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Elementary Number Theory Problems. Part XIII

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Summary. This paper formalizes problems 41, 92, 121–123, 172, 182, 183, 191, 192 and 192a from "250 Problems in Elementary Number Theory" by Wacław Sierpiński [8].

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Introduction

In this paper, Problems 41 from Section I, 92, 121, 122, 123 from Section IV, 172, 182, 183, 191, 192, and 192a from Section V of [8] are formalized, using the Mizar formalism [2], [1]. The paper is a part of the project *Formalization of Elementary Number Theory in Mizar* [7], [4], [5], [6], [3].

In the preliminary section, we proved some trivial but useful facts about numbers.

In problem 92 the inequality $p_{k+1} + p_{k+2} \leqslant p_1 \cdot p_2 \cdot \ldots \cdot p_k$ should be justified for any integer $k \geqslant 3$, where p_k denotes the k-th prime. Because we count primes starting from the index 0, we formulated the fact as:

```
3 <= k implies
   primenumber(k) + primenumber(k+1) <= Product primesFinS(k);
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where `primesFinS(k)` denotes the finite sequence of primes of the length `k`, and elements of finite sequences are indexed from 1.

Problem 121 about finding the least positive integer n for which $k \cdot 2^{2^n} + 1$ is composite is represented as separated theorems for every positive $k \leqslant 10$.

Problem 122 requires finding all positive integers $k \leqslant 10$ such that every number $k \cdot 2^{2^n} + 1$ $(n = 1, 2, \dots)$ is composite. The proof lies in the fact that numbers $(3 \cdot t + 2) \cdot 2^{2^n} + 1$ are all divisible by 3 and greater than 3, for every natural t, and every positive natural n. In the book, there are minor misprints in the proof, where $2 \cdot 2^{2^2} + 1$ should be $2 \cdot 2^{2^n} + 1$ and $5 \cdot 2^{2^2} + 1$ should be $5 \cdot 2^{2^n} + 1$.

Problems 191 and 192 are generalized from positive integers to non-zero integers.

Problem 192a is formulated incorrectly in the book. It asks to prove that the system of two equations $x^2 + 7y^2 = z^2$ and $7x^2 + y^2 = t^2$ has no solutions in positive integers x, y, z, and t. However, it has solutions, for instance, $x = 3$, $y = 1$, $z = 4$, and $t = 8$. The example is provided in the book.

Proofs of other problems are straightforward formalizations of solutions given in the book.

1. Preliminaries

From now on a, b, c, k, m, n denote natural numbers, i, j denote integers, and p denotes a prime number.

Now we state the propositions:

(1) If $n < 3$, then $n = 0$ or $n = 1$ or $n = 2$.

(2) If $n < 4$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$.

(3) If $n < 5$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$.

Let us note that $\frac{1}{2}$ is non integer and there exists a rational number which is non natural and there exists a rational number which is non integer.

Now we state the proposition:

(4) If $j \neq 0$ and $\frac{i}{j}$ is integer, then $j \mid i$.

Let q be a non integer rational number. One can verify that $q^{\mathbf{2}}$ is non integer.

Now we state the proposition:

(5) If $\frac{a}{b} \cdot c$ is natural and $b \neq 0$ and a and b are relatively prime, then there exists a natural number d such that $c = b \cdot d$.

2. Problem 41

Let us consider an integer k. Now we state the propositions:

(6) $2 \cdot k + 1$ and $9 \cdot k + 4$ are relatively prime.

(7) $\gcd(2 \cdot k - 1, 9 \cdot k + 4) = \gcd(k + 8, 17)$.

3. Problem 92

Now we state the proposition:

(8) If $m > 1$ and $n > 1$ and m and n are relatively prime, then there exist prime numbers p, q such that $p \mid m$ and $p \nmid n$ and $q \mid n$ and $q \nmid m$ and $p \neq q$.

Let us consider k. The functor primesFinS(k) yielding a finite sequence of elements of $\mathbb{N}$ is defined by

(Def. 1) $\operatorname{len} it = k$ and for every natural number i such that $i < k$ holds $it(i+1) = \operatorname{pr}(i)$.

Let us observe that primesFinS(0) is empty.

Now we state the propositions:

(9) primesFinS(1) $= \langle 2 \rangle$.

(10) primesFinS(2) $= \langle 2, 3 \rangle$.

(11) primesFinS(3) $= \langle 2, 3, 5 \rangle$.

(12) $p < \operatorname{pr}(k)$ if and only if primeindex(p) $< k$.

(13) If primeindex(p) $< k$, then $1 +$ primeindex(p) $\in$ dom(primesFinS(k)).

(14) If primeindex(p) $< k$, then (primesFinS(k))($1 +$ primeindex(p)) $= p$.

(15) If $p < \operatorname{pr}(k)$, then $p \in \operatorname{rng}$ primesFinS(k). The theorem is a consequence of (13), (12), and (14).

(16) If p and $\prod$ primesFinS(k) are relatively prime, