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, r.

(61) Suppose $]a, b[\subseteq \operatorname{dom} f$ and f is improper integrable on a and b. Then

(i) $r \cdot f$ is improper integrable on a and b, and

(ii) improper-integral$(r \cdot f, a, b) = r \cdot$ improper-integral(f, a, b).

The theorem is a consequence of (48), (53), and (54).

(62) Suppose $]a, b[\subseteq \operatorname{dom} f$ and f is improper integrable on a and b. Then

(i) $-f$ is improper integrable on a and b, and

(ii) improper-integral$(-f, a, b) = -$improper-integral(f, a, b).

The theorem is a consequence of (61).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(63) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $]a, b[\subseteq \operatorname{dom} g$ and f is improper integrable on a and b and g is improper integrable on a and b and it is not true that improper-integral$(f, a, b) = +\infty$ and improper-integral$(g, a, b) = -\infty$ and it is not true that improper-integral$(f, a, b) = -\infty$ and improper-integral$(g, a, b) = +\infty$. Then

(i) $f + g$ is improper integrable on a and b, and

(ii) improper-integral$(f+g, a, b) =$ improper-integral$(f, a, b) +$ improper-integral(g, a, b).

The theorem is a consequence of (37), (38), (43), (42), (48), (57), and (58).

(64) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $]a, b[\subseteq \operatorname{dom} g$ and f is improper integrable on a and b and g is improper integrable on a and b and it is not true that improper-integral$(f, a, b) = +\infty$ and improper-integral$(g, a, b) = +\infty$ and it is not true that improper-integral$(f, a, b) = -\infty$ and improper-integral$(g, a, b) = -\infty$. Then

(i) $f - g$ is improper integrable on a and b, and

(ii) improper-integral$(f-g, a, b) =$ improper-integral$(f, a, b) -$ improper-integral(g, a, b).

The theorem is a consequence of (62) and (63).

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Prime Representing Polynomial

Karol Pąk
Institute of Computer Science
University of Białystok
Poland

Summary. The main purpose of formalization is to prove that the set of prime numbers is diophantine, i.e., is representable by a polynomial formula. We formalize this problem, using the Mizar system [1], [2], in two independent ways, proving the existence of a polynomial without formulating it explicitly as well as with its indication.

First, we reuse nearly all the techniques invented to prove the MRDP-theorem [11]. Applying a trick with Mizar schemes that go beyond first-order logic we give a short sophisticated proof for the existence of such a polynomial but without formulating it explicitly. Then we formulate the polynomial proposed in [6] that has 26 variables in the Mizar language as follows

$(w\cdot z+h+j-q)^{\mathbf{2}}+((g\cdot k+g+k)\cdot(h+j)+h-z)^{\mathbf{2}}+(2\cdot k^3\cdot(2\cdot k+2)\cdot(n+1)^2+1-f^{\mathbf{2}})^{\mathbf{2}}+$
$(p+q+z+2\cdot n-e)^{\mathbf{2}}+(e^3\cdot(e+2)\cdot(a+1)^2+1-o^{\mathbf{2}})^{\mathbf{2}}+(x^{\mathbf{2}}-(a^{\mathbf{2}}-'1)\cdot y^{\mathbf{2}}-1)^{\mathbf{2}}+$
$(16\cdot(a^{\mathbf{2}}-1)\cdot r^{\mathbf{2}}\cdot y^{\mathbf{2}}\cdot y^{\mathbf{2}}+1-u^{\mathbf{2}})^{\mathbf{2}}+(((a+u^{\mathbf{2}}\cdot(u^{\mathbf{2}}-a))^{\mathbf{2}}-1)\cdot(n+4\cdot d\cdot y)^{\mathbf{2}}+$
$1-(x+c\cdot u)^{\mathbf{2}})^{\mathbf{2}}+$
$(m^{\mathbf{2}}-(a^{\mathbf{2}}-'1)\cdot l^{\mathbf{2}}-1)^{\mathbf{2}}+(k+i\cdot(a-1)-l)^{\mathbf{2}}+(n+l+v-y)^{\mathbf{2}}+$
$(p+l\cdot(a-n-1)+b\cdot(2\cdot a\cdot(n+1)-(n+1)^{\mathbf{2}}-1)-m)^{\mathbf{2}}+$
$(q+y\cdot(a-p-1)+s\cdot(2\cdot a\cdot(p+1)-(p+1)^{\mathbf{2}}-1)-x)^{\mathbf{2}}+(z+p\cdot l\cdot(a-p)+$
$t\cdot(2\cdot a\cdot p-p^{\mathbf{2}}-1)-p\cdot m)^{\mathbf{2}}$

and we prove that that for any positive integer k so that $k+1$ is prime it is necessary and sufficient that there exist other natural variables a-z for which the polynomial equals zero. 26 variables is not the best known result in relation to the set of prime numbers, since any diophantine equation over $\mathbb{N}$ can be reduced to one in 13 unknowns [8] or even less [5], [13]. The best currently known result for all prime numbers, where the polynomial is explicitly constructed is 10 [7] or even 7 in the case of Fermat as well as Mersenne prime number [4]. We are currently focusing our formalization efforts in this direction.

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1. The Prime Number Set as a Diophantine Set

From now on n denotes a natural number, i, j, i_1, i_2, i_3, i_4, i_5, i_6 denote elements of n, and p, q, r denote n-element finite 0-sequences of $\mathbb{N}$.

Now we state the propositions:

(1) $\{p : p(i) > 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(i) > 0 \cdot \$_1(i) + 1$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i) > 1$. $\{q : \mathcal{Q}[q]\} = \{r : \mathcal{R}[r]\}$. $\square$

(2) $\{p : p(i) = (p(j) -' 1)! + 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: For every n, i_1, and i_2, $\{p : p(i_1) = p(i_2) -' 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, and i_2, $\{p : p(i_1) = (p(i_2) -' 1)!\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$ by [10, (32)]. Define $\mathcal{P}$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv \$_4 = 1 \cdot \$_3 + 1$. Define $\mathcal{F}$(natural number, natural number, natural number) $= (\$_2 -' 1)!$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) = 1 \cdot ((\$_1(i_2) -' 1)!) + 1$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) = (\$_1(i_2) -' 1)! + 1$. $\{q : \mathcal{Q}[q]\} = \{r : \mathcal{R}[r]\}$. $\square$

(3) $\{p : (p(i) -' 1)! + 1 \bmod p(i) = 0$ and $p(i) > 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{P}$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv 1 \cdot \$_3 \equiv 0 \cdot \$_4 \pmod{1 \cdot \$_4}$. Define $\mathcal{F}$(natural number, natural number, natural number) $= (\$_2 -' 1)! + 1$. For every n, i_1, i_2, i_3, and i_4, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = p(i_4)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{Q}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot ((\$_1(i) -' 1)! + 1) \equiv 0 \cdot \$_1(i) \pmod{1 \cdot \$_1(i)}$.

Define $\mathcal{Q}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i) > 1$. Define $\mathcal{Q}_{12}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{Q}_1[\$_1]$ and $\mathcal{Q}_2[\$_1]$. $\{q : \mathcal{Q}_2[q]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. $\{q : \mathcal{Q}_1[q]$ and $\mathcal{Q}_2[q]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv (\$_1(i) -' 1)! + 1 \bmod \$_1(i) = 0$ and $\$_1(i) > 1$ by [12, (11)]. $\mathcal{Q}_{12}[q]$ iff $\mathcal{R}[q]$. $\{q : \mathcal{Q}_{12}[q]\} = \{r : \mathcal{R}[r]\}$. $\square$

(4) Let us consider a natural number n, and an element i of n. Then $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N}$: $p(i)$ is prime$\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i)$ is prime. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv (\$_1(i) -' 1)! + 1 \bmod \$_1(i) = 0$ and $\$_1(i) > 1$. $\{q : \mathcal{Q}[q]\} = \{r : \mathcal{R}[r]\}$. $\square$

2. Special Case of Pell's Equation - Selected Properties

In the sequel i, j, n, n_1, n_2, m, k, l, u, e, p, t denote natural numbers, a, b denote non trivial natural numbers, x, y denote integers, and r, q denote real numbers.

Now we state the propositions:

(5) If $2 \leqslant e$ and there exists i such that $e^{\mathbf{2}} \cdot e \cdot (e+2) \cdot (n+1)^{\mathbf{2}} + 1 = i^{\mathbf{2}}$, then $e - 1 + e^{e -' 2} \leqslant n$.

Proof: Set $a = e+1$. Set $n_1 = n+1$. Reconsider $e_2 = e-2$ as a natural number. Consider j such that $i = \mathtt{x}_a(j)$ and $e \cdot n_1 = \mathtt{y}_a(j)$. $(a-2) \cdot e + e^{e_2+1} < (2 \cdot a - 1)^{e_2+1}$ by [14, (103)]. □

(6) If $2 \leqslant e$ and $0 < t$, then there exists n and there exists i such that $t \mid n+1$ and $e^{\mathbf{2}} \cdot e \cdot (e+2) \cdot (n+1)^{\mathbf{2}} + 1 = i^{\mathbf{2}}$.

(7) If $n \geqslant k$, then $\binom{n}{k} \geqslant \frac{(n+1-k)^k}{k!}$.

Proof: Set $n_1 = n+1$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then $\binom{n}{\$_1} \geqslant \frac{(n_1 - \$_1)^{\$_1}}{\$_1!}$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$. $\mathcal{P}[i]$. □

(8) If $n \geqslant k$, then $\binom{n}{k} \leqslant \frac{n^k}{k!}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then $\binom{n}{\$_1} \leqslant \frac{n^{\$_1}}{\$_1!}$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$. $\mathcal{P}[i]$. □

(9) If $i \leqslant j$ and $2 \cdot j \leqslant n+1$, then $\binom{n}{i} \leqslant \binom{n}{j}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $i \leqslant \$_1$ and $2 \cdot \$_1 \leqslant n+1$, then $\binom{n}{i} \leqslant \binom{n}{\$_1}$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$. $\mathcal{P}[k]$. □

(10) If $k \leqslant n$, then $n! \leqslant k! \cdot (n^{n -' k})$.

Proof: Define $\mathcal{P}$[natural number] $\equiv (k + \$_1)! \leqslant k! \cdot (k+\$_1)^{\$_1}$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$. $\mathcal{P}[i]$. □

(11) Suppose $0 < k$ and $2 \cdot k^k \leqslant n$ and $n^k < p$. Then

(i) $(p+1)^n \bmod p^{k+1} > 0$, and

(ii) $k! < \frac{(n+1)^k \cdot (p^k)}{(p+1)^n \bmod p^{k+1}} < k! + 1$.

Proof: Set $k_1 = k+1$. Set $n_1 = n+1$. Reconsider $K = k-1$, $n_3 = n-k$ as a natural number. Set $P = \langle \binom{n}{0} 1^0 p^n, \ldots, \binom{n}{n} 1^n p^0 \rangle$. $\sum(P \restriction k_1) \equiv \sum P \pmod{p^{k_1}}$. $\sum(P \restriction k_1) \neq 0$. $\sum(P \restriction k_1) < p^{k_1}$. $\binom{n}{k} \leqslant \frac{n^k}{k!}$. $\sum(P \restriction k) \leqslant \frac{n^k}{k!} \cdot (p^K) \cdot k$. $\binom{n}{k} \geqslant \frac{(n_1 - k)^k}{k!}$. $k \cdot k \leqslant n$ and $2 \cdot k \cdot k \leqslant n_1$. $1 \cdot (2 \cdot k^k) \geqslant 2 \cdot k^{\mathbf{2}} \cdot (k!)$. □

(12) (i) $\mathtt{x}_a(n+2) = 2 \cdot a \cdot \mathtt{x}_a(n+1) - \mathtt{x}_a(n)$, and

(ii) $\mathtt{y}_a(n+2) = 2 \cdot a \cdot \mathtt{y}_a(n+1) - \mathtt{y}_a(n)$.

(13) $\mathtt{x}_a(n) \equiv p^n + \mathtt{y}_a(n) \cdot (a - p) \pmod{2 \cdot a \cdot p - p^{\mathbf{2}} - 1}$.
PROOF: Set $P = 2 \cdot a \cdot p - p^{\mathbf{2}} - 1$. Define $\mathcal{T}$[natural number] $\equiv \mathtt{x}_a(\$_1) - \mathtt{y}_a(\$_1) \cdot (a - p) \equiv p^{\$_1} \pmod{P}$. Define $\mathcal{P}$[natural number] $\equiv \mathcal{T}[\$_1]$ and $\mathcal{T}[\$_1 + 1]$. $\mathcal{P}[0]$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$. $\mathcal{P}[k]$. □

(14) If $0 < p^n < a$, then $p^n + \mathtt{y}_a(n) \cdot (a - p) \leqslant \mathtt{x}_a(n)$.

(15) If $a \leqslant b$, then $\mathtt{x}_a(n) \leqslant \mathtt{x}_b(n)$ and $\mathtt{y}_a(n) \leqslant \mathtt{y}_b(n)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathtt{x}_a(\$_1) \leqslant \mathtt{x}_b(\$_1)$ and $\mathtt{y}_a(\$_1) \leqslant \mathtt{y}_b(\$_1)$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$. $\mathcal{P}[k]$. □

(16) If $a \equiv b \pmod{k}$, then $\mathtt{x}_a(n) \equiv \mathtt{x}_b(n) \pmod{k}$.

(17) $\mathtt{x}_a(|2 \cdot x + y|) \equiv -\mathtt{x}_a(|y|) \pmod{\mathtt{x}_a(|x|)}$.
PROOF: Set $i = x$. Set $j = y$. Set $A = a^{\mathbf{2}} -' 1$. $A \cdot \mathrm{sgn}(i) \cdot \mathtt{y}_a(|i|) \cdot (\mathrm{sgn}(i) \cdot \mathtt{y}_a(|i|) \cdot \mathtt{x}_a(|j|)) = (A \cdot (\mathtt{y}_a(|i|) \cdot \mathtt{y}_a(|i|))) \cdot \mathtt{x}_a(|j|)$. □

(18) $\mathtt{x}_a(|4 \cdot x + y|) \equiv \mathtt{x}_a(|y|) \pmod{\mathtt{x}_a(|x|)}$. The theorem is a consequence of (17).

(19) If $k < n$, then $\mathtt{x}_a(k) < \mathtt{x}_a(n)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 > 0$, then $\mathtt{x}_a(k) < \mathtt{x}_a(k + \$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. $\mathcal{P}[n_1]$. □

(20) If $\mathtt{x}_a(k) = \mathtt{x}_a(n)$, then $k = n$. The theorem is a consequence of (19).

(21) If $i \leqslant j \leqslant 2 \cdot n$ and $\mathtt{x}_a(i) \equiv \mathtt{x}_a(j) \pmod{\mathtt{x}_a(n)}$, then $i = 0$ and $j = 2$ and $a = 2$ and $n = 1$ or $i = j$. The theorem is a consequence of (19), (17), and (20).

(22) If $0 < i \leqslant n$ and $0 \leqslant j < 4 \cdot n$ and $\mathtt{x}_a(i) \equiv \mathtt{x}_a(j) \pmod{\mathtt{x}_a(n)}$, then $j = i$ or $j + i = 4 \cdot n$. The theorem is a consequence of (18) and (21).

(23) $\mathtt{x}_a(|4 \cdot x \cdot n + y|) \equiv \mathtt{x}_a(|y|) \pmod{\mathtt{x}_a(|x|)}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathtt{x}_a(|4 \cdot x \cdot \$_1 + y|) \equiv \mathtt{x}_a(|y|) \pmod{\mathtt{x}_a(|x|)}$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$. $\mathcal{P}[k]$. □

(24) Suppose $0 < i \leqslant n$ and $\mathtt{x}_a(i) \equiv \mathtt{x}_a(j) \pmod{\mathtt{x}_a(n)}$. Then

(i) $j \equiv i \pmod{4 \cdot n}$, or

(ii) $j \equiv -i \pmod{4 \cdot n}$.

The theorem is a consequence of (23) and (22).

(25) $\mathtt{y}_a(2 \cdot n) = 2 \cdot \mathtt{y}_a(n) \cdot \mathtt{x}_a(n)$.

3. Special Case of Pell's Equation - Diophantine Polynomial with 8 Variables

Now we state the propositions:

(26) Let us consider a non trivial natural number a, and natural numbers y, n, b, c, d, r, s, t, u, v, x. Suppose $1 \leqslant n$ and $\langle x, y\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $\langle u, v\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $\langle s, t\rangle$ is a Pell's solution of $b^{\mathbf{2}} -' 1$ and $v = 4 \cdot r \cdot y^{\mathbf{2}}$ and $b = a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a)$ and $s = x + c \cdot u$ and $t = n + 4 \cdot d \cdot y$ and $n \leqslant y$. Then

(i) b is not trivial, and

(ii) $u^{\mathbf{2}} > a$, and

(iii) $y = \mathtt{y}_a(n)$.

Proof: Consider i being a natural number such that $x = \mathtt{x}_a(i)$ and $y = \mathtt{y}_a(i)$. Consider n_1 being a natural number such that $u = \mathtt{x}_a(n_1)$ and $v = \mathtt{y}_a(n_1)$. $v \neq 0$ by [3, (1)]. Reconsider $B = b$ as a non trivial natural number. Consider j being a natural number such that $s = \mathtt{x}_B(j)$ and $t = \mathtt{y}_B(j)$. $\mathtt{x}_B(j) \equiv \mathtt{x}_a(j) \pmod{\mathtt{x}_a(n_1)}$. $j \equiv i \pmod{4 \cdot n_1}$ or $j \equiv -i \pmod{4 \cdot n_1}$. Consider d_1 being a natural number such that $\mathtt{y}_a(i) \cdot d_1 = n_1$. $n = i$ by [9, (13)]. □

(27) Let us consider a non trivial natural number a, and natural numbers y, n. Suppose $1 \leqslant n$. Suppose $y = \mathtt{y}_a(n)$. Then there exist natural numbers b, c, d, r, s, t, u, v, x such that

(i) $\langle x, y\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$, and

(ii) $\langle u, v\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$, and

(iii) $\langle s, t\rangle$ is a Pell's solution of $b^{\mathbf{2}} -' 1$, and

(iv) $v = 4 \cdot r \cdot y^{\mathbf{2}}$, and

(v) $b = a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a)$, and

(vi) $s = x + c \cdot u$, and

(vii) $t = n + 4 \cdot d \cdot y$, and

(viii) $n \leqslant y$.

The theorem is a consequence of (25), (16), and (15).

(28) Let us consider natural numbers y, n. Suppose $1 \leqslant n$. Then $y = \mathtt{y}_a(n)$ if and only if there exist natural numbers c, d, r, u, x such that $\langle x, y\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $u^{\mathbf{2}} = 16 \cdot (a^{\mathbf{2}} - 1) \cdot r^{\mathbf{2}} \cdot y^{\mathbf{2}} \cdot y^{\mathbf{2}} + 1$ and $(x + c \cdot u)^{\mathbf{2}} = ((a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a))^{\mathbf{2}} - 1) \cdot (n + 4 \cdot d \cdot y)^{\mathbf{2}} + 1$ and $n \leqslant y$.

Proof: If $y = \mathtt{y}_a(n)$, then there exist natural numbers c, d, r, u, x such that $\langle x, y\rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $u^{\mathbf{2}} = 16 \cdot (a^{\mathbf{2}} - 1) \cdot r^{\mathbf{2}} \cdot y^{\mathbf{2}} \cdot y^{\mathbf{2}} + 1$

and $(x + c \cdot u)^{\mathbf{2}} = ((a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a))^{\mathbf{2}} - 1) \cdot (n + 4 \cdot d \cdot y)^{\mathbf{2}} + 1$ and $n \leqslant y$. Consider k such that $x = \mathtt{x}_a(k)$ and $y = \mathtt{y}_a(k)$. $r \neq 0$. □

(29) Let us consider positive natural numbers f, k. Then $f = k!$ if and only if there exist natural numbers j, h, w and there exist positive natural numbers n, p, q, z such that $q = w \cdot z + h + j$ and $z = f \cdot (h + j) + h$ and $2 \cdot k^3 \cdot (2 \cdot k + 2) \cdot (n + 1)^2 + 1$ is a square and $p = (n + 1)^k$ and $q = (p + 1)^n$ and $z = p^{k+1}$.

Proof: Set $k_2 = 2 \cdot k$. If $f = k!$, then there exist natural numbers j, h, w and there exist positive natural numbers n, p, q, z such that $q = w \cdot z + h + j$ and $z = f \cdot (h + j) + h$ and $2 \cdot k^3 \cdot (k_2 + 2) \cdot (n + 1)^2 + 1$ is a square and $p = (n + 1)^k$ and $q = (p + 1)^n$ and $z = p^{k+1}$. ${k_2}^k \leqslant n$. $h + j \neq z$. $k! < \frac{z}{h+j} < k! + 1$. □

(30) Let us consider a positive natural number k. Then $k + 1$ is prime if and only if there exist natural numbers a, b, c, d, e, f, g, h, i, j, l, m, n, o, p, q, r, s, t, u, w, v, x, y, z such that $q = w \cdot z + h + j$ and $z = (g \cdot k + g + k) \cdot (h + j) + h$ and $2 \cdot k^3 \cdot (2 \cdot k + 2) \cdot (n + 1)^2 + 1 = f^{\mathbf{2}}$ and $e = p + q + z + 2 \cdot n$ and $e^3 \cdot (e + 2) \cdot (a + 1)^2 + 1 = o^{\mathbf{2}}$ and $\langle x, y \rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $u^{\mathbf{2}} = 16 \cdot (a^{\mathbf{2}} - 1) \cdot r^{\mathbf{2}} \cdot y^{\mathbf{2}} \cdot y^{\mathbf{2}} + 1$ and $(x + c \cdot u)^{\mathbf{2}} = ((a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a))^{\mathbf{2}} - 1) \cdot (n + 4 \cdot d \cdot y)^{\mathbf{2}} + 1$ and $\langle m, l \rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $l = k + i \cdot (a - 1)$ and $n + l + v = y$ and $m = p + l \cdot (a - n - 1) + b \cdot (2 \cdot a \cdot (n + 1) - (n + 1)^{\mathbf{2}} - 1)$ and $x = q + y \cdot (a - p - 1) + s \cdot (2 \cdot a \cdot (p + 1) - (p + 1)^{\mathbf{2}} - 1)$ and $p \cdot m = z + p \cdot l \cdot (a - p) + t \cdot (2 \cdot a \cdot p - p^{\mathbf{2}} - 1)$.

Proof: If $k + 1$ is prime, then there exist natural numbers a, b, c, d, e, f, g, h, i, j, l, m, n, o, p, q, r, s, t, u, w, v, x, y, z such that $q = w \cdot z + h + j$ and $z = (g \cdot k + g + k) \cdot (h + j) + h$ and $2 \cdot k^3 \cdot (2 \cdot k + 2) \cdot (n + 1)^2 + 1 = f^{\mathbf{2}}$ and $e = p + q + z + 2 \cdot n$ and $e^3 \cdot (e + 2) \cdot (a + 1)^2 + 1 = o^{\mathbf{2}}$ and $\langle x, y \rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $u^{\mathbf{2}} = 16 \cdot (a^{\mathbf{2}} - 1) \cdot r^{\mathbf{2}} \cdot y^{\mathbf{2}} \cdot y^{\mathbf{2}} + 1$ and $(x + c \cdot u)^{\mathbf{2}} = ((a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a))^{\mathbf{2}} - 1) \cdot (n + 4 \cdot d \cdot y)^{\mathbf{2}} + 1$ and $\langle m, l \rangle$ is a Pell's solution of $a^{\mathbf{2}} -' 1$ and $l = k + i \cdot (a - 1)$ and $n + l + v = y$ and $m = p + l \cdot (a - n - 1) + b \cdot (2 \cdot a \cdot (n + 1) - (n + 1)^{\mathbf{2}} - 1)$ and $x = q + y \cdot (a - p - 1) + s \cdot (2 \cdot a \cdot (p + 1) - (p + 1)^{\mathbf{2}} - 1)$ and $p \cdot m = z + p \cdot l \cdot (a - p) + t \cdot (2 \cdot a \cdot p - p^{\mathbf{2}} - 1)$. $2 \cdot k - 1 + 2 \cdot k^{2 \cdot k -' 2} \leqslant n$. $e - 1 + e^{e -' 2} \leqslant a$. $e - 1 + e^{e -' 2} \leqslant a$. $y = \mathtt{y}_a(n)$.

Consider n_2 being a natural number such that $x = \mathtt{x}_a(n_2)$ and $y = \mathtt{y}_a(n_2)$. Consider k_1 being a natural number such that $m = \mathtt{x}_a(k_1)$ and $l = \mathtt{y}_a(k_1)$. $(n + 1)^k < a$. $(n + 1)^k + (\mathtt{y}_a(k)) \cdot (a - (n + 1)) \equiv \mathtt{x}_a(k) \pmod{2 \cdot a \cdot (n + 1) - (n + 1)^{\mathbf{2}} - 1}$. $(p + 1)^n < a$. $(p + 1)^n + (\mathtt{y}_a(n)) \cdot (a - (p + 1)) \equiv \mathtt{x}_a(n) \pmod{2 \cdot a \cdot (p + 1) - (p + 1)^{\mathbf{2}} - 1}$. $p^{k+1} < a$. $p^k + (\mathtt{y}_a(k)) \cdot (a - p) \equiv \mathtt{x}_a(k) \pmod{2 \cdot a \cdot p - p^{\mathbf{2}} - 1}$. $g \cdot k + g + k = k!$. □

4. Prime Representing Polynomial with 26 Variables

Now we state the proposition:

(31) Prime Representing Polynomial:

Let us consider a positive natural number k. Then $k+1$ is prime if and only if there exist natural numbers a, b, c, d, e, f, g, h, i, j, l, m, n, o, p, q, r, s, t, u, w, v, x, y, z such that:

$0 = (w \cdot z + h + j - q)^{\mathbf{2}} + ((g \cdot k + g + k) \cdot (h + j) + h - z)^{\mathbf{2}} + (2 \cdot k^3 \cdot (2 \cdot k + 2) \cdot (n+1)^2 + 1 - f^{\mathbf{2}})^{\mathbf{2}} + (p+q+z+2 \cdot n - e)^{\mathbf{2}} + (e^3 \cdot (e+2) \cdot (a+1)^2 + 1 - o^{\mathbf{2}})^{\mathbf{2}} + (x^{\mathbf{2}} - (a^{\mathbf{2}} -' 1) \cdot y^{\mathbf{2}} - 1)^{\mathbf{2}} + (16 \cdot (a^{\mathbf{2}} - 1) \cdot r^{\mathbf{2}} \cdot y^{\mathbf{2}} \cdot y^{\mathbf{2}} + 1 - u^{\mathbf{2}})^{\mathbf{2}} + (((a + u^{\mathbf{2}} \cdot (u^{\mathbf{2}} - a))^{\mathbf{2}} - 1) \cdot (n + 4 \cdot d \cdot y)^{\mathbf{2}} + 1 - (x + c \cdot u)^{\mathbf{2}})^{\mathbf{2}} + (m^{\mathbf{2}} - (a^{\mathbf{2}} -' 1) \cdot l^{\mathbf{2}} - 1)^{\mathbf{2}} + (k + i \cdot (a-1) - l)^{\mathbf{2}} + (n + l + v - y)^{\mathbf{2}} + (p + l \cdot (a - n - 1) + b \cdot (2 \cdot a \cdot (n+1) - (n+1)^{\mathbf{2}} - 1) - m)^{\mathbf{2}} + (q + y \cdot (a - p - 1) + s \cdot (2 \cdot a \cdot (p+1) - (p+1)^{\mathbf{2}} - 1) - x)^{\mathbf{2}} + (z + p \cdot l \cdot (a - p) + t \cdot (2 \cdot a \cdot p - p^{\mathbf{2}} - 1) - p \cdot m)^{\mathbf{2}}$. The theorem is a consequence of (30).

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Quadratic Extensions

Christoph Schwarzweller
Institute of Informatics
University of Gdańsk
Poland

Agnieszka Rowińska-Schwarzweller
Sopot, Poland

Summary. In this article we further develop field theory [6], [7], [12] in Mizar [1], [2], [3]: we deal with quadratic polynomials and quadratic extensions [5], [4]. First we introduce quadratic polynomials, their discriminants and prove the midnight formula. Then we show that - in case the discriminant of p being non square - adjoining a root of p's discriminant results in a splitting field of p. Finally we prove that these are the only field extensions of degree 2, e.g. that an extension E of F is quadratic if and only if there is a non square Element $a \in F$ such that E and $F(\sqrt{a})$ are isomorphic over F.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers a, b. If $a \leqslant b$, then $a -' 1 \leqslant b -' 1$.

Let i be an integer. One can check that $i^{\mathbf{2}}$ is integer.

Let R be a ring, S be a ring extension of R, and a be an R-membered element of S. The functor ${}^{@}a$ yielding an element of R is defined by the term

(Def. 1) a.

One can verify that $-a$ is R-membered.

Let a, b be R-membered elements of S. One can verify that $a + b$ is R-membered and $a \cdot b$ is R-membered and 0_S is R-membered.

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Let R be a non degenerated ring. One can check that 1_S is non zero and R-membered and there exists an element of S which is non zero and R-membered.

Let F be a field, E be an extension of F, and a be a non zero, F-membered element of E. Let us observe that a^{-1} is F-membered.

Let R be a ring and a, b, c be elements of R. One can check that $\langle a, b, c\rangle$ is (the carrier of R)-valued and there exists a field which is strict and has not characteristic 2.

Let R be a ring. One can check that $(0_R)^{\mathbf{2}}$ reduces to 0_R and $(1_R)^{\mathbf{2}}$ reduces to 1_R and $(-1_R)^{\mathbf{2}}$ reduces to 1_R.

Now we state the propositions:

(2) Let us consider a commutative ring R, and elements a, b of R. Then $(a \cdot b)^{\mathbf{2}} = a^{\mathbf{2}} \cdot b^{\mathbf{2}}$.

(3) Let us consider a field F, an element a of F, a non zero element b of F, and an integer i. Suppose $i \star a \neq 0_F$ and $i \star b \neq 0_F$. Then $(i \star a) \cdot (i \star b)^{-1} = a \cdot b^{-1}$.

(4) Let us consider a commutative ring R, an element a of R, and an integer i. Then $(i \star a)^{\mathbf{2}} = i^{\mathbf{2}} \star a^{\mathbf{2}}$.

Let us consider an integral domain R with non characteristic 2 and an element a of R. Now we state the propositions:

(5) $2 \star a = 0_R$ if and only if $a = 0_R$.

(6) $4 \star a = 0_R$ if and only if $a = 0_R$. The theorem is a consequence of (5).

(7) Let us consider a ring R, a ring extension S of R, an element a of R, and an element b of S. If $b = a$, then for every integer i, $i \star a = i \star b$.

PROOF: Define $\mathcal{P}[\text{integer}] \equiv$ for every integer k such that $k = \$_1$ holds $k \star a = k \star b$. For every integer u such that $\mathcal{P}[u]$ holds $\mathcal{P}[u-1]$ and $\mathcal{P}[u+1]$ by [11, (62), (64)], [8, (15)]. For every integer i, $\mathcal{P}[i]$. □

(8) Let us consider an integral domain R, a domain ring extension S of R, an element a of R, and an element b of S. If $b^{\mathbf{2}} = a^{\mathbf{2}}$, then $b = a$ or $b = -a$.

Let us consider a field F, an extension E of F, and an element a of E. Now we state the propositions:

(9) $\mathrm{FAdj}(F, \{a, -a\}) = \mathrm{FAdj}(F, \{a\})$.

(10) $\mathrm{FAdj}(F, \{a\}) = \mathrm{FAdj}(F, \{-a\})$. The theorem is a consequence of (9).

One can check that there exists a polynomial-disjoint field which is non algebraic closed.

Let F be a non algebraic closed field. One can verify that there exists an element of the carrier of $\mathrm{PolyRing}(F)$ which is irreducible and non linear.

Let F be a field. One can verify that every element of the carrier of $\mathrm{PolyRing}(F)$ which is irreducible and non linear and has also not roots and every element of the carrier of PolyRing(F) which is irreducible and has roots is also linear.

Let F be a polynomial-disjoint field and p be an irreducible element of the carrier of PolyRing(F). Note that KrRootP(p) is F-algebraic.

Let F be a non algebraic closed, polynomial-disjoint field and p be an irreducible, non linear element of the carrier of PolyRing(F). Let us note that KrRootP(p) is non zero and non F-membered.

2. More on Polynomials

Now we state the proposition:

(11) Let us consider a non degenerated ring R, a non zero polynomial p over R, and a polynomial q over R. Then $\deg(p * q) \leqslant \deg p + \deg q$.

Let L be a well unital, non degenerated double loop structure, k be a non zero element of $\mathbb{N}$, and a be an element of L. Let us note that rpoly(k, a) is monic.

Let R be a non degenerated ring, a be a non zero element of R, and b be an element of R. Let us note that $\langle b, a \rangle$ is linear and $\langle b, 1_R \rangle$ is monic and linear.

Now we state the propositions:

(12) Let us consider a ring R, and elements a, b, x of R. Then $x \cdot \langle b, a \rangle = \langle x \cdot b, x \cdot a \rangle$.

(13) Let us consider a ring R, and a polynomial p over R. Suppose $\deg p < 2$. Let us consider an element a of R. Then there exist elements y, z of R such that $p = \langle y, z \rangle$.

(14) Let us consider a commutative ring R, and a polynomial p over R. Suppose $\deg p < 2$. Let us consider an element a of R. Then there exist elements y, z of R such that $\text{eval}(p, a) = y + a \cdot z$. The theorem is a consequence of (13).

(15) Let us consider a field F, an extension E of F, and a polynomial p over F. Suppose $\deg p < 2$. Let us consider an element a of E. Then there exist F-membered elements y, z of E such that $\text{ExtEval}(p, a) = y + a \cdot z$. The theorem is a consequence of (13).

Let R be a ring and a be an element of R. The functors: X- a and X+ a yielding elements of the carrier of PolyRing(R) are defined by terms

(Def. 2) rpoly$(1, a)$,

(Def. 3) rpoly$(1, -a)$,

respectively. Let R be a non degenerated ring. Let us observe that X- a is linear and monic and X+ a is linear and monic.

3. Quadratic Polynomials

Let R be a ring and p be a polynomial over R. We say that p is quadratic if and only if

(Def. 4) $\deg p = 2$.

Let R be a non degenerated ring. Note that there exists a polynomial over R which is monic and quadratic and there exists an element of the carrier of PolyRing(R) which is monic and quadratic and every quadratic polynomial over R is non constant and every quadratic element of the carrier of PolyRing(R) is non constant.

Let L be a non empty zero structure and a, b, c be elements of L. The functor $\langle c, b, a\rangle$ yielding a sequence of L is defined by the term

(Def. 5) $((\mathbf{0}.L +\cdot (0, c)) +\cdot (1, b)) +\cdot (2, a)$.

Note that $\langle c, b, a\rangle$ is finite-Support.

Let us consider a non empty zero structure L and elements a, b, c of L. Now we state the propositions:

(16) (i) $\langle c, b, a\rangle(0) = c$, and

(ii) $\langle c, b, a\rangle(1) = b$, and

(iii) $\langle c, b, a\rangle(2) = a$, and

(iv) for every natural number n such that $n \geqslant 3$ holds $\langle c, b, a\rangle(n) = 0_L$.

(17) $\deg\langle c, b, a\rangle \leqslant 2$.

(18) $\deg\langle c, b, a\rangle = 2$ if and only if $a \neq 0_L$.

Let R be a non degenerated ring, a be a non zero element of R, and b, c be elements of R. One can check that $\langle c, b, a\rangle$ is quadratic and $\langle c, b, 1_R\rangle$ is quadratic and monic.

Let R be an integral domain and a, x be non zero elements of R. Observe that $x \cdot \langle c, b, a\rangle$ is quadratic.

Let us consider a ring R and elements a, b, c, x of R. Now we state the propositions:

(19) $x \cdot \langle c, b, a\rangle = \langle x \cdot c, x \cdot b, x \cdot a\rangle$.

(20) $\mathrm{eval}(\langle c, b, a\rangle, x) = c + b \cdot x + a \cdot x^{\mathbf{2}}$.

(21) Let us consider a non degenerated ring R, and a polynomial p over R. Then p is quadratic if and only if there exists a non zero element a of R and there exist elements b, c of R such that $p = \langle c, b, a\rangle$.

(22) Let us consider a non degenerated ring R, and a monic polynomial p over R. Then p is quadratic if and only if there exist elements b, c of R such that $p = \langle c, b, 1_R\rangle$. The theorem is a consequence of (21).

(23) Let us consider a non degenerated ring R, a ring extension S of R, elements a_1, b_1, c_1 of R, and elements a_2, b_2, c_2 of S. Suppose $a_1 = a_2$ and $b_1 = b_2$ and $c_1 = c_2$. Then $\langle c_2, b_2, a_2\rangle = \langle c_1, b_1, a_1\rangle$.

Let R be a non degenerated ring and p be a polynomial over R. We say that p is purely quadratic if and only if

(Def. 6) there exists a non zero element a of R and there exists an element c of R such that $p = \langle c, 0_R, a\rangle$.

Let a be a non zero element of R and c be an element of R. Let us note that $\langle c, 0_R, a\rangle$ is purely quadratic and there exists a polynomial over R which is monic and purely quadratic and every polynomial over R which is purely quadratic is also quadratic.

Let R be a ring and a be an element of R. The functors: $\mathrm{X}^2\text{-}\,a$ and $\mathrm{X}^2{+}\,a$ yielding elements of the carrier of PolyRing(R) are defined by terms

(Def. 7) $\langle -a, 0_R, 1_R\rangle$,

(Def. 8) $\langle a, 0_R, 1_R\rangle$,

respectively. Let R be a non degenerated ring. One can check that every polynomial over R which is linear is also non quadratic and every polynomial over R which is quadratic is also non linear.

Let a be an element of R. One can verify that $\mathrm{X}^2\text{-}\,a$ is purely quadratic, monic, and non constant and $\mathrm{X}^2{+}\,a$ is purely quadratic, monic, and non constant.

Now we state the propositions:

(24) Let us consider a field F, and elements b_1, c_1, b_2, c_2 of F. Then $\langle c_1, b_1\rangle * \langle c_2, b_2\rangle = \langle c_1 \cdot c_2, b_1 \cdot c_2 + b_2 \cdot c_1, b_1 \cdot b_2\rangle$. The theorem is a consequence of (1).

(25) Let us consider a field F with non characteristic 2, a non zero element a of F, elements b, c of F, and an element w of F. Suppose $w^{\mathbf{2}} = b^{\mathbf{2}} - (4 \star a) \cdot c$. Then

(i) $\mathrm{eval}(\langle c, b, a\rangle, (-b + w) \cdot (2 \star a)^{-1}) = 0_F$, and

(ii) $\mathrm{eval}(\langle c, b, a\rangle, (-b - w) \cdot (2 \star a)^{-1}) = 0_F$.

The theorem is a consequence of (5), (2), (4), and (20).

(26) Let us consider a field F, a non zero element a of F, and elements b, c of F. Suppose $\mathrm{Roots}(\langle c, b, a\rangle) \neq \emptyset$. Then $b^{\mathbf{2}} - (4 \star a) \cdot c$ is a square. The theorem is a consequence of (20), (4), and (2).

(27) Let us consider a field F with non characteristic 2, a non zero element a of F, elements b, c of F, and an element w of F. Suppose $w^{\mathbf{2}} = b^{\mathbf{2}} - (4 \star a) \cdot c$. Then $\mathrm{Roots}(\langle c, b, a\rangle) = \{(-b + w) \cdot (2 \star a)^{-1}, (-b - w) \cdot (2 \star a)^{-1}\}$. The theorem is a consequence of (5), (20), (4), (2), and (25).

(28) Let us consider a field F with non characteristic 2, a non zero element a of F, elements b, c of F, and an element w of F. Suppose $w^{\mathbf{2}} = b^{\mathbf{2}} - (4 \star a) \cdot c$. Let us consider elements r_1, r_2 of F. Suppose $r_1 = (-b + w) \cdot (2 \star a)^{-1}$ and $r_2 = (-b - w) \cdot (2 \star a)^{-1}$. Then $\langle c, b, a \rangle = a \cdot (\mathrm{X}\text{-}\, r_1 * \mathrm{X}\text{-}\, r_2)$.
Proof: $\langle a \cdot r_1 \cdot r_2, a \cdot (-(r_1 + r_2)), a \cdot (1_F) \rangle = a \cdot (\mathrm{rpoly}(1, r_1) * \mathrm{rpoly}(1, r_2))$. $2 \star a \neq 0_F$ and $4 \star a \neq 0_F$ and $a \neq 0_F$. $a \cdot r_1 \cdot r_2 = c$ by [9, (5),(9)]. $a \cdot (-(r_1 + r_2)) = b$ by [10, (2)],(3). □

Let R be a non degenerated ring and p be a quadratic polynomial over R. The functor Discriminant(p) yielding an element of R is defined by

(Def. 9) there exists a non zero element a of R and there exist elements b, c of R such that $p = \langle c, b, a \rangle$ and $it = b^{\mathbf{2}} - (4 \star a) \cdot c$.

We introduce the notation DC(p) as a synonym of Discriminant(p).

Let p be a monic, quadratic polynomial over R. Observe that the functor Discriminant(p) is defined by

(Def. 10) there exist elements b, c of R such that $p = \langle c, b, 1_R \rangle$ and $it = b^{\mathbf{2}} - 4 \star c$.

Let p be a monic, purely quadratic polynomial over R. One can check that the functor Discriminant(p) is defined by

(Def. 11) there exists an element c of R such that $p = \langle c, 0_R, 1_R \rangle$ and $it = -4 \star c$.

Let us consider a field F with non characteristic 2 and a quadratic polynomial p over F. Now we state the propositions:

(29) Roots(p) $\neq \emptyset$ if and only if DC(p) is a square. The theorem is a consequence of (21), (25), and (26).

(30) $\overline{\overline{\mathrm{Roots}(p)}} = 1$ if and only if DC(p) $= 0_F$. The theorem is a consequence of (21), (27), (5), and (29).

(31) $\overline{\overline{\mathrm{Roots}(p)}} = 2$ if and only if DC(p) is non zero and a square. The theorem is a consequence of (21), (5), (29), and (27).

(32) Let us consider a field F with non characteristic 2, and a quadratic element p of the carrier of PolyRing(F). Then p is reducible if and only if DC(p) is a square. The theorem is a consequence of (21), (28), and (19).

(33) Let us consider a field F with non characteristic 2, and an element a of F. Then $\mathrm{X}^2\text{-}\, a$ is reducible if and only if a is a square. The theorem is a consequence of (5), (6), and (32).

4. QUADRATIC POLYNOMIALS OVER $\mathbb{Z}/2$

Now we state the propositions:

(34) The carrier of $\mathbb{Z}/2 = \{0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\}$.

(35) $-1_{\mathbb{Z}/2} = 1_{\mathbb{Z}/2}$.

One can verify that $\mathbb{Z}/2$ is polynomial-disjoint and every element of $\mathbb{Z}/2$ is a square and every non zero polynomial over $\mathbb{Z}/2$ is monic and every non zero element of the carrier of PolyRing($\mathbb{Z}/2$) is monic.

The functors: X^2, X^2+1, $\mathrm{X}^2+\mathrm{X}$, and $\mathrm{X}^2+\mathrm{X}+1$ yielding quadratic elements of the carrier of PolyRing($\mathbb{Z}/2$) are defined by terms

(Def. 12) $\langle 0_{\mathbb{Z}/2}, 0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

(Def. 13) $\langle 1_{\mathbb{Z}/2}, 0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

(Def. 14) $\langle 0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

(Def. 15) $\langle 1_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

respectively. The functors: X- and X-1 yielding linear elements of the carrier of PolyRing($\mathbb{Z}/2$) are defined by terms

(Def. 16) $\langle 0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

(Def. 17) $\langle 1_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\rangle$,

respectively. Now we state the propositions:

(36) the set of all p where p is a quadratic polynomial over $\mathbb{Z}/2 = \{\mathrm{X}^2, \mathrm{X}^2+1, \mathrm{X}^2+\mathrm{X}, \mathrm{X}^2+\mathrm{X}+1\}$. The theorem is a consequence of (22) and (34).

(37) $\overline{\overline{\text{the set of all } p \text{ where } p \text{ is a quadratic polynomial over } \mathbb{Z}/2}} = 4$. The theorem is a consequence of (36).

(38) Let us consider a quadratic polynomial p over $\mathbb{Z}/2$. Then DC(p) is a square.

(39) (i) $\mathrm{X}^2 = \text{X-} * \text{X-}$, and

(ii) $\mathrm{Roots}(\mathrm{X}^2) = \{0_{\mathbb{Z}/2}\}$.

(40) (i) $\mathrm{X}^2+1 = \text{X-1} * \text{X-1}$, and

(ii) $\mathrm{Roots}(\mathrm{X}^2+1) = \{1_{\mathbb{Z}/2}\}$.

The theorem is a consequence of (35).

(41) (i) $\mathrm{X}^2+\mathrm{X} = \text{X-} * \text{X-1}$, and

(ii) $\mathrm{Roots}(\mathrm{X}^2+\mathrm{X}) = \{0_{\mathbb{Z}/2}, 1_{\mathbb{Z}/2}\}$.

The theorem is a consequence of (35).

(42) $\mathrm{Roots}(\mathrm{X}^2+\mathrm{X}+1) = \emptyset$. The theorem is a consequence of (34) and (20).

Let us note that X^2 is reducible and X^2+1 is reducible and $\mathrm{X}^2+\mathrm{X}$ is reducible and $\mathrm{X}^2+\mathrm{X}+1$ is irreducible. Now we state the propositions:

(43) $\mathbb{Z}/2$ is a splitting field of X^2.

(44) $\mathbb{Z}/2$ is a splitting field of X^2+1.

(45) $\mathbb{Z}/2$ is a splitting field of $\mathrm{X}^2+\mathrm{X}$.

The functor α yielding an element of embField(canHomP($\mathrm{X}^2+\mathrm{X}+1$)) is defined by the term

(Def. 18) KrRootP($\mathrm{X}^2+\mathrm{X}+1$).

The functor $\alpha-1$ yielding an element of embField(canHomP($\mathrm{X}^2+\mathrm{X}+1$)) is defined by the term

(Def. 19) $\alpha - 1_{\mathrm{embField(canHomP(X^2+X+1))}}$.

Let us observe that α is non zero and ($\mathbb{Z}/2$)-algebraic.

Now we state the propositions:

(46) (i) $-\alpha = \alpha$, and

(ii) $(\alpha)^{-1} = \alpha - 1$, and

(iii) $(\alpha)^{-1} \neq \alpha$.

(47) $\mathrm{X}^2+\mathrm{X}+1 = \mathrm{X}\text{-}\,\alpha * \mathrm{X}\text{-}(\alpha)^{-1} = \mathrm{X}\text{-}\,\alpha * \mathrm{X}\text{-}\,\alpha - 1$.

(48) Roots(FAdj($\mathbb{Z}/2, \{\alpha\}$), $\mathrm{X}^2+\mathrm{X}+1$) $= \{\alpha, \alpha-1\}$. The theorem is a consequence of (46).

(49) $\overline{\overline{\mathrm{Roots(FAdj}(\mathbb{Z}/2,\{\alpha\}),\mathrm{X}^2+\mathrm{X}+1)}} = 2$.

(50) MinPoly($\alpha, \mathbb{Z}/2$) $= \mathrm{X}^2+\mathrm{X}+1$.

(51) deg(FAdj($\mathbb{Z}/2, \{\alpha\}$), $\mathbb{Z}/2$) $= 2$. The theorem is a consequence of (50) and (18).

(52) FAdj($\mathbb{Z}/2, \{\alpha\}$) is a splitting field of $\mathrm{X}^2+\mathrm{X}+1$. The theorem is a consequence of (48).

5. Fields with Non Squares

Let R be a ring. We say that R is quadratic complete if and only if

(Def. 20) the carrier of $R \subseteq \mathrm{SQ}(R)$.

Let us observe that $-1_{\mathbb{R}_\mathrm{F}}$ is non square and $-1_{\mathbb{F}_\mathbb{Q}}$ is non square and every non degenerated ring which is algebraic closed is also quadratic complete and every non degenerated ring which is preordered is also non quadratic complete and $\mathbb{F}_\mathbb{Q}$ is non quadratic complete and $\mathbb{R}_\mathrm{F}$ is non quadratic complete and $\mathbb{C}_\mathrm{F}$ is quadratic complete and there exists a field which is non quadratic complete, polynomial-disjoint, and strict and there exists a field which is quadratic complete and strict and every ring which is non quadratic complete is also non degenerated.

Let R be a non quadratic complete ring. One can check that there exists an element of R which is non square and there exists a field which is strict, polynomial-disjoint, and non quadratic complete and has not characteristic 2.

Let F be a non quadratic complete field without characteristic 2. Let us note that there exists an element of the carrier of PolyRing(F) which is monic, quadratic, and irreducible.

Let F be a field with non characteristic 2 and a be square element of F. One can verify that $\mathrm{X}^2\text{-}\, a$ is reducible.

Let F be a non quadratic complete field without characteristic 2 and a be a non square element of F. Note that $\mathrm{X}^2\text{-}\, a$ is irreducible.

Let F be a non quadratic complete, polynomial-disjoint field without characteristic 2. The functor $\sqrt{a}$ yielding an element of embField(canHomP($\mathrm{X}^2\text{-}\, a$)) is defined by the term

(Def. 21) KrRootP($\mathrm{X}^2\text{-}\, a$).

One can verify that $\sqrt{a}$ is non zero and F-algebraic and embField(canHomP ($\mathrm{X}^2\text{-}\, a$)) is (FAdj($F, \{\sqrt{a}\}$))-extending and $\sqrt{a}$ is (FAdj($F, \{\sqrt{a}\}$))-membered and non F-membered.

From now on F denotes a non quadratic complete, polynomial-disjoint field without characteristic 2.

Let us consider a non square element a of F. Now we state the propositions:

(53) $\sqrt{a} \cdot \sqrt{a} = a$. The theorem is a consequence of (20).

(54) $\mathrm{MinPoly}(\sqrt{a}, F) = \mathrm{X}^2\text{-}\, a$.

(55) $\deg(\mathrm{FAdj}(F, \{\sqrt{a}\}), F) = 2$.

(56) $\mathrm{X}\text{-}\, \sqrt{a} * \mathrm{X}{+}\, \sqrt{a} = \mathrm{X}^2\text{-}\, a$. The theorem is a consequence of (53).

(57) $\mathrm{Roots}(\mathrm{FAdj}(F, \{\sqrt{a}\}), \mathrm{X}^2\text{-}\, a) = \{\sqrt{a}, -\sqrt{a}\}$. The theorem is a consequence of (56).

(58) FAdj($F, \{\sqrt{a}\}$) is a splitting field of $\mathrm{X}^2\text{-}\, a$. The theorem is a consequence of (56) and (57).

(59) $\{1_F, \sqrt{a}\}$ is a basis of VecSp(FAdj($F, \{\sqrt{a}\}$), F).

(60) The carrier of FAdj($F, \{\sqrt{a}\}$) = the set of all $y + ({}^{@}\sqrt{a}) \cdot z$ where y, z are F-membered elements of FAdj($F, \{\sqrt{a}\}$).

(61) Let us consider a non square element a of F, and F-membered elements a_1, a_2, b_1, b_2 of FAdj($F, \{\sqrt{a}\}$). Suppose $a_1 + ({}^{@}\sqrt{a}) \cdot b_1 = a_2 + ({}^{@}\sqrt{a}) \cdot b_2$. Then

(i) $a_1 = a_2$, and

(ii) $b_1 = b_2$.

6. Splittingfields for Quadratic Polynomials

Let F be a field with non characteristic 2 and p be a quadratic element of the carrier of PolyRing(F). We say that p is DC-square if and only if

(Def. 22) DC(p) is a square.

Note that there exists a quadratic element of the carrier of PolyRing(F) which is monic and DC-square.

Let F be a non quadratic complete field without characteristic 2. One can check that there exists a quadratic element of the carrier of PolyRing(F) which is monic and non DC-square.

Let p be a non DC-square, quadratic element of the carrier of PolyRing(F). One can verify that DC(p) is non square and X^2- DC(p) is irreducible.

Let F be a field with non characteristic 2 and p be a DC-square, quadratic element of the carrier of PolyRing(F). One can verify that X^2- DC(p) is reducible.

Now we state the proposition:

(62) Let us consider a field F with non characteristic 2, and a quadratic element p of the carrier of PolyRing(F). Then F is a splitting field of p if and only if DC(p) is a square. The theorem is a consequence of (21), (28), and (26).

Let F be a non quadratic complete, polynomial-disjoint field without characteristic 2 and p be a non DC-square, quadratic element of the carrier of PolyRing(F). Observe that $\sqrt{\mathrm{DC}(p)}$ is non zero and F-algebraic.

The functor RootDC(p) yielding an element of $\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\})$ is defined by the term

(Def. 23) $\sqrt{\mathrm{DC}(p)}$.

The functors: Root1(p) and Root2(p) yielding elements of $\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\})$ are defined by terms

(Def. 24) $(-({}^{@}(p(1), \mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}))) +$
$\mathrm{RootDC}(p)) \cdot (2 \star ({}^{@}(p(2), \mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}))))^{-1}$,

(Def. 25) $(-({}^{@}(p(1), \mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}))) -$
$\mathrm{RootDC}(p)) \cdot (2 \star ({}^{@}(p(2), \mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}))))^{-1}$,

respectively. In the sequel p denotes a non DC-square, quadratic element of the carrier of PolyRing(F).

Now we state the propositions:

(63) $\mathrm{RootDC}(p) \cdot \mathrm{RootDC}(p) = \mathrm{DC}(p)$. The theorem is a consequence of (53).

(64) Let us consider a non zero element a of $\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\})$, and elements b, c of $\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\})$. Suppose $p = \langle c, b, a \rangle$. Then

(i) $\mathrm{Root1}(p) = (-b + \mathrm{RootDC}(p)) \cdot (2 \star a)^{-1}$, and

(ii) $\mathrm{Root2}(p) = (-b - \mathrm{RootDC}(p)) \cdot (2 \star a)^{-1}$.

(65) $p = ({}^{@}(\mathrm{LC}\, p, \mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}))) \cdot (\mathrm{X}\text{-}\,\mathrm{Root1}(p) * \mathrm{X}\text{-}\,\mathrm{Root2}(p))$. The theorem is a consequence of (28), (21), (23), (64), (63), and (7).

(66) $\mathrm{Roots}(\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}), p) = \{\mathrm{Root1}(p), \mathrm{Root2}(p)\}$. The theorem is a consequence of (65).

(67) $\mathrm{Root1}(p) \neq \mathrm{Root2}(p)$. The theorem is a consequence of (21), (23), (5), and (64).

(68) $\deg(\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\}), F) = 2$.

(69) $\mathrm{FAdj}(F, \{\sqrt{\mathrm{DC}(p)}\})$ is a splitting field of p. The theorem is a consequence of (65), (66), (21), (5), (23), (64), and (7).

7. Quadratic Extensions

Let F be a field and E be an extension of F. We say that E is F-quadratic if and only if

(Def. 26) $\deg(E, F) = 2$.

Let F be a non quadratic complete, polynomial-disjoint field without characteristic 2. Let us observe that there exists an extension of F which is F-quadratic.

Let F be a field. One can check that every extension of F which is F-quadratic is also F-finite.

Let F be a non quadratic complete, polynomial-disjoint field without characteristic 2 and a be a non square element of F. Let us observe that $\mathrm{FAdj}(F, \{\sqrt{a}\})$ is F-quadratic.

Now we state the propositions:

(70) Let us consider a field F, and elements a, b of F. If $b^{\mathbf{2}} = a$, then $\mathrm{eval}(\mathrm{X}^2\text{-}\,a, b) = 0_F$.

(71) Let us consider a field F with non characteristic 2, an extension E of F, and an element a of F. Suppose there exists no element b of F such that $a = b^{\mathbf{2}}$. Let us consider an element b of E. Suppose $b^{\mathbf{2}} = a$. Then

(i) $\mathrm{FAdj}(F, \{b\})$ is a splitting field of $\mathrm{X}^2\text{-}\,a$, and

(ii) $\deg(\mathrm{FAdj}(F, \{b\}), F) = 2$.

The theorem is a consequence of (9), (70), and (33).

(72) Let us consider a field F with non characteristic 2, and an extension E of F. Then $\deg(E, F) = 2$ if and only if there exists an element a of F such that there exists no element b of F such that $a = b^{\mathbf{2}}$ and there exists an element b of E such that $a = b^{\mathbf{2}}$ and $E \approx \mathrm{FAdj}(F, \{b\})$. The theorem is a consequence of (22), (23), (7), (26), (27), (5), (8), and (71).

(73) Let us consider an extension E of F. Then E is F-quadratic if and only if there exists a non square element a of F such that E and $\mathrm{FAdj}(F, \{\sqrt{a}\})$ are isomorphic over F. The theorem is a consequence of (22), (23), (7), (26), (27), (5), (8), (58), and (71).

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The 3-Fold Product Space of Real Normed Spaces and its Properties

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Summary. In this article, we formalize in Mizar [1], [2] the 3-fold product space of real normed spaces for usefulness in application fields such as engineering, although the formalization of the 2-fold product space of real normed spaces has been stored in the Mizar Mathematical Library [3].

First, we prove some theorems about the 3-variable function and 3-fold Cartesian product for preparation. Then we formalize the definition of 3-fold product space of real linear spaces. Finally, we formulate the definition of 3-fold product space of real normed spaces. We referred to [7] and [6] in the formalization.

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1. 3-Variable Function & 3-Fold Cartesian Product

From now on v, x, x_1, x_2, y, z denote objects and X, X_1, X_2, X_3 denote sets.

The scheme *FuncEx3A* deals with sets X, Y, W, Z and a 4-ary predicate P and states that

(Sch. 1) There exists a function f from $X \times Y \times W$ into Z such that for every objects x, y, w such that $x, y, w \in W$ holds $P[x, y, w, f(x, y, w)]$

provided

- for every objects x, y, w such that $x, y, w \in W$ there exists z such that $z \in Z$ and $P[x, y, w, z]$.

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Now we state the propositions:

(1) Let us consider non empty sets X, Y, Z, and a function D. Suppose $\operatorname{dom} D = \{1, 2, 3\}$ and $D(1) = X$ and $D(2) = Y$ and $D(3) = Z$. Then there exists a function I from $X \times Y \times Z$ into $\prod D$ such that

(i) I is one-to-one and onto, and

(ii) for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $I(x, y, z) = \langle x, y, z \rangle$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}, \text{object}, \text{object}] \equiv \$_4 = \langle \$_1, \$_2, \$_3 \rangle$. For every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ there exists an object w such that $w \in \prod D$ and $\mathcal{P}[x, y, z, w]$. Consider I being a function from $X \times Y \times Z$ into $\prod D$ such that for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $\mathcal{P}[x, y, z, I(x, y, z)]$. $\square$

(2) Let us consider non empty sets X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod \langle X, Y, Z \rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $I(x, y, z) = \langle x, y, z \rangle$.

The theorem is a consequence of (1).

2. 3-Fold Product Space of Real Linear Spaces

Let E, F, G be non empty additive loop structures. The functor $E \times F \times G$ yielding a strict, non empty additive loop structure is defined by the term

(Def. 1) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. One can verify that the functor $\langle e, f, g \rangle$ yields an element of $E \times F \times G$. Let E, F, G be Abelian, non empty additive loop structures. Observe that $E \times F \times G$ is Abelian.

Let E, F, G be add-associative, non empty additive loop structures. One can verify that $E \times F \times G$ is add-associative. Let E, F, G be right zeroed, non empty additive loop structures. Note that $E \times F \times G$ is right zeroed.

Let E, F, G be right complementable, non empty additive loop structures. Let us note that $E \times F \times G$ is right complementable.

Now we state the propositions:

(3) Let us consider non empty additive loop structures E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3 \rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$ by [5, (7)]. □

(4) Let us consider add-associative, right zeroed, right complementable, non empty additive loop structures E, F, G, a point x_1 of E, a point x_2 of F, and a point x_3 of G. Then $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$.

Let E, F, G be non empty RLS structures. The functor $E \times F \times G$ yielding a strict, non empty RLS structure is defined by the term

(Def. 2) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. Let us note that the functor $\langle e, f, g\rangle$ yields an element of $E \times F \times G$. Let E, F, G be Abelian, non empty RLS structures. One can check that $E \times F \times G$ is Abelian.

Let E, F, G be add-associative, non empty RLS structures. Let us note that $E \times F \times G$ is add-associative.

Let E, F, G be right zeroed, non empty RLS structures. Let us observe that $E \times F \times G$ is right zeroed. Let E, F, G be right complementable, non empty RLS structures. One can verify that $E \times F \times G$ is right complementable.

Now we state the propositions:

(5) Let us consider non empty RLS structures E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. □

(6) Let us consider add-associative, right zeroed, right complementable, non empty RLS structures E, F, G, a point x_1 of E, a point x_2 of F, and a point x_3 of G. Then $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$.

Let E, F, G be vector distributive, non empty RLS structures. Let us observe that $E \times F \times G$ is vector distributive.

Let E, F, G be scalar distributive, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar distributive.

Let E, F, G be scalar associative, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar associative.

Let E, F, G be scalar unital, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar unital.

Let E, F, G be Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty RLS structures. One can verify that $\langle E, F, G\rangle$ is real-linear-space-yielding. Now we state the proposition:

(7) Let us consider real linear spaces X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every point x of X and for every point y of Y and for every point z of Z, $I(x, y, z) = \langle x, y, z\rangle$, and

(iii) for every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$, and

(iv) for every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X\times Y\times Z}) = 0_{\prod\langle X,Y,Z\rangle}$.

PROOF: Set $C_1 =$ the carrier of X. Set $C_2 =$ the carrier of Y. Set $C_3 =$ the carrier of Z. Consider I being a function from $C_1 \times C_2 \times C_3$ into $\prod\langle C_1, C_2, C_3\rangle$ such that I is one-to-one and onto and for every objects x, y, z such that $x \in C_1$ and $y \in C_2$ and $z \in C_3$ holds $I(x, y, z) = \langle x, y, z\rangle$. For every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$. For every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$. □

Let E, F, G be real linear spaces, e be a point of E, f be a point of F, and g be a point of G. Note that the functor $\langle e, f, g\rangle$ yields an element of $\prod\langle E, F, G\rangle$. Now we state the proposition:

(8) Let us consider real linear spaces E, F, G. Then

(i) for every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and

(v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$.

PROOF: Consider I being a function from $E \times F \times G$ into $\prod\langle E, F, G\rangle$ such that I is one-to-one and onto and for every point x of E and for every point y of F and for every point z of G, $I(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $E \times F \times G$, $I(v + w) = I(v) + I(w)$ and for every point v of $E \times F \times G$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle E,F,G\rangle} = I(0_{E\times F\times G})$.

For every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$. For every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$. $I(a \cdot \langle x_1, x_2, x_3\rangle) = I(a \cdot x_1, a \cdot x_2, a \cdot x_3)$. □

3. 3-Fold Product Space of Real Normed Spaces

Let E, F, G be non empty normed structures. The functor $E \times F \times G$ yielding a strict, non empty normed structure is defined by the term

(Def. 3) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. One can verify that the functor $\langle e, f, g\rangle$ yields an element of $E \times F \times G$. Let E, F, G be real normed spaces. Let us note that $E \times F \times G$ is reflexive, discernible, real normed space-like, scalar distributive, vector distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable and $\langle E, F, G\rangle$ is real-norm-space-yielding.

Now we state the propositions:

(9) Let us consider real normed spaces E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$, and

(v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and

(vi) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^{\mathbf{2}} + \|x_2\|^{\mathbf{2}} + \|x_3\|^{\mathbf{2}}}$ and there exists an element w of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a\cdot x_1, a\cdot x_2, a\cdot x_3\rangle$. Consider v_{10} being an element of $\mathcal{R}^2$ such that $v_{10} = \langle \|\langle x_1, y_1\rangle\|, \|z1\|\rangle$ and $(\mathrm{prodnorm}(E \times F, G))(\langle x_1, y_1\rangle, z1) = |v_{10}|$. Consider v_{20} being an element of $\mathcal{R}^2$ such that $v_{20} = \langle \|x_1\|, \|y_1\|\rangle$ and $(\mathrm{prodnorm}(E, F))(x_1, y_1) = |v_{20}|$. $\square$

(10) Let us consider real normed spaces X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every point x of X and for every point y of Y and for every point z of Z, $I(x, y, z) = \langle x, y, z\rangle$, and

(iii) for every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$, and

(iv) for every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$, and

(v) $0_{\prod\langle X,Y,Z\rangle} = I(0_{X\times Y\times Z})$, and

(vi) for every point v of $X \times Y \times Z$, $\|I(v)\| = \|v\|$.

PROOF: Reconsider $X_0 = X$, $Y_0 = Y$, $Z_0 = Z$ as a real linear space. Consider I_0 being a function from $X_0 \times Y_0 \times Z_0$ into $\prod\langle X_0, Y_0, Z_0\rangle$ such that I_0 is one-to-one and onto and for every point x of X and for every point y of Y and for every point z of Z, $I_0(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $X_0 \times Y_0 \times Z_0$, $I_0(v + w) = I_0(v) + I_0(w)$ and for every point v of $X_0 \times Y_0 \times Z_0$ and for every real number r, $I_0(r \cdot v) = r \cdot I_0(v)$ and $0_{\prod\langle X_0,Y_0,Z_0\rangle} = I_0(0_{X_0 \times Y_0 \times Z_0})$.

Reconsider $I = I_0$ as a function from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$. For every points g_1, g_2 of $X_0 \times Y_0$ and for every points f_1, f_2 of Z_0, $(\mathrm{prodadd}(X \times Y, Z))(\langle g_1, f_1\rangle, \langle g_2, f_2\rangle) = \langle g_1 + g_2, f_1 + f_2\rangle$. For every real number r and for every point g of $X_0 \times Y_0$ and for every point f of Z_0, $(\mathrm{prodmlt}(X \times Y, Z))(r, \langle g, f\rangle) = \langle r \cdot g, r \cdot f\rangle$. For every point v of $X \times Y \times Z$, $\|I(v)\| = \|v\|$ by [4, (11)]. $\square$

Let E, F, G be real normed spaces, e be a point of E, f be a point of F, and g be a point of G. One can check that the functor $\langle e, f, g\rangle$ yields an element of $\prod\langle E, F, G\rangle$. Now we state the proposition:

(11) Let us consider real normed spaces E, F, G. Then

(i) for every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and

(v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$, and

(vi) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^{\mathbf{2}} + \|x_2\|^{\mathbf{2}} + \|x_3\|^{\mathbf{2}}}$ and there exists an element w of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$.

PROOF: Consider I being a function from $E \times F \times G$ into $\prod\langle E, F, G\rangle$ such that I is one-to-one and onto and for every point x of E and for every point y of F and for every point z of G, $I(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $E \times F \times G$, $I(v + w) = I(v) + I(w)$ and for every point v of $E \times F \times G$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle E,F,G\rangle} = I(0_{E \times F \times G})$ and for every point v of $E \times F \times G$, $\|I(v)\| = \|v\|$. For every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1,$

$y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$. $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^{\mathbf{2}} + \|x_2\|^{\mathbf{2}} + \|x_3\|^{\mathbf{2}}}$. Consider w being an element of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$. □

Let E, F, G be complete real normed spaces. Let us note that $E \times F \times G$ is complete.

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About Graph Sums

Sebastian Koch
Johannes Gutenberg University
Mainz, Germany[1]

Summary. In this article the sum (or disjoint union) of graphs is formalized in the Mizar system [4], [1], based on the formalization of graphs in [9].

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0. Introduction

The sum of graphs has already been formalized in Mizar to a certain extent in [7], in the case where the vertices and edges of the graphs are disjoint. This disjoint union matches the definitions often given in the literature (cf. [2], [10], [11], [3]). However, graphs are added together most of the time without much concern about what kind of objects actually constitute the vertices and edges. This article's goal is to formalize that practice. Naturally, in this paper the sum is generalized to families of multidigraphs, i.e. the graphs of [9].

The first section introduces functors to replace the concrete objects behind vertices and edges of a graph with other objects, which will later be used in section 5.

In the second section graph selector variants for `Graph-yielding` functions are described in a similar way as it was done for `Graph-membered` sets in section 1 of [7].

[1]The author is enrolled in the Johannes Gutenberg University in Mayence, Germany, mailto: `skoch02@students.uni-mainz.de`

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Isomorphisms between two `Graph-membered` sets or two `Graph-yielding` functions are formalized in section 3. They are the foundation for isomorphisms between unions (section 4) and sums (section 6) of graphs.

Section 4 introduces attributes `vertex-disjoint` and `edge-disjoint` for sets or functions of graphs. A lot of attention is given to graph unions of vertex-disjoint sets of graphs, since these essentially are the graph sums.

The rest of the article then focuses on graph sums, that are vertex-disjoint unions of the range of a function of graphs, which is isomorphic to a given graph function not necessarily vertex-disjoint, so that in future articles authors do not need to create a vertex-disjoint function themselves. This "canonical" distinction function is formalized in section 5. A second distinction function is provided that leaves exactly one graph of the original graph function as it was. Isomorphism theorems between these two distinction functions and the original functions are provided as well and needed for the sum isomorphisms in the next section.

Section 6 introduces the mode `GraphSum` of a (not necessarily vertex-disjoint) graph function as a graph (directed) isomorphic to the union of the range of the distinction function. The second distinction function is used to provide a graph sum that is a supergraph of a given graph in the graph function.

Finally the last section defines the graph sum of two graph as a supergraph of the first graph using the general definition from section 6.

1. Replacing Vertices and Edges

Let G be a graph, V be a non empty, one-to-one many sorted set indexed by the vertices of G, and E be a one-to-one many sorted set indexed by the edges of G. The functor replaceVerticesEdges(V, E) yielding a plain graph is defined by

(Def. 1) there exist functions S, T from $\operatorname{rng} E$ into $\operatorname{rng} V$ such that $S = V \cdot$ (the source of G) $\cdot (E^{-1})$ and $T = V \cdot$ (the target of G) $\cdot (E^{-1})$ and $it = \operatorname{createGraph}(\operatorname{rng} V, \operatorname{rng} E, S, T)$.

The functor replaceVertices(V) yielding a plain graph is defined by the term

(Def. 2) replaceVerticesEdges(V, id_α), where α is the edges of G.

Let E be a one-to-one many sorted set indexed by the edges of G. The functor replaceEdges(E) yielding a plain graph is defined by the term

(Def. 3) replaceVerticesEdges(id_α, E), where α is the vertices of G.

Now we state the propositions:

(1) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and a one-to-one many sorted set E indexed

by the edges of G. Then

(i) the vertices of $\mathrm{replaceVerticesEdges}(V, E) = \mathrm{rng}\, V$, and

(ii) the edges of $\mathrm{replaceVerticesEdges}(V, E) = \mathrm{rng}\, E$, and

(iii) the source of $\mathrm{replaceVerticesEdges}(V, E) = V \cdot (\text{the source of } G) \cdot (E^{-1})$, and

(iv) the target of $\mathrm{replaceVerticesEdges}(V, E) = V \cdot (\text{the target of } G) \cdot (E^{-1})$.

(2) Let us consider a graph G, and a non empty, one-to-one many sorted set V indexed by the vertices of G. Then

(i) the vertices of $\mathrm{replaceVertices}(V) = \mathrm{rng}\, V$, and

(ii) the edges of $\mathrm{replaceVertices}(V) =$ the edges of G, and

(iii) the source of $\mathrm{replaceVertices}(V) = V \cdot (\text{the source of } G)$, and

(iv) the target of $\mathrm{replaceVertices}(V) = V \cdot (\text{the target of } G)$.

The theorem is a consequence of (1).

(3) Let us consider a graph G, and a one-to-one many sorted set E indexed by the edges of G. Then

(i) the vertices of $\mathrm{replaceEdges}(E) =$ the vertices of G, and

(ii) the edges of $\mathrm{replaceEdges}(E) = \mathrm{rng}\, E$, and

(iii) the source of $\mathrm{replaceEdges}(E) = (\text{the source of } G) \cdot (E^{-1})$, and

(iv) the target of $\mathrm{replaceEdges}(E) = (\text{the target of } G) \cdot (E^{-1})$.

The theorem is a consequence of (1).

(4) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. Suppose e joins v to w in G. Then $E(e)$ joins $V(v)$ to $V(w)$ in $\mathrm{replaceVerticesEdges}(V, E)$. The theorem is a consequence of (1).

(5) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and objects e, v, w. Suppose e joins v to w in G. Then e joins $V(v)$ to $V(w)$ in $\mathrm{replaceVertices}(V)$. The theorem is a consequence of (4).

(6) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. If e joins v to w in G, then $E(e)$ joins v to w in $\mathrm{replaceEdges}(E)$. The theorem is a consequence of (4).

(7) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by

the edges of G, and objects e, v, w. Suppose e joins v and w in G. Then $E(e)$ joins $V(v)$ and $V(w)$ in replaceVerticesEdges(V, E). The theorem is a consequence of (4).

(8) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and objects e, v, w. Suppose e joins v and w in G. Then e joins $V(v)$ and $V(w)$ in replaceVertices(V). The theorem is a consequence of (5).

(9) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. If e joins v and w in G, then $E(e)$ joins v and w in replaceEdges(E). The theorem is a consequence of (6).

(10) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. Suppose $e \in \operatorname{dom} E$ and v, $w \in \operatorname{dom} V$ and $E(e)$ joins $V(v)$ to $V(w)$ in replaceVerticesEdges(V, E). Then e joins v to w in G. The theorem is a consequence of (1).

(11) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and objects e, v, w. Suppose v, $w \in \operatorname{dom} V$ and e joins $V(v)$ to $V(w)$ in replaceVertices(V). Then e joins v to w in G. The theorem is a consequence of (2) and (10).

(12) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. Suppose $e \in \operatorname{dom} E$ and $E(e)$ joins v to w in replaceEdges(E). Then e joins v to w in G. The theorem is a consequence of (3) and (10).

(13) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. Suppose $e \in \operatorname{dom} E$ and v, $w \in \operatorname{dom} V$ and $E(e)$ joins $V(v)$ and $V(w)$ in replaceVerticesEdges(V, E). Then e joins v and w in G. The theorem is a consequence of (10).

(14) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and objects e, v, w. Suppose v, $w \in \operatorname{dom} V$ and e joins $V(v)$ and $V(w)$ in replaceVertices(V). Then e joins v and w in G. The theorem is a consequence of (11).

(15) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and objects e, v, w. Suppose $e \in \operatorname{dom} E$ and $E(e)$ joins v and w in replaceEdges(E). Then e joins v and w in G. The theorem is a consequence of (12).

Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and a one-to-one many sorted set E indexed by the edges of G. Now we state the propositions:

(16) There exists a partial graph mapping F from G to replaceVerticesEdges (V, E) such that

(i) $F_{\mathbb{V}} = V$, and

(ii) $F_{\mathbb{E}} = E$, and

(iii) F is directed-isomorphism.

The theorem is a consequence of (1) and (4).

(17) replaceVerticesEdges(V, E) is G-directed-isomorphic.

The theorem is a consequence of (16).

Let G be a loopless graph, V be a non empty, one-to-one many sorted set indexed by the vertices of G, and E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceVerticesEdges(V, E) is loopless and replaceVertices(V) is loopless.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is loopless.

Let G be a non loopless graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can verify that replaceVerticesEdges(V, E) is non loopless and replaceVertices(V) is non loopless.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us note that replaceEdges(E) is non loopless.

Let G be a non-multi graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Observe that replaceVerticesEdges(V, E) is non-multi and replaceVertices(V) is non-multi.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us note that replaceEdges(E) is non-multi.

Let G be a non non-multi graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Observe that replaceVerticesEdges(V, E) is non non-multi and replaceVertices(V) is non non-multi.

Let E be a one-to-one many sorted set indexed by the edges of G. One can check that replaceEdges(E) is non non-multi.

Let G be a non-directed-multi graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Note that replaceVerticesEdges(V, E) is non-directed-multi and replaceVertices(V) is non-directed-multi.

Let E be a one-to-one many sorted set indexed by the edges of G. One can check that replaceEdges(E) is non-directed-multi.

Let G be a non non-directed-multi graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Note that replaceVerticesEdges$(V,$

$E)$ is non non-directed-multi and replaceVertices(V) is non non-directed-multi.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is non non-directed-multi.

Let G be a simple graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can verify that replaceVerticesEdges(V, E) is simple and replaceVertices(V) is simple.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is simple.

Let G be a directed-simple graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can check that replaceVerticesEdges (V, E) is directed-simple and replaceVertices(V) is directed-simple.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is directed-simple.

Let G be a trivial graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Note that replaceVerticesEdges(V, E) is trivial and replaceVertices(V) is trivial.

Let E be a one-to-one many sorted set indexed by the edges of G. One can check that replaceEdges(E) is trivial.

Let G be a non trivial graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Note that replaceVerticesEdges(V, E) is non trivial and replaceVertices(V) is non trivial.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is non trivial.

Let G be a vertex-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can verify that replaceVerticesEdges(V, E) is vertex-finite and replaceVertices(V) is vertex-finite.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is vertex-finite.

Let G be a non vertex-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can verify that replaceVerticesEdges (V, E) is non vertex-finite and replaceVertices(V) is non vertex-finite.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us note that replaceEdges(E) is non vertex-finite.

Let G be an edge-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Observe that replaceVerticesEdges(V, E) is edge-finite and replaceVertices(V) is edge-finite.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us note that replaceEdges(E) is edge-finite.

Let G be a non edge-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Observe that replaceVerticesEdges(V, E) is non edge-finite and replaceVertices(V) is non edge-finite.

Let E be a one-to-one many sorted set indexed by the edges of G. One can check that replaceEdges(E) is non edge-finite.

Let G be a finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Note that replaceVerticesEdges(V, E) is finite and replaceVertices(V) is finite.

Let E be a one-to-one many sorted set indexed by the edges of G. One can check that replaceEdges(E) is finite.

Let G be an acyclic graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us note that replaceVerticesEdges(V, E) is acyclic and replaceVertices(V) is acyclic.

Let E be a one-to-one many sorted set indexed by the edges of G. Note that replaceEdges(E) is acyclic.

Let G be a non acyclic graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us note that replaceVerticesEdges(V, E) is non acyclic and replaceVertices(V) is non acyclic.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is non acyclic.

Let G be a connected graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can check that replaceVerticesEdges(V, E) is connected and replaceVertices(V) is connected.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is connected.

Let G be a non connected graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can check that replaceVerticesEdges(V, E) is non connected and replaceVertices(V) is non connected.

Let E be a one-to-one many sorted set indexed by the edges of G. Observe that replaceEdges(E) is non connected.

Let G be a tree-like graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us observe that replaceVerticesEdges(V, E) is tree-like and replaceVertices(V) is tree-like.

Let E be a one-to-one many sorted set indexed by the edges of G. Observe that replaceEdges(E) is tree-like.

Let G be a chordal graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can verify that replaceVerticesEdges(V, E) is chordal and replaceVertices(V) is chordal.

Let E be a one-to-one many sorted set indexed by the edges of G. Let us observe that replaceEdges(E) is chordal.

Let G be an edgeless graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can check that replaceVerticesEdges(V, E) is edgeless and replaceVertices(V) is edgeless.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is edgeless.

Let G be a non edgeless graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. One can check that replaceVerticesEdges(V, E) is non edgeless and replaceVertices(V) is non edgeless.

Let E be a one-to-one many sorted set indexed by the edges of G. Observe that replaceEdges(E) is non edgeless.

Let G be a loopfull graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us observe that replaceVerticesEdges(V, E) is loopfull and replaceVertices(V) is loopfull. Let E be a one-to-one many sorted set indexed by the edges of G. Observe that replaceEdges(E) is loopfull.

Let G be a non loopfull graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us observe that replaceVerticesEdges(V, E) is non loopfull and replaceVertices(V) is non loopfull.

Let E be a one-to-one many sorted set indexed by the edges of G. Note that replaceEdges(E) is non loopfull.

Let G be a locally-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us note that replaceVerticesEdges(V, E) is locally-finite and replaceVertices(V) is locally-finite.

Let E be a one-to-one many sorted set indexed by the edges of G. Note that replaceEdges(E) is locally-finite.

Let G be a non locally-finite graph and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us note that replaceVerticesEdges(V, E) is non locally-finite and replaceVertices(V) is non locally-finite. Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is non locally-finite.

Let c be a non zero cardinal number, G be a c-vertex graph, and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us observe that replaceVerticesEdges(V, E) is c-vertex and replaceVertices(V) is c-vertex.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is c-vertex.

Let c be a cardinal number, G be a c-edge graph, and V be a non empty, one-to-one many sorted set indexed by the vertices of G. Let us observe that replaceVerticesEdges(V, E) is c-edge and replaceVertices(V) is c-edge.

Let E be a one-to-one many sorted set indexed by the edges of G. One can verify that replaceEdges(E) is c-edge. Now we state the propositions:

(18) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by the edges of G, and a walk W_1 of G. Then there exists a walk W_2 of replaceVerticesEdges(V, E) such that

(i) $V \cdot W_1$.vertexSeq() $= W_2$.vertexSeq(), and

(ii) $E \cdot W_1$.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (16).

(19) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and a walk W_1 of G. Then there exists a walk W_2 of replaceVertices(V) such that

(i) $V \cdot W_1$.vertexSeq() $= W_2$.vertexSeq(), and

(ii) W_1.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (18).

(20) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and a walk W_1 of G. Then there exists a walk W_2 of replaceEdges(E) such that

(i) W_1.vertexSeq() $= W_2$.vertexSeq(), and

(ii) $E \cdot W_1$.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (18).

(21) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, a one-to-one many sorted set E indexed by the edges of G, and a walk W_2 of replaceVerticesEdges(V, E). Then there exists a walk W_1 of G such that

(i) $V \cdot W_1$.vertexSeq() $= W_2$.vertexSeq(), and

(ii) $E \cdot W_1$.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (16).

(22) Let us consider a graph G, a non empty, one-to-one many sorted set V indexed by the vertices of G, and a walk W_2 of replaceVertices(V). Then there exists a walk W_1 of G such that

(i) $V \cdot W_1$.vertexSeq() $= W_2$.vertexSeq(), and

(ii) W_1.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (21).

(23) Let us consider a graph G, a one-to-one many sorted set E indexed by the edges of G, and a walk W_2 of replaceEdges(E). Then there exists a walk W_1 of G such that

(i) W_1.vertexSeq() $= W_2$.vertexSeq(), and

(ii) $E \cdot W_1$.edgeSeq() $= W_2$.edgeSeq().

The theorem is a consequence of (21).

2. Graph Selectors of Graph-yielding Functions

Let F be a graph-yielding function. The functors: the vertices of F, the edges of F, the source of F, and the target of F yielding functions are defined by conditions

(Def. 4) dom the vertices of $F = \operatorname{dom} F$ and for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $G = F(x)$ and (the vertices of $F)(x) =$ the vertices of G,

(Def. 5) dom the edges of $F = \operatorname{dom} F$ and for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $G = F(x)$ and (the edges of $F)(x) =$ the edges of G,

(Def. 6) dom the source of $F = \operatorname{dom} F$ and for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $G = F(x)$ and (the source of $F)(x) =$ the source of G,

(Def. 7) dom the target of $F = \operatorname{dom} F$ and for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $G = F(x)$ and (the target of $F)(x) =$ the target of G,

respectively. Let us observe that the source of F is function yielding and the target of F is function yielding.

Let F be an empty, graph-yielding function. One can verify that the vertices of F is empty and the edges of F is empty and the source of F is empty and the target of F is empty.

Let F be a non empty, graph-yielding function. One can verify that the vertices of F is non empty and the edges of F is non empty and the source of F is non empty and the target of F is non empty.

Let F be a graph-yielding function. One can check that the vertices of F is non-empty.

Let F be a non empty, graph-yielding function. The functors: the vertices of F, the edges of F, the source of F, and the target of F are defined by conditions

(Def. 8) dom the vertices of $F = \operatorname{dom} F$ and for every element x of $\operatorname{dom} F$, (the vertices of $F)(x) =$ the vertices of $F(x)$,

(Def. 9) dom the edges of $F = \operatorname{dom} F$ and for every element x of $\operatorname{dom} F$, (the edges of $F)(x) =$ the edges of $F(x)$,

(Def. 10) dom the source of $F = \operatorname{dom} F$ and for every element x of $\operatorname{dom} F$, (the source of $F)(x) =$ the source of $F(x)$,

(Def. 11) dom the target of $F = \operatorname{dom} F$ and for every element x of $\operatorname{dom} F$, (the target of $F)(x) =$ the target of $F(x)$,

respectively.

Let us consider a graph-yielding function F. Now we state the propositions:

(24) The vertices of $\operatorname{rng} F = \operatorname{rng}$(the vertices of F).

(25) The edges of $\operatorname{rng} F = \operatorname{rng}$(the edges of F).

(26) The source of $\operatorname{rng} F = \operatorname{rng}$(the source of F).

(27) The target of $\operatorname{rng} F = \operatorname{rng}$(the target of F).

3. Isomorphisms between Graph-membered Sets or Graph-yielding Functions

Let S_1, S_2 be graph-membered sets. We say that S_1 and S_2 are directed-isomorphic if and only if

(Def. 12) there exists a one-to-one function f such that $\operatorname{dom} f = S_1$ and $\operatorname{rng} f = S_2$ and for every graph G such that $G \in S_1$ holds $f(G)$ is a G-directed-isomorphic graph.

One can check that the predicate is reflexive and symmetric. We say that S_1 and S_2 are isomorphic if and only if

(Def. 13) there exists a one-to-one function f such that $\operatorname{dom} f = S_1$ and $\operatorname{rng} f = S_2$ and for every graph G such that $G \in S_1$ holds $f(G)$ is a G-isomorphic graph.

Let us note that the predicate is reflexive and symmetric.

Let us consider graph-membered sets S_1, S_2, S_3. Now we state the propositions:

(28) If S_1 and S_2 are directed-isomorphic and S_2 and S_3 are directed-isomorphic, then S_1 and S_3 are directed-isomorphic.

(29) If S_1 and S_2 are isomorphic and S_2 and S_3 are isomorphic, then S_1 and S_3 are isomorphic.

Let us consider graph-membered sets S_1, S_2. Now we state the propositions:

(30) If S_1 and S_2 are directed-isomorphic, then S_1 and S_2 are isomorphic.

(31) If S_1 and S_2 are directed-isomorphic, then $\overline{\overline{S_1}} = \overline{\overline{S_2}}$.

(32) If S_1 and S_2 are isomorphic, then $\overline{\overline{S_1}} = \overline{\overline{S_2}}$.

(33) Let us consider empty, graph-membered sets S_1, S_2. Then S_1 and S_2 are directed-isomorphic.

Let us consider graphs G_1, G_2. Now we state the propositions:

(34) $\{G_1\}$ and $\{G_2\}$ are directed-isomorphic if and only if G_2 is G_1-directed-isomorphic.

(35) $\{G_1\}$ and $\{G_2\}$ are isomorphic if and only if G_2 is G_1-isomorphic.

Let us consider graph-membered sets S_1, S_2. Now we state the propositions:

(36) Suppose S_1 and S_2 are isomorphic. Then

(i) if S_1 is empty, then S_2 is empty, and

(ii) if S_1 is loopless, then S_2 is loopless, and

(iii) if S_1 is non-multi, then S_2 is non-multi, and

(iv) if S_1 is simple, then S_2 is simple, and

(v) if S_1 is acyclic, then S_2 is acyclic, and

(vi) if S_1 is connected, then S_2 is connected, and

(vii) if S_1 is tree-like, then S_2 is tree-like, and

(viii) if S_1 is chordal, then S_2 is chordal, and

(ix) if S_1 is edgeless, then S_2 is edgeless, and

(x) if S_1 is loopfull, then S_2 is loopfull.

(37) Suppose S_1 and S_2 are directed-isomorphic. Then

(i) if S_1 is non-directed-multi, then S_2 is non-directed-multi, and

(ii) if S_1 is directed-simple, then S_2 is directed-simple.

Let F_1, F_2 be graph-yielding functions. We say that F_1 and F_2 are directed-isomorphic if and only if

(Def. 14) there exists a one-to-one function p such that $\operatorname{dom} p = \operatorname{dom} F_1$ and $\operatorname{rng} p = \operatorname{dom} F_2$ and for every object x such that $x \in \operatorname{dom} F_1$ there exist graphs G_1, G_2 such that $G_1 = F_1(x)$ and $G_2 = F_2(p(x))$ and G_2 is G_1-directed-isomorphic.

Let us observe that the predicate is reflexive and symmetric. We say that F_1 and F_2 are isomorphic if and only if

(Def. 15) there exists a one-to-one function p such that $\operatorname{dom} p = \operatorname{dom} F_1$ and $\operatorname{rng} p = \operatorname{dom} F_2$ and for every object x such that $x \in \operatorname{dom} F_1$ there exist graphs G_1, G_2 such that $G_1 = F_1(x)$ and $G_2 = F_2(p(x))$ and G_2 is G_1-isomorphic.

Observe that the predicate is reflexive and symmetric.

Let us consider non empty, graph-yielding functions F_1, F_2. Now we state the propositions:

(38) Suppose $\operatorname{dom} F_1 = \operatorname{dom} F_2$ and for every element x_1 of $\operatorname{dom} F_1$ and for every element x_2 of $\operatorname{dom} F_2$ such that $x_1 = x_2$ holds $F_2(x_2)$ is $F_1(x_1)$-directed-isomorphic. Then F_1 and F_2 are directed-isomorphic.

(39) Suppose $\operatorname{dom} F_1 = \operatorname{dom} F_2$ and for every element x_1 of $\operatorname{dom} F_1$ and for every element x_2 of $\operatorname{dom} F_2$ such that $x_1 = x_2$ holds $F_2(x_2)$ is $F_1(x_1)$-isomorphic. Then F_1 and F_2 are isomorphic.

Let us consider graph-yielding functions F_1, F_2, F_3. Now we state the propositions:

(40) If F_1 and F_2 are directed-isomorphic and F_2 and F_3 are directed-isomorphic, then F_1 and F_3 are directed-isomorphic.

(41) If F_1 and F_2 are isomorphic and F_2 and F_3 are isomorphic, then F_1 and F_3 are isomorphic.

(42) Let us consider graph-yielding functions F_1, F_2. If F_1 and F_2 are directed-isomorphic, then F_1 and F_2 are isomorphic.

(43) Let us consider empty, graph-yielding functions F_1, F_2. Then

  (i) F_1 and F_2 are directed-isomorphic, and

  (ii) F_1 and F_2 are isomorphic.

Let us consider graph-yielding functions F_1, F_2. Now we state the propositions:

(44) If F_1 and F_2 are directed-isomorphic, then $\overline{\overline{F_1}} = \overline{\overline{F_2}}$.

(45) If F_1 and F_2 are isomorphic, then $\overline{\overline{F_1}} = \overline{\overline{F_2}}$.

Let us consider graphs G_1, G_2 and objects x, y. Now we state the propositions:

(46) $x \stackrel{\cdot}{\longmapsto} G_1$ and $y \stackrel{\cdot}{\longmapsto} G_2$ are directed-isomorphic if and only if G_2 is G_1-directed-isomorphic.

(47) $x \stackrel{\cdot}{\longmapsto} G_1$ and $y \stackrel{\cdot}{\longmapsto} G_2$ are isomorphic if and only if G_2 is G_1-isomorphic.

Let us consider graph-yielding functions F_1, F_2. Now we state the propositions:

(48) Suppose F_1 and F_2 are isomorphic. Then

  (i) if F_1 is empty, then F_2 is empty, and

  (ii) if F_1 is loopless, then F_2 is loopless, and

  (iii) if F_1 is non-multi, then F_2 is non-multi, and

  (iv) if F_1 is simple, then F_2 is simple, and

  (v) if F_1 is acyclic, then F_2 is acyclic, and

  (vi) if F_1 is connected, then F_2 is connected, and

  (vii) if F_1 is tree-like, then F_2 is tree-like, and

  (viii) if F_1 is chordal, then F_2 is chordal, and

  (ix) if F_1 is edgeless, then F_2 is edgeless, and

  (x) if F_1 is loopfull, then F_2 is loopfull.

(49) Suppose F_1 and F_2 are directed-isomorphic. Then

  (i) if F_1 is non-directed-multi, then F_2 is non-directed-multi, and

  (ii) if F_1 is directed-simple, then F_2 is directed-simple.

Let I be a set and F_1, F_2 be graph-yielding many sorted sets indexed by I. Note that F_1 and F_2 are directed-isomorphic if and only if the condition (Def. 16) is satisfied.

(Def. 16) there exists a permutation p of I such that for every object x such that $x \in I$ there exist graphs G_1, G_2 such that $G_1 = F_1(x)$ and $G_2 = F_2(p(x))$ and G_2 is G_1-directed-isomorphic.

One can check that the predicate is reflexive and symmetric. Let us note that F_1 and F_2 are isomorphic if and only if the condition (Def. 17) is satisfied.

(Def. 17) there exists a permutation p of I such that for every object x such that $x \in I$ there exist graphs G_1, G_2 such that $G_1 = F_1(x)$ and $G_2 = F_2(p(x))$ and G_2 is G_1-isomorphic.

Note that the predicate is reflexive and symmetric.

4. Distinguishing the Vertex and Edge Sets of Several Graphs from Each Other

Let S be a graph-membered set. We say that S is vertex-disjoint if and only if

(Def. 18) for every graphs G_1, G_2 such that G_1, $G_2 \in S$ and $G_1 \neq G_2$ holds the vertices of G_1 misses the vertices of G_2.

We say that S is edge-disjoint if and only if

(Def. 19) for every graphs G_1, G_2 such that G_1, $G_2 \in S$ and $G_1 \neq G_2$ holds the edges of G_1 misses the edges of G_2.

Now we state the proposition:

(50) Let us consider a graph-membered set S. Then S is vertex-disjoint and edge-disjoint if and only if for every graphs G_1, G_2 such that G_1, $G_2 \in S$ and $G_1 \neq G_2$ holds the vertices of G_1 misses the vertices of G_2 and the edges of G_1 misses the edges of G_2.

Let us note that every graph-membered set which is trivial is also vertex-disjoint and edge-disjoint and every graph-membered set which is edgeless is also edge-disjoint and every graph-membered set which is edge-disjoint is also $\cup$-tolerating and every graph-membered set which is vertex-disjoint and $\cup$-tolerating is also edge-disjoint.

Let G be a graph. One can check that $\{G\}$ is vertex-disjoint and edge-disjoint.

Let us consider graphs G_1, G_2. Now we state the propositions:

(51) $\{G_1, G_2\}$ is vertex-disjoint if and only if $G_1 = G_2$ or the vertices of G_1 misses the vertices of G_2.

(52) $\{G_1, G_2\}$ is edge-disjoint if and only if $G_1 = G_2$ or the edges of G_1 misses the edges of G_2.

One can verify that there exists a graph-membered set which is non empty, $\cup$-tolerating, vertex-disjoint, edge-disjoint, acyclic, simple, directed-simple, loopless, non-multi, and non-directed-multi.

Let S be a vertex-disjoint, graph-membered set. Note that the vertices of S is mutually-disjoint.

Let S be an edge-disjoint, graph-membered set. One can verify that the edges of S is mutually-disjoint.

Let S be a vertex-disjoint, graph-membered set. Observe that every subset of S is vertex-disjoint.

Let S_1 be a vertex-disjoint, graph-membered set and S_2 be a set. Let us note that $S_1 \cap S_2$ is vertex-disjoint and $S_1 \setminus S_2$ is vertex-disjoint.

Let S be an edge-disjoint, graph-membered set. One can verify that every subset of S is edge-disjoint.

Let S_1 be an edge-disjoint, graph-membered set and S_2 be a set. Let us observe that $S_1 \cap S_2$ is edge-disjoint and $S_1 \setminus S_2$ is edge-disjoint.

Let us consider graph-membered sets S_1, S_2. Now we state the propositions:

(53) If $S_1 \cup S_2$ is vertex-disjoint, then S_1 is vertex-disjoint and S_2 is vertex-disjoint.

(54) If $S_1 \cup S_2$ is edge-disjoint, then S_1 is edge-disjoint and S_2 is edge-disjoint.

Let us consider vertex-disjoint graph union sets S_1, S_2, a graph union G_1 of S_1, and a graph union G_2 of S_2. Now we state the propositions:

(55) If S_1 and S_2 are directed-isomorphic, then G_2 is G_1-directed-isomorphic.
Proof: Consider h being a one-to-one function such that $\mathrm{dom}\, h = S_1$ and $\mathrm{rng}\, h = S_2$ and for every graph G such that $G \in S_1$ holds $h(G)$ is a G-directed-isomorphic graph. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element G of S_1 and there exists a partial graph mapping F from G to $h(G)$ such that $\$_1 = G$ and $\$_2 = F$ and F is directed-isomorphism. For every element G of S_1, there exists an object F such that $\mathcal{Q}[G, F]$.

Consider H being a many sorted set indexed by S_1 such that for every element G of S_1, $\mathcal{Q}[G, H(G)]$. For every element G of S_1, there exists a partial graph mapping F from G to $h(G)$ such that $H(G) = F$ and F is directed-isomorphism. Set $V = \mathrm{rng}\, \mathrm{pr1}(H)$. Set $E = \mathrm{rng}\, \mathrm{pr2}(H)$. For every object y such that $y \in V$ holds y is a function. For every functions f_1, f_2 such that f_1, $f_2 \in V$ holds f_1 tolerates f_2. For every object y such that $y \in E$ holds y is a function. For every functions g_1, g_2 such that g_1, $g_2 \in E$ holds g_1 tolerates g_2. $\square$

(56) Suppose S_1 and S_2 are isomorphic. Then there exists a vertex-disjoint

graph union set S_3 and there exists a subset E of the edges of G_2 and there exists a graph union G_3 of S_3 such that S_1 and S_3 are directed-isomorphic and G_3 is a graph given by reversing directions of the edges E of G_2.

PROOF: Consider h being a one-to-one function such that $\operatorname{dom} h = S_1$ and $\operatorname{rng} h = S_2$ and for every graph G such that $G \in S_1$ holds $h(G)$ is a G-isomorphic graph. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element G of S_1 and there exists a partial graph mapping F from G to $h(G)$ such that $\$_1 = G$ and $\$_2 = F$ and F is isomorphism. For every element G of S_1, there exists an object F such that $\mathcal{Q}[G, F]$. Consider H being a many sorted set indexed by S_1 such that for every element G of S_1, $\mathcal{Q}[G, H(G)]$. For every element G of S_1, there exists a partial graph mapping F from G to $h(G)$ such that $H(G) = F$ and F is isomorphism. Define $\mathcal{R}[\text{object}, \text{object}] \equiv$ there exists an element G of S_1 and there exists a subset E of the edges of $h(G)$ such that $\$_1 = G$ and $\$_2 = E$ and for every graph G' given by reversing directions of the edges E of $h(G)$, there exists a partial graph mapping F from G to G' such that $F = H(G)$ and F is directed-isomorphism.

For every element G of S_1, there exists an object E such that $\mathcal{R}[G, E]$ by [5, (89)]. Consider A being a many sorted set indexed by S_1 such that for every element G of S_1, $\mathcal{R}[G, A(G)]$. For every element G of S_1, $A(G)$ is a subset of the edges of $h(G)$. For every element G of S_1 and for every graph G' given by reversing directions of the edges $A(G)$ of $h(G)$, there exists a partial graph mapping F from G to G' such that $F = H(G)$ and F is directed-isomorphism. Define $\mathcal{U}(\text{element of } S_1) =$ the graph given by reversing directions of the edges $A(\$_1)$ of $h(\$_1)$. Consider B being a many sorted set indexed by S_1 such that for every element G of S_1, $B(G) = \mathcal{U}(G)$. For every object y such that $y \in \bigcup \operatorname{rng} A$ holds $y \in$ the edges of G_2. $\square$

(57) If S_1 and S_2 are isomorphic, then G_2 is G_1-isomorphic. The theorem is a consequence of (56) and (55).

(58) Let us consider a vertex-disjoint graph union set S, a graph union G of S, and a walk W of G. Then there exists an element H of S such that W is a walk of H.

PROOF: Define $\mathcal{P}[\text{walk of } G] \equiv$ there exists an element H of S such that $\$_1$ is a walk of H. For every trivial walk W of G, $\mathcal{P}[W]$ by [8, (128)]. For every walk W of G and for every object e such that $e \in W.\text{last}().\text{edgesInOut}()$ and $\mathcal{P}[W]$ holds $\mathcal{P}[W.\text{addEdge}(e)]$ by [7, (21)], [8, (16)], [9, (67)], [6, (117)]. For every walk W of G, $\mathcal{P}[W]$ by [8, Sch.1]. $\square$

Let us consider a vertex-disjoint graph union set S and a graph union G of S. Now we state the propositions:

(59) If G is connected, then there exists a graph H such that $S = \{H\}$. The theorem is a consequence of (58).

(60) (i) S is non-multi iff G is non-multi, and

(ii) S is non-directed-multi iff G is non-directed-multi, and

(iii) S is acyclic iff G is acyclic.

The theorem is a consequence of (58).

(61) (i) S is simple iff G is simple, and

(ii) S is directed-simple iff G is directed-simple.

The theorem is a consequence of (60).

Let S be a vertex-disjoint, non-multi graph union set. Let us note that every graph union of S is non-multi.

Let S be a vertex-disjoint, non-directed-multi graph union set. One can check that every graph union of S is non-directed-multi.

Let S be a vertex-disjoint, simple graph union set. Let us observe that every graph union of S is simple.

Let S be a vertex-disjoint, directed-simple graph union set. Observe that every graph union of S is directed-simple.

Let S be a vertex-disjoint, acyclic graph union set. Let us note that every graph union of S is acyclic.

Now we state the propositions:

(62) Let us consider a vertex-disjoint graph union set S, an element H of S, and a graph union G of S. Then H is a subgraph of G induced by the vertices of H.

(63) Let us consider a vertex-disjoint graph union set S, and a graph union G of S. Then

(i) S is chordal iff G is chordal, and

(ii) S is loopfull iff G is loopfull.

The theorem is a consequence of (58) and (62).

(64) Let us consider a vertex-disjoint graph union set S, a graph union G of S, an element H of S, a vertex v of G, and a vertex w of H. If $v = w$, then $G.\mathrm{reachableFrom}(v) = H.\mathrm{reachableFrom}(w)$. The theorem is a consequence of (58).

(65) Let us consider a vertex-disjoint graph union set S, and a graph union G of S. Then $G.\mathrm{componentSet}() = \bigcup$ the set of all $H.\mathrm{componentSet}()$ where H is an element of S. The theorem is a consequence of (64).

(66) Let us consider a vertex-disjoint, non empty, graph-membered set S. Then the set of all $H.\mathrm{componentSet}()$ where H is an element of S is mutually-disjoint.

(67) Let us consider a non empty, connected, graph-membered set S. Then the set of all H.componentSet() where H is an element of $S =$ SmallestPartition(the vertices of S).

Let us consider a vertex-disjoint graph union set S and a graph union G of S. Now we state the propositions:

(68) $\overline{\overline{S}} \subseteq G$.numComponents(). The theorem is a consequence of (66) and (65).

(69) If S is connected, then $\overline{\overline{S}} = G$.numComponents(). The theorem is a consequence of (67) and (65).

Let F be a graph-yielding function. We say that F is vertex-disjoint if and only if

(Def. 20) for every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ there exist graphs G_1, G_2 such that $G_1 = F(x_1)$ and $G_2 = F(x_2)$ and the vertices of G_1 misses the vertices of G_2.

We say that F is edge-disjoint if and only if

(Def. 21) for every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ there exist graphs G_1, G_2 such that $G_1 = F(x_1)$ and $G_2 = F(x_2)$ and the edges of G_1 misses the edges of G_2.

Observe that every graph-yielding function which is trivial is also vertex-disjoint and edge-disjoint and every graph-yielding function which is vertex-disjoint is also one-to-one.

Let F be a non empty, graph-yielding function. Let us observe that F is vertex-disjoint if and only if the condition (Def. 22) is satisfied.

(Def. 22) for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds the vertices of $F(x_1)$ misses the vertices of $F(x_2)$.

Observe that F is edge-disjoint if and only if the condition (Def. 23) is satisfied.

(Def. 23) for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds the edges of $F(x_1)$ misses the edges of $F(x_2)$.

Let us consider a non empty, graph-yielding function F. Now we state the propositions:

(70) F is vertex-disjoint if and only if for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds (the vertices of F)(x_1) misses (the vertices of F)(x_2).

(71) F is edge-disjoint if and only if for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds (the edges of F)(x_1) misses (the edges of F)(x_2).

(72) F is vertex-disjoint and edge-disjoint if and only if for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds the vertices of $F(x_1)$ misses the vertices of $F(x_2)$ and the edges of $F(x_1)$ misses the edges of $F(x_2)$.

(73) F is vertex-disjoint and edge-disjoint if and only if for every elements x_1, x_2 of $\operatorname{dom} F$ such that $x_1 \neq x_2$ holds (the vertices of $F)(x_1)$ misses (the vertices of $F)(x_2)$ and (the edges of $F)(x_1)$ misses (the edges of $F)(x_2)$. The theorem is a consequence of (70) and (71).

Let x be an object and G be a graph. One can check that $x \dot{\longmapsto} G$ is vertex-disjoint and edge-disjoint and $\langle G \rangle$ is vertex-disjoint and edge-disjoint and there exists a graph-yielding function which is non empty, vertex-disjoint, and edge-disjoint.

Let F be a vertex-disjoint, graph-yielding function. Observe that $\operatorname{rng} F$ is vertex-disjoint.

Let F be an edge-disjoint, graph-yielding function. Let us note that $\operatorname{rng} F$ is edge-disjoint.

Let us consider non empty, one-to-one, graph-yielding functions F_1, F_2. Now we state the propositions:

(74) If F_1 and F_2 are directed-isomorphic, then $\operatorname{rng} F_1$ and $\operatorname{rng} F_2$ are directed-isomorphic.

(75) If F_1 and F_2 are isomorphic, then $\operatorname{rng} F_1$ and $\operatorname{rng} F_2$ are isomorphic.

Let us consider graphs G_1, G_2. Now we state the propositions:

(76) $\langle G_1, G_2 \rangle$ is vertex-disjoint if and only if the vertices of G_1 misses the vertices of G_2.

(77) $\langle G_1, G_2 \rangle$ is edge-disjoint if and only if the edges of G_1 misses the edges of G_2.

5. Distinguishing the Range of a Graph-yielding Function

Let f be a function and x be an object. The functor $\coprod(f, x)$ yielding a many sorted set indexed by $f(x)$ is defined by the term

(Def. 24) $\langle f(x) \longmapsto \langle\!\langle f, x \rangle\!\rangle, \mathrm{id}_{f(x)} \rangle$.

Now we state the propositions:

(78) Let us consider a function f, and objects x, y. Suppose $x \in \operatorname{dom} f$ and $y \in f(x)$. Then $\coprod(f, x)(y) = \langle\!\langle f, x, y \rangle\!\rangle$.

(79) Let us consider a function f, and objects x, z. Suppose $x \in \operatorname{dom} f$ and $z \in \operatorname{rng} \coprod(f, x)$. Then there exists an object y such that

(i) $y \in f(x)$, and

(ii) $z = \langle\!\langle f, x, y \rangle\!\rangle$.

The theorem is a consequence of (78).

(80) Let us consider a function f, and an object x. Then $\operatorname{rng} \coprod(f, x) = \{\langle f, x\rangle\} \times f(x)$. The theorem is a consequence of (79) and (78).

Let us consider a function f and objects x_1, x_2. Now we state the propositions:

(81) $\operatorname{rng} \coprod(f, x_1)$ misses $f(x_2)$. The theorem is a consequence of (79).

(82) If $x_1 \neq x_2$, then $\operatorname{rng} \coprod(f, x_1)$ misses $\operatorname{rng} \coprod(f, x_2)$. The theorem is a consequence of (79).

Let f be a function and x be an object. One can verify that $\coprod(f, x)$ is one-to-one.

Let f be an empty function. One can verify that $\coprod(f, x)$ is empty.

Let f be a non empty, non-empty function and x be an element of $\operatorname{dom} f$. One can verify that $\coprod(f, x)$ is non empty.

Let F be a non empty, graph-yielding function and x be an element of $\operatorname{dom} F$. One can check that $\coprod$(the vertices of F, x) is non empty and (the vertices of $F(x)$)-defined and $\coprod$(the edges of F, x) is (the edges of $F(x)$)-defined and $\coprod$(the vertices of F, x) is total as a (the vertices of $F(x)$)-defined function and $\coprod$(the edges of F, x) is total as a (the edges of $F(x)$)-defined function.

The functor $\coprod F$ yielding a graph-yielding function is defined by

(Def. 25) $\operatorname{dom} it = \operatorname{dom} F$ and for every element x of $\operatorname{dom} F$, $it(x) =$ replaceVerticesEdges($\coprod$(the vertices of F, x), $\coprod$(the edges of F, x)).

Note that $\coprod F$ is non empty and $\coprod F$ is plain.

Let us consider a non empty, graph-yielding function F and an element x of $\operatorname{dom} F$. Now we state the propositions:

(83) (The vertices of $\coprod F)(x) = \{\langle$the vertices of F, $x\rangle\} \times$ (the vertices of $F)(x)$. The theorem is a consequence of (1) and (80).

(84) (The edges of $\coprod F)(x) = \{\langle$the edges of F, $x\rangle\} \times$ (the edges of $F)(x)$. The theorem is a consequence of (1) and (80).

Let F be a non empty, graph-yielding function. Note that $\coprod F$ is vertex-disjoint and edge-disjoint.

Let us consider a non empty, graph-yielding function F, an element x of $\operatorname{dom} F$, and an element x' of $\operatorname{dom}(\coprod F)$. Now we state the propositions:

(85) Suppose $x = x'$. Then there exists a partial graph mapping G from $F(x)$ to $(\coprod F)(x')$ such that

(i) $G_{\mathbb{V}} = \coprod$(the vertices of F, x), and

(ii) $G_{\mathbb{E}} = \coprod$(the edges of F, x), and

(iii) G is directed-isomorphism.

The theorem is a consequence of (16).

(86) If $x = x'$, then $(\coprod F)(x')$ is $F(x)$-directed-isomorphic. The theorem is a consequence of (85).

(87) Let us consider a non empty, graph-yielding function F. Then F and $\coprod F$ are directed-isomorphic. The theorem is a consequence of (86) and (38).

Let us consider non empty, graph-yielding functions F_1, F_2. Now we state the propositions:

(88) If F_1 and F_2 are directed-isomorphic, then $\coprod F_1$ and $\coprod F_2$ are directed-isomorphic. The theorem is a consequence of (87) and (40).

(89) If F_1 and F_2 are isomorphic, then $\coprod F_1$ and $\coprod F_2$ are isomorphic. The theorem is a consequence of (42), (87), and (41).

Let us consider a non empty, graph-yielding function F, an element x of $\operatorname{dom} F$, an element x' of $\operatorname{dom}(\coprod F)$, and objects v, e, w. Now we state the propositions:

(90) Suppose $x = x'$. Then suppose e joins v to w in $F(x)$. Then $\langle$the edges of F, x, $e\rangle$ joins $\langle$the vertices of F, x, $v\rangle$ to $\langle$the vertices of F, x, $w\rangle$ in $(\coprod F)(x')$. The theorem is a consequence of (85) and (78).

(91) Suppose $x = x'$. Then suppose e joins v and w in $F(x)$. Then $\langle$the edges of F, x, $e\rangle$ joins $\langle$the vertices of F, x, $v\rangle$ and $\langle$the vertices of F, x, $w\rangle$ in $(\coprod F)(x')$. The theorem is a consequence of (90).

Let us consider a non empty, graph-yielding function F, an element x of $\operatorname{dom} F$, an element x' of $\operatorname{dom}(\coprod F)$, and objects v', e', w'. Now we state the propositions:

(92) Suppose $x = x'$ and e' joins v' to w' in $(\coprod F)(x')$. Then there exist objects v, e, w such that

(i) e joins v to w in $F(x)$, and

(ii) $e' = \langle$the edges of F, x, $e\rangle$, and

(iii) $v' = \langle$the vertices of F, x, $v\rangle$, and

(iv) $w' = \langle$the vertices of F, x, $w\rangle$.

The theorem is a consequence of (85), (83), (80), (79), (84), and (78).

(93) Suppose $x = x'$ and e' joins v' and w' in $(\coprod F)(x')$. Then there exist objects v, e, w such that

(i) e joins v and w in $F(x)$, and

(ii) $e' = \langle$the edges of F, x, $e\rangle$, and

(iii) $v' = \langle$the vertices of F, x, $v\rangle$, and

(iv) $w' = \langle$the vertices of F, x, $w\rangle$.

The theorem is a consequence of (92).

Let F be a non empty, loopless, graph-yielding function. One can verify that $\coprod F$ is loopless.

Let F be a non empty, non loopless, graph-yielding function. Note that $\coprod F$ is non loopless.

Let F be a non empty, non-multi, graph-yielding function. Observe that $\coprod F$ is non-multi.

Let F be a non empty, non non-multi, graph-yielding function. One can verify that $\coprod F$ is non non-multi.

Let F be a non empty, non-directed-multi, graph-yielding function. Note that $\coprod F$ is non-directed-multi.

Let F be a non empty, non non-directed-multi, graph-yielding function. One can verify that $\coprod F$ is non non-directed-multi.

Let F be a non empty, simple, graph-yielding function. Observe that $\coprod F$ is simple.

Let F be a non empty, directed-simple, graph-yielding function. One can check that $\coprod F$ is directed-simple.

Let F be a non empty, acyclic, graph-yielding function. Let us observe that $\coprod F$ is acyclic.

Let F be a non empty, non acyclic, graph-yielding function. One can check that $\coprod F$ is non acyclic.

Let F be a non empty, connected, graph-yielding function. Let us note that $\coprod F$ is connected.

Let F be a non empty, non connected, graph-yielding function. Let us observe that $\coprod F$ is non connected.

Let F be a non empty, tree-like, graph-yielding function. One can check that $\coprod F$ is tree-like.

Let F be a non empty, edgeless, graph-yielding function. Observe that $\coprod F$ is edgeless.

Let F be a non empty, non edgeless, graph-yielding function. One can verify that $\coprod F$ is non edgeless.

Let F be a non empty, graph-yielding function and z be an element of $\operatorname{dom} F$. The functor $\coprod(F, z)$ yielding a graph-yielding function is defined by the term

(Def. 26) $\coprod F +\cdot (z, F(z) \restriction (\text{the graph selectors}))$.

Let us note that $\coprod(F, z)$ is non empty. Now we state the propositions:

(94) Let us consider a non empty, graph-yielding function F, and an element z of $\operatorname{dom} F$. Then $\operatorname{dom} F = \operatorname{dom}(\coprod(F, z))$.

(95) Let us consider a non empty, graph-yielding function F, an element z of $\operatorname{dom} F$, and a graph-yielding function G. Then $G = \coprod(F, z)$ if and only

if $\operatorname{dom} G = \operatorname{dom} F$ and $G(z) = F(z)\restriction$(the graph selectors) and for every element x of $\operatorname{dom} F$ such that $x \neq z$ holds $G(x) =$ replaceVerticesEdges($\coprod$(the vertices of F, x), $\coprod$(the edges of F, x)).

(96) Let us consider a non empty, graph-yielding function F, and an element z of $\operatorname{dom} F$. Then $\coprod(F, z)(z) = F(z)\restriction$(the graph selectors).

Let F be a non empty, graph-yielding function and z be an element of $\operatorname{dom} F$. Observe that $\coprod(F, z)$ is plain. Now we state the propositions:

(97) Let us consider a non empty, graph-yielding function F, and an element z of $\operatorname{dom} F$. Then (the vertices of $\coprod(F, z))(z) =$ (the vertices of $F)(z)$. The theorem is a consequence of (94) and (96).

(98) Let us consider a non empty, graph-yielding function F, and elements x, z of $\operatorname{dom} F$. Suppose $x \neq z$. Then (the vertices of $\coprod(F, z))(x) =$ (the vertices of $\coprod F)(x)$. The theorem is a consequence of (95).

Let us consider a non empty, graph-yielding function F and an element z of $\operatorname{dom} F$. Now we state the propositions:

(99) The vertices of $\coprod(F, z) =$ (the vertices of $\coprod F) +\cdot (z,$ the vertices of $F(z))$. The theorem is a consequence of (97) and (98).

(100) (The edges of $\coprod(F, z))(z) =$ (the edges of $F)(z)$. The theorem is a consequence of (94) and (96).

(101) Let us consider a non empty, graph-yielding function F, and elements x, z of $\operatorname{dom} F$. Suppose $x \neq z$. Then (the edges of $\coprod(F, z))(x) =$ (the edges of $\coprod F)(x)$. The theorem is a consequence of (95).

(102) Let us consider a non empty, graph-yielding function F, and an element z of $\operatorname{dom} F$. Then the edges of $\coprod(F, z) =$ (the edges of $\coprod F) +\cdot (z,$ the edges of $F(z))$. The theorem is a consequence of (100) and (101).

Let F be a non empty, graph-yielding function and z be an element of $\operatorname{dom} F$. Let us note that $\coprod(F, z)$ is vertex-disjoint and edge-disjoint.

Let us consider a non empty, graph-yielding function F, elements x, z of $\operatorname{dom} F$, and an element x' of $\operatorname{dom}(\coprod(F, z))$. Now we state the propositions:

(103) Suppose $x \neq z$ and $x = x'$. Then there exists a partial graph mapping G from $F(x)$ to $\coprod(F, z)(x')$ such that

- (i) $G_{\mathbb{V}} = \coprod$(the vertices of F, x), and
- (ii) $G_{\mathbb{E}} = \coprod$(the edges of F, x), and
- (iii) G is directed-isomorphism.

The theorem is a consequence of (85).

(104) If $x = x'$, then $\coprod(F, z)(x')$ is $(F(x))$-directed-isomorphic. The theorem is a consequence of (96) and (103).

Let us consider a non empty, graph-yielding function F and an element z of $\operatorname{dom} F$. Now we state the propositions:

(105) F and $\coprod(F, z)$ are directed-isomorphic. The theorem is a consequence of (104) and (38).

(106) $\coprod F$ and $\coprod(F, z)$ are directed-isomorphic. The theorem is a consequence of (87), (105), and (40).

(107) Let us consider non empty, graph-yielding functions F_1, F_2, an element z_1 of $\operatorname{dom} F_1$, and an element z_2 of $\operatorname{dom} F_2$. Suppose F_1 and F_2 are directed-isomorphic. Then $\coprod(F_1, z_1)$ and $\coprod(F_2, z_2)$ are directed-isomorphic. The theorem is a consequence of (105) and (40).

Let us consider a non empty, graph-yielding function F, an element z of $\operatorname{dom} F$, an element z' of $\operatorname{dom}(\coprod(F, z))$, and objects v, e, w. Now we state the propositions:

(108) If $z = z'$, then e joins v to w in $F(z)$ iff e joins v to w in $\coprod(F, z)(z')$. The theorem is a consequence of (96).

(109) If $z = z'$, then e joins v and w in $F(z)$ iff e joins v and w in $\coprod(F, z)(z')$. The theorem is a consequence of (96).

Let us consider a non empty, graph-yielding function F, elements x, z of $\operatorname{dom} F$, an element x' of $\operatorname{dom}(\coprod(F, z))$, and objects v, e, w. Now we state the propositions:

(110) Suppose $x \neq z$ and $x = x'$. Then suppose e joins v to w in $F(x)$. Then $\langle$the edges of F, x, $e\rangle$ joins $\langle$the vertices of F, x, $v\rangle$ to $\langle$the vertices of F, x, $w\rangle$ in $\coprod(F, z)(x')$. The theorem is a consequence of (90).

(111) Suppose $x \neq z$ and $x = x'$. Then suppose e joins v and w in $F(x)$. Then $\langle$the edges of F, x, $e\rangle$ joins $\langle$the vertices of F, x, $v\rangle$ and $\langle$the vertices of F, x, $w\rangle$ in $\coprod(F, z)(x')$. The theorem is a consequence of (91).

Let us consider a non empty, graph-yielding function F, elements x, z of $\operatorname{dom} F$, an element x' of $\operatorname{dom}(\coprod(F, z))$, and objects v', e', w'. Now we state the propositions:

(112) Suppose $x \neq z$ and $x = x'$ and e' joins v' to w' in $\coprod(F, z)(x')$. Then there exist objects v, e, w such that

(i) e joins v to w in $F(x)$, and

(ii) $e' = \langle$the edges of F, x, $e\rangle$, and

(iii) $v' = \langle$the vertices of F, x, $v\rangle$, and

(iv) $w' = \langle$the vertices of F, x, $w\rangle$.

The theorem is a consequence of (92).

(113) Suppose $x \neq z$ and $x = x'$ and e' joins v' and w' in $\coprod(F, z)(x')$. Then there exist objects v, e, w such that

(i) e joins v and w in $F(x)$, and

(ii) $e' = \langle$the edges of F, x, $e\rangle$, and

(iii) $v' = \langle$the vertices of F, x, $v\rangle$, and

(iv) $w' = \langle$the vertices of F, x, $w\rangle$.

The theorem is a consequence of (93).

Let F be a non empty, loopless, graph-yielding function and z be an element of $\operatorname{dom} F$. One can check that $\coprod(F, z)$ is loopless.

Let F be a non empty, non loopless, graph-yielding function. Let us observe that $\coprod(F, z)$ is non loopless.

Let F be a non empty, non-multi, graph-yielding function. Let us note that $\coprod(F, z)$ is non-multi.

Let F be a non empty, non non-multi, graph-yielding function. One can check that $\coprod(F, z)$ is non non-multi.

Let F be a non empty, non-directed-multi, graph-yielding function. Let us observe that $\coprod(F, z)$ is non-directed-multi.

Let F be a non empty, non non-directed-multi, graph-yielding function. Let us observe that $\coprod(F, z)$ is non non-directed-multi.

Let F be a non empty, simple, graph-yielding function. Let us observe that $\coprod(F, z)$ is simple.

Let F be a non empty, directed-simple, graph-yielding function. Note that $\coprod(F, z)$ is directed-simple.

Let F be a non empty, acyclic, graph-yielding function. Let us observe that $\coprod(F, z)$ is acyclic.

Let F be a non empty, non acyclic, graph-yielding function. Let us note that $\coprod(F, z)$ is non acyclic.

Let F be a non empty, connected, graph-yielding function. One can check that $\coprod(F, z)$ is connected.

Let F be a non empty, non connected, graph-yielding function. Let us observe that $\coprod(F, z)$ is non connected.

Let F be a non empty, tree-like, graph-yielding function. Let us note that $\coprod(F, z)$ is tree-like.

Let F be a non empty, edgeless, graph-yielding function. One can verify that $\coprod(F, z)$ is edgeless.

Let F be a non empty, non edgeless, graph-yielding function. Observe that $\coprod(F, z)$ is non edgeless.

Let us consider graphs G_2, H and a partial graph mapping F from G_2 to H. Now we state the propositions:

(114) If F is directed and weak subgraph embedding, then there exists a supergraph G_1 of G_2 such that G_1 is H-directed-isomorphic.
PROOF: Set $c =$ (the vertices of H) $\longmapsto$ (the vertices of G_2). $\operatorname{rng}\langle c, \mathrm{id}_\alpha\rangle \cap \operatorname{rng}(F_{\mathbb{V}})^{-1} = \emptyset$, where α is the vertices of H. Set $d =$ (the edges of H) $\longmapsto$ (the edges of G_2). $\operatorname{rng}\langle d, \mathrm{id}_\alpha\rangle \cap \operatorname{rng}(F_{\mathbb{E}})^{-1} = \emptyset$, where α is the edges of H. $\square$

(115) If F is weak subgraph embedding, then there exists a supergraph G_1 of G_2 such that G_1 is H-isomorphic. The theorem is a consequence of (114).

6. The Sum of Graphs

Let F be a non empty, graph-yielding function.

A graph sum of F is a graph defined by

(Def. 27) there exists a graph union G' of $\operatorname{rng} \coprod F$ such that *it* is G'-directed-isomorphic.

Now we state the proposition:

(116) Let us consider a non empty, graph-yielding function F, a graph sum S of F, and a graph union G' of $\operatorname{rng} \coprod F$. Then S is G'-directed-isomorphic.

Let us consider non empty, graph-yielding functions F_1, F_2, a graph sum S_1 of F_1, and a graph sum S_2 of F_2. Now we state the propositions:

(117) If F_1 and F_2 are directed-isomorphic, then S_2 is S_1-directed-isomorphic. The theorem is a consequence of (74), (88), (55), and (116).

(118) If F_1 and F_2 are isomorphic, then S_2 is S_1-isomorphic. The theorem is a consequence of (89), (57), (75), and (116).

Now we state the propositions:

(119) Let us consider a non empty, graph-yielding function F, and graph sums S_1, S_2 of F. Then S_2 is S_1-directed-isomorphic.

(120) Let us consider an object x, and a graph G. Then every graph sum of $x \dot{\longmapsto} G$ is G-directed-isomorphic. The theorem is a consequence of (17).

(121) Let us consider a non empty, graph-yielding function F, and a graph sum S of F. Suppose S is connected. Then there exists an object x and there exists a connected graph G such that $F = x \dot{\longmapsto} G$. The theorem is a consequence of (59) and (120).

Let X be a non empty set. Observe that there exists a graph-yielding many sorted set indexed by X which is non empty, vertex-disjoint, and edge-disjoint.

Now we state the propositions:

(122) Let us consider a non empty, graph-yielding function F, an element x of $\operatorname{dom} F$, and a graph sum S of F. Then there exists a partial graph

mapping M from $F(x)$ to S such that M is strong subgraph embedding. The theorem is a consequence of (62) and (17).

(123) Let us consider a non empty, graph-yielding function F, and an element z of $\operatorname{dom} F$. Then there exists a graph sum S of F such that S is supergraph of $F(z)$ and graph union of $\operatorname{rng} \coprod(F, z)$. The theorem is a consequence of (106), (55), (74), (94), and (95).

(124) Let us consider a non empty, graph-yielding function F, and a graph sum S of F. Then

(i) F is loopless iff S is loopless, and

(ii) F is non-multi iff S is non-multi, and

(iii) F is non-directed-multi iff S is non-directed-multi, and

(iv) F is simple iff S is simple, and

(v) F is directed-simple iff S is directed-simple, and

(vi) F is chordal iff S is chordal, and

(vii) F is edgeless iff S is edgeless, and

(viii) F is loopfull iff S is loopfull.

Let F be a non empty, loopless, graph-yielding function. Observe that every graph sum of F is loopless.

Let F be a non empty, non loopless, graph-yielding function. Note that every graph sum of F is non loopless.

Let F be a non empty, non-directed-multi, graph-yielding function. One can verify that every graph sum of F is non-directed-multi.

Let F be a non empty, non non-directed-multi, graph-yielding function. Observe that every graph sum of F is non non-directed-multi.

Let F be a non empty, non-multi, graph-yielding function. Note that every graph sum of F is non-multi.

Let F be a non empty, non non-multi, graph-yielding function. One can verify that every graph sum of F is non non-multi.

Let F be a non empty, simple, graph-yielding function. Observe that every graph sum of F is simple.

Let F be a non empty, directed-simple, graph-yielding function. Observe that every graph sum of F is directed-simple.

Let F be a non empty, edgeless, graph-yielding function. Observe that every graph sum of F is edgeless.

Let F be a non empty, non edgeless, graph-yielding function. Note that every graph sum of F is non edgeless.

Let F be a non empty, loopfull, graph-yielding function. One can verify that every graph sum of F is loopfull.

Let F be a non empty, non loopfull, graph-yielding function. Observe that every graph sum of F is non loopfull. Now we state the proposition:

(125) Let us consider a non empty, graph-yielding function F, and a graph sum S of F. Then

(i) F is acyclic iff S is acyclic, and

(ii) F is chordal iff S is chordal.

The theorem is a consequence of (87), (42), (60), (48), and (63).

Let F be a non empty, acyclic, graph-yielding function. Let us note that every graph sum of F is acyclic.

Let F be a non empty, non acyclic, graph-yielding function. One can check that every graph sum of F is non acyclic.

Now we state the propositions:

(126) Let us consider a non empty, graph-yielding function F, and a graph sum S of F. Then $\overline{\overline{F}} \subseteq S$.numComponents(). The theorem is a consequence of (68).

(127) Let us consider a non empty, connected, graph-yielding function F, and a graph sum S of F. Then $\overline{\overline{F}} = S$.numComponents(). The theorem is a consequence of (69).

7. The Sum of two Graphs

Let G_1, G_2 be graphs.

A graph sum of G_1 and G_2 is a supergraph of G_1 defined by

(Def. 28) *it* is a graph sum of $\langle G_1, G_2 \rangle$.

Now we state the proposition:

(128) Let us consider graphs G_1, G_2, and a graph sum S of G_1 and G_2. Then

(i) G_1 is loopless and G_2 is loopless iff S is loopless, and

(ii) G_1 is non-multi and G_2 is non-multi iff S is non-multi, and

(iii) G_1 is non-directed-multi and G_2 is non-directed-multi iff S is non-directed-multi, and

(iv) G_1 is simple and G_2 is simple iff S is simple, and

(v) G_1 is directed-simple and G_2 is directed-simple iff S is directed-simple, and

(vi) G_1 is acyclic and G_2 is acyclic iff S is acyclic, and

(vii) G_1 is chordal and G_2 is chordal iff S is chordal, and

(viii) G_1 is edgeless and G_2 is edgeless iff S is edgeless, and

(ix) G_1 is loopfull and G_2 is loopfull iff S is loopfull.

The theorem is a consequence of (124).

Let G_1, G_2 be loopless graphs. Note that every graph sum of G_1 and G_2 is loopless.

Let G_1, G_2 be non loopless graphs. Let us observe that every graph sum of G_1 and G_2 is non loopless.

Let G_1, G_2 be non-directed-multi graphs. Let us note that every graph sum of G_1 and G_2 is non-directed-multi.

Let G_1, G_2 be non non-directed-multi graphs. One can verify that every graph sum of G_1 and G_2 is non non-directed-multi.

Let G_1, G_2 be non-multi graphs. Observe that every graph sum of G_1 and G_2 is non-multi.

Let G_1, G_2 be non non-multi graphs. One can check that every graph sum of G_1 and G_2 is non non-multi.

Let G_1, G_2 be simple graphs. Let us observe that every graph sum of G_1 and G_2 is simple.

Let G_1, G_2 be directed-simple graphs. Observe that every graph sum of G_1 and G_2 is directed-simple.

Let G_1, G_2 be acyclic graphs. Let us note that every graph sum of G_1 and G_2 is acyclic.

Let G_1, G_2 be non acyclic graphs. One can verify that every graph sum of G_1 and G_2 is non acyclic.

Let G_1, G_2 be edgeless graphs. Observe that every graph sum of G_1 and G_2 is edgeless.

Let G_1, G_2 be non edgeless graphs. One can check that every graph sum of G_1 and G_2 is non edgeless.

Let G_1, G_2 be loopfull graphs. Let us observe that every graph sum of G_1 and G_2 is loopfull.

Let G_1, G_2 be non loopfull graphs. Note that every graph sum of G_1 and G_2 is non loopfull.

Let us consider graphs G_1, G_2 and a graph sum S of G_1 and G_2. Now we state the propositions:

(129) $S.\mathrm{order}() = G_1.\mathrm{order}() + G_2.\mathrm{order}()$.

(130) $S.\mathrm{size}() = G_1.\mathrm{size}() + G_2.\mathrm{size}()$.

(131) $S.\mathrm{numComponents}() = G_1.\mathrm{numComponents}() + G_2.\mathrm{numComponents}()$.

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Improper Integral. Part II

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. In this article, using the Mizar system [2], [3], we deal with Riemann's improper integral on infinite interval [1]. As with [4], we referred to [6], which discusses improper integrals of finite values.

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1. Properties of Extended Riemann Integral on Infinite Interval

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(1) If f is divergent in $-\infty$ to $+\infty$, then f is not convergent in $-\infty$ and f is not divergent in $-\infty$ to $-\infty$.

(2) If f is divergent in $-\infty$ to $-\infty$, then f is not convergent in $-\infty$ and f is not divergent in $-\infty$ to $+\infty$.

(3) If f is divergent in $+\infty$ to $+\infty$, then f is not convergent in $+\infty$ and f is not divergent in $+\infty$ to $-\infty$.

(4) If f is divergent in $+\infty$ to $-\infty$, then f is not convergent in $+\infty$ and f is not divergent in $+\infty$ to $+\infty$.

(5) Suppose f is convergent in $-\infty$. Then

(i) there exists a real number r such that $f{\restriction}]-\infty, r[$ is lower bounded, and

(ii) there exists a real number r such that $f{\restriction}]-\infty, r[$ is upper bounded.

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Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]{-\infty}, r[)$ holds $-1 + g < (f{\restriction}]{-\infty}, r[)(r_1)$. Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]{-\infty}, r[)$ holds $(f{\restriction}]{-\infty}, r[)(r_1) < g + 1$. □

(6) Suppose f is convergent in $+\infty$. Then

(i) there exists a real number r such that $f{\restriction}]r, +\infty[$ is lower bounded, and

(ii) there exists a real number r such that $f{\restriction}]r, +\infty[$ is upper bounded.

Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]r, +\infty[)$ holds $-1 + g < (f{\restriction}]r, +\infty[)(r_1)$. Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]r, +\infty[)$ holds $(f{\restriction}]r, +\infty[)(r_1) < g + 1$. □

(7) Suppose f is divergent in $-\infty$ to $+\infty$. Then there exists a real number r such that $f{\restriction}]{-\infty}, r[$ is lower bounded.
Proof: Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $1 < f(r_1)$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]{-\infty}, r[)$ holds $1 < (f{\restriction}]{-\infty}, r[)(r_1)$. □

(8) Suppose f is divergent in $-\infty$ to $-\infty$. Then there exists a real number r such that $f{\restriction}]{-\infty}, r[$ is upper bounded.
Proof: Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $f(r_1) < 1$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]{-\infty}, r[)$ holds $(f{\restriction}]{-\infty}, r[)(r_1) < 1$. □

(9) Suppose f is divergent in $+\infty$ to $+\infty$. Then there exists a real number r such that $f{\restriction}]r, +\infty[$ is lower bounded.
Proof: Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $1 < f(r_1)$. For every object r_1 such that $r_1 \in \mathrm{dom}(f{\restriction}]r, +\infty[)$ holds $1 < (f{\restriction}]r, +\infty[)(r_1)$. □

(10) Suppose f is divergent in $+\infty$ to $-\infty$. Then there exists a real number r such that $f{\restriction}]r, +\infty[$ is upper bounded.

Proof: Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]r, +\infty[)$ holds $(f \restriction]r, +\infty[)(r_1) < 1$. □

Let us consider partial functions f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(11) Suppose f_1 is divergent in $-\infty$ to $-\infty$ and for every real number r, there exists a real number g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $f_2 \restriction]-\infty, r[$ is upper bounded. Then $f_1 + f_2$ is divergent in $-\infty$ to $-\infty$.

(12) Suppose f_1 is divergent in $+\infty$ to $-\infty$ and for every real number r, there exists a real number g such that $r < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $f_2 \restriction]r, +\infty[$ is upper bounded. Then $f_1 + f_2$ is divergent in $+\infty$ to $-\infty$.

(13) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number d. Suppose $]-\infty, d] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, d. Let us consider real numbers b, c. Suppose $b < c \leqslant d$. Then f is right extended Riemann integrable on b, c and left extended Riemann integrable on b, c.

(14) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Let us consider real numbers b, c. Suppose $a \leqslant b < c$. Then f is right extended Riemann integrable on b, c and left extended Riemann integrable on b, c.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a real number b. Now we state the propositions:

(15) Suppose $]-\infty, a] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, a. Then if $b \leqslant a$, then f is extended Riemann integrable on $-\infty$, b.

(16) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Then if $a \leqslant b$, then f is extended Riemann integrable on b, $+\infty$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(17) Suppose $a \leqslant b$ and $]-\infty, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and f is extended Riemann integrable on $-\infty$, a. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $(R^{<}) \int\limits_{-\infty}^{b} f(x)dx = (R^{<}) \int\limits_{-\infty}^{a} f(x)dx + \int\limits_{a}^{b} f(x)dx$.

Proof: For every real number c such that $c \leqslant b$ holds f is integrable on

$[c,b]$ and $f{\restriction}[c,b]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]-\infty, a]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{x}^{a} f(x)dx$ and I is convergent in $-\infty$. Reconsider $B =]-\infty, b]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } B) = (\int\limits_{\$_1}^{b} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from B into $\mathbb{R}$ such that for every element x of B, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. For every real number r, there exists a real number g such that $g < r$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_{a}^{b} f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. $\square$

(18) Suppose $a \leqslant b$ and $[a, +\infty[\subseteq \operatorname{dom} f$ and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and f is extended Riemann integrable on b, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $(R^{>}) \int\limits_{a}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx + \int\limits_{a}^{b} f(x)dx$.

PROOF: For every real number c such that $a \leqslant c$ holds f is integrable on $[a,c]$ and $f{\restriction}[a,c]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [b, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{b}^{x} f(x)dx$ and I is convergent in $+\infty$. Reconsider $A = [a, +\infty[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_{a}^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. For every real number r, there exists a real number g such that $r < g$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_a^b f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$ by [5, (17)]. □

(19) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$. Then f is ∞-extended Riemann integrable if and only if for every real number a, f is extended Riemann integrable on a, $+\infty$ and extended Riemann integrable on $-\infty$, a. The theorem is a consequence of (16), (17), (18), and (15).

2. Improper Integral on Infinite Interval

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. We say that f is improper integrable on $]-\infty, b]$ if and only if

(Def. 1) for every real number a such that $a \leqslant b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and (I_1 is convergent in $-\infty$ or divergent in $-\infty$ to $+\infty$ or I_1 is divergent in $-\infty$ to $-\infty$).

Let a be a real number. We say that f is improper integrable on $[a, +\infty[$ if and only if

(Def. 2) for every real number b such that $a \leqslant b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and (I_1 is convergent in $+\infty$ or divergent in $+\infty$ to $+\infty$ or I_1 is divergent in $+\infty$ to $-\infty$).

Let b be a real number. Assume f is improper integrable on $]-\infty, b]$. The functor $\int\limits_{-\infty}^b f(x)dx$ yielding an extended real is defined by

(Def. 3) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$

and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and (I_1 is convergent in $-\infty$ and $it = \lim_{-\infty} I_1$ or I_1 is divergent in $-\infty$ to $+\infty$ and $it = +\infty$ or I_1 is divergent in $-\infty$ to $-\infty$ and $it = -\infty$).

Let a be a real number. Assume f is improper integrable on $[a, +\infty[$. The functor $\int\limits_a^{+\infty} f(x)dx$ yielding an extended real is defined by

(Def. 4) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and (I_1 is convergent in $+\infty$ and $it = \lim_{+\infty} I_1$ or I_1 is divergent in $+\infty$ to $+\infty$ and $it = +\infty$ or I_1 is divergent in $+\infty$ to $-\infty$ and $it = -\infty$).

Now we state the propositions:

(20) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is extended Riemann integrable on $-\infty$, b. Then f is improper integrable on $]-\infty, b]$.

(21) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose f is extended Riemann integrable on a, $+\infty$. Then f is improper integrable on $[a, +\infty[$.

(22) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $]-\infty, b]$. Then

(i) f is extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = (R^<) \int\limits_{-\infty}^b f(x)dx$, or

(ii) f is not extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = +\infty$, or

(iii) f is not extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = -\infty$.

The theorem is a consequence of (1) and (2).

(23) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) =$

$\int\limits_x^b f(x)dx$ and I_1 is divergent in $-\infty$ to $+\infty$ or divergent in $-\infty$ to $-\infty$. Then f is not extended Riemann integrable on $-\infty$, b. The theorem is a consequence of (1) and (2).

(24) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $]-\infty, b]$ and $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and I_1 is convergent in $-\infty$. Then $\int\limits_{-\infty}^b f(x)dx = \lim_{-\infty} I_1$. The theorem is a consequence of (22).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers b, c. Now we state the propositions:

(25) Suppose $b \leqslant c$ and $]-\infty, c] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, c]$. Then

(i) f is improper integrable on $]-\infty, b]$, and

(ii) if $\int\limits_{-\infty}^c f(x)dx = (R^<) \int\limits_{-\infty}^c f(x)dx$, then $\int\limits_{-\infty}^b f(x)dx = (R^<) \int\limits_{-\infty}^b f(x)dx$, and

(iii) if $\int\limits_{-\infty}^c f(x)dx = +\infty$, then $\int\limits_{-\infty}^b f(x)dx = +\infty$, and

(iv) if $\int\limits_{-\infty}^c f(x)dx = -\infty$, then $\int\limits_{-\infty}^b f(x)dx = -\infty$.

The theorem is a consequence of (22).

(26) Suppose $b \leqslant c$ and $]-\infty, c] \subseteq \operatorname{dom} f$ and $f \restriction [b, c]$ is bounded and f is improper integrable on $]-\infty, b]$ and f is integrable on $[b, c]$. Then

(i) f is improper integrable on $]-\infty, c]$, and

(ii) if $\int\limits_{-\infty}^b f(x)dx = (R^<) \int\limits_{-\infty}^b f(x)dx$, then

$\int\limits_{-\infty}^c f(x)dx = \int\limits_{-\infty}^b f(x)dx + \int\limits_b^c f(x)dx$, and

(iii) if $\int\limits_{-\infty}^{b} f(x)dx = +\infty$, then $\int\limits_{-\infty}^{c} f(x)dx = +\infty$, and

(iv) if $\int\limits_{-\infty}^{b} f(x)dx = -\infty$, then $\int\limits_{-\infty}^{c} f(x)dx = -\infty$.

The theorem is a consequence of (22).

(27) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $[b, +\infty[$. Then

(i) f is extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, or

(ii) f is not extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, or

(iii) f is not extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (3) and (4).

(28) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [b, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{b}^{x} f(x)dx$ and I_1 is divergent in $+\infty$ to $+\infty$ or divergent in $+\infty$ to $-\infty$. Then f is not extended Riemann integrable on b, $+\infty$. The theorem is a consequence of (3) and (4).

(29) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $[b, +\infty[$ and $\operatorname{dom} I_1 = [b, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{b}^{x} f(x)dx$ and I_1 is convergent in $+\infty$. Then $\int\limits_{b}^{+\infty} f(x)dx = \lim\limits_{+\infty} I_1$. The theorem is a consequence of (27).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers b, c. Now we state the propositions:

(30) Suppose $b \geqslant c$ and $[c, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[c, +\infty[$. Then

(i) f is improper integrable on $[b, +\infty[$, and

(ii) if $\int\limits_{c}^{+\infty} f(x)dx = (R^{>}) \int\limits_{c}^{+\infty} f(x)dx$, then $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, and

(iii) if $\int\limits_{c}^{+\infty} f(x)dx = +\infty$, then $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, and

(iv) if $\int\limits_{c}^{+\infty} f(x)dx = -\infty$, then $\int\limits_{b}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (27).

(31) Suppose $b \geqslant c$ and $[c, +\infty[\subseteq \operatorname{dom} f$ and $f \restriction [c, b]$ is bounded and f is improper integrable on $[b, +\infty[$ and f is integrable on $[c, b]$. Then

(i) f is improper integrable on $[c, +\infty[$, and

(ii) if $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, then $\int\limits_{c}^{+\infty} f(x)dx = \int\limits_{b}^{+\infty} f(x)dx + \int\limits_{c}^{b} f(x)dx$, and

(iii) if $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, then $\int\limits_{c}^{+\infty} f(x)dx = +\infty$, and

(iv) if $\int\limits_{b}^{+\infty} f(x)dx = -\infty$, then $\int\limits_{c}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (27).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is improper integrable on $\mathbb{R}$ if and only if

(Def. 5) there exists a real number r such that f is improper integrable on $]-\infty, r]$ and f is improper integrable on $[r, +\infty[$ and it is not true that $\int\limits_{-\infty}^{r} f(x)dx = -\infty$ and $\int\limits_{r}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{r} f(x)dx = +\infty$ and $\int\limits_{r}^{+\infty} f(x)dx = -\infty$.

Now we state the propositions:

(32) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is improper integrable on $\mathbb{R}$. Then there exists a real number b such that $\int\limits_{-\infty}^{b} f(x)dx = (R^{<}) \int\limits_{-\infty}^{b} f(x)dx$ and $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$ or $\int\limits_{-\infty}^{b} f(x)dx + \int\limits_{b}^{+\infty} f(x)dx = +\infty$ or $\int\limits_{-\infty}^{b} f(x)dx + \int\limits_{b}^{+\infty} f(x)dx = -\infty$. The theorem is a consequence of (22) and (27).

(33) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $]-\infty, b]$ and f is improper integrable on $[b, +\infty[$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = -\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = +\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = -\infty$. Let us consider a real number b_1. Suppose $b_1 \leqslant b$. Then $\int\limits_{-\infty}^{b} f(x)dx + \int\limits_{b}^{+\infty} f(x)dx = \int\limits_{-\infty}^{b_1} f(x)dx + \int\limits_{b_1}^{+\infty} f(x)dx$. The theorem is a consequence of (22), (27), and (31).

(34) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $]-\infty, b]$ and f is improper integrable on $[b, +\infty[$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = -\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = +\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = -\infty$. Let us consider a real number b_2. Suppose $b \leqslant b_2$. Then $\int\limits_{-\infty}^{b} f(x)dx + \int\limits_{b}^{+\infty} f(x)dx = \int\limits_{-\infty}^{b_2} f(x)dx + \int\limits_{b_2}^{+\infty} f(x)dx$. The theorem is

a consequence of (27), (30), (31), and (22).

(35) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Let us consider real numbers b_1, b_2. Then $\int\limits_{-\infty}^{b_1} f(x)dx + \int\limits_{b_1}^{+\infty} f(x)dx = \int\limits_{-\infty}^{b_2} f(x)dx + \int\limits_{b_2}^{+\infty} f(x)dx$. The theorem is a consequence of (33) and (34).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Assume $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. The functor $\int\limits_{-\infty}^{+\infty} f(x)dx$ yielding an extended real is defined by

(Def. 6) there exists a real number c such that f is improper integrable on $]-\infty, c]$ and f is improper integrable on $[c, +\infty[$ and $it = \int\limits_{-\infty}^{c} f(x)dx + \int\limits_{c}^{+\infty} f(x)dx$.

Now we state the proposition:

(36) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) f is improper integrable on $]-\infty, b]$, and

(ii) f is improper integrable on $[b, +\infty[$, and

(iii) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int\limits_{-\infty}^{b} f(x)dx + \int\limits_{b}^{+\infty} f(x)dx$.

The theorem is a consequence of (25), (31), (35), (26), and (30).

3. Linearity of Improper Integral on Infinite Interval

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(37) Suppose f is improper integrable on $]-\infty, b]$ and $\int\limits_{-\infty}^{b} f(x)dx = +\infty$. Then suppose $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. Then I_1 is divergent in $-\infty$ to $+\infty$.

(38) Suppose f is improper integrable on $]-\infty, b]$ and $\int\limits_{-\infty}^{b} f(x)dx = -\infty$.

Then suppose $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. Then I_1 is divergent in $-\infty$ to $-\infty$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(39) Suppose f is improper integrable on $[a, +\infty[$ and $\int\limits_{a}^{+\infty} f(x)dx = +\infty$. Then suppose $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. Then I_1 is divergent in $+\infty$ to $+\infty$.

(40) Suppose f is improper integrable on $[a, +\infty[$ and $\int\limits_{a}^{+\infty} f(x)dx = -\infty$. Then suppose $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. Then I_1 is divergent in $+\infty$ to $-\infty$.

(41) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers b, r. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$. Then

(i) $r \cdot f$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b} (r \cdot f)(x)dx = r \cdot \int\limits_{-\infty}^{b} f(x)dx$.

Proof: For every real number d such that $d \leqslant b$ holds $r \cdot f$ is integrable on $[d, b]$ and $(r \cdot f)\restriction[d, b]$ is bounded. □

(42) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, r. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$. Then

(i) $r \cdot f$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty} (r \cdot f)(x)dx = r \cdot \int\limits_{a}^{+\infty} f(x)dx$.

Proof: For every real number d such that $a \leqslant d$ holds $r \cdot f$ is integrable on $[a, d]$ and $(r \cdot f)\restriction[a, d]$ is bounded. □

(43) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$. Then

(i) $-f$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(-f)(x)dx = -\int\limits_{-\infty}^{b}f(x)dx.$

The theorem is a consequence of (41).

(44) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$. Then

(i) $-f$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(-f)(x)dx = -\int\limits_{a}^{+\infty}f(x)dx.$

The theorem is a consequence of (42).

(45) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $]-\infty, b] \subseteq \operatorname{dom} g$ and f is improper integrable on $]-\infty, b]$ and g is improper integrable on $]-\infty, b]$ and it is not true that $\int\limits_{-\infty}^{b}f(x)dx = +\infty$ and $\int\limits_{-\infty}^{b}g(x)dx = -\infty$ and it is not true that $\int\limits_{-\infty}^{b}f(x)dx = -\infty$ and $\int\limits_{-\infty}^{b}g(x)dx = +\infty$. Then

(i) $f+g$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(f+g)(x)dx = \int\limits_{-\infty}^{b}f(x)dx + \int\limits_{-\infty}^{b}g(x)dx.$

Proof: For every real number d such that $d \leqslant b$ holds $f+g$ is integrable on $[d, b]$ and $(f+g)\restriction[d, b]$ is bounded. □

(46) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $[a, +\infty[\subseteq \operatorname{dom} g$ and f is improper integrable on $[a, +\infty[$ and g is improper integrable on $[a, +\infty[$ and it is not true that $\int\limits_{a}^{+\infty}f(x)dx = +\infty$ and $\int\limits_{a}^{+\infty}g(x)dx = -\infty$ and it is not true that $\int\limits_{a}^{+\infty}f(x)dx = -\infty$ and $\int\limits_{a}^{+\infty}g(x)dx = +\infty$. Then

(i) $f+g$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(f+g)(x)dx = \int\limits_{a}^{+\infty}f(x)dx + \int\limits_{a}^{+\infty}g(x)dx.$

Proof: For every real number d such that $a \leqslant d$ holds $f+g$ is integrable on $[a,d]$ and $(f+g)\restriction[a,d]$ is bounded. □

(47) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty,b] \subseteq \operatorname{dom} f$ and $]-\infty,b] \subseteq \operatorname{dom} g$ and f is improper integrable on $]-\infty, b]$ and g is improper integrable on $]-\infty, b]$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = +\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = -\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = -\infty$. Then

(i) $f-g$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(f-g)(x)dx = \int\limits_{-\infty}^{b} f(x)dx - \int\limits_{-\infty}^{b} g(x)dx$.

The theorem is a consequence of (43) and (45).

(48) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a,+\infty[\subseteq \operatorname{dom} f$ and $[a,+\infty[\subseteq \operatorname{dom} g$ and f is improper integrable on $[a, +\infty[$ and g is improper integrable on $[a, +\infty[$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = +\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = +\infty$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = -\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = -\infty$. Then

(i) $f-g$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(f-g)(x)dx = \int\limits_{a}^{+\infty} f(x)dx - \int\limits_{a}^{+\infty} g(x)dx$.

The theorem is a consequence of (44) and (46).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number r. Now we state the propositions:

(49) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) $r \cdot f$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(r \cdot f)(x)dx = r \cdot \int\limits_{-\infty}^{+\infty} f(x)dx$.

The theorem is a consequence of (36), (41), and (42).

(50) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) $-f$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(-f)(x)dx = -\int\limits_{-\infty}^{+\infty}f(x)dx$.

The theorem is a consequence of (49).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(51) Suppose $\operatorname{dom} f = \mathbb{R}$ and $\operatorname{dom} g = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and g is improper integrable on $\mathbb{R}$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = +\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = -\infty$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = -\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = +\infty$. Then

(i) $f + g$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(f+g)(x)dx = \int\limits_{-\infty}^{+\infty}f(x)dx + \int\limits_{-\infty}^{+\infty}g(x)dx$.

The theorem is a consequence of (25), (26), (31), (30), (36), (45), and (46).

(52) Suppose $\operatorname{dom} f = \mathbb{R}$ and $\operatorname{dom} g = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and g is improper integrable on $\mathbb{R}$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = +\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = -\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = -\infty$. Then

(i) $f - g$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(f-g)(x)dx = \int\limits_{-\infty}^{+\infty}f(x)dx - \int\limits_{-\infty}^{+\infty}g(x)dx$.

The theorem is a consequence of (50) and (51).

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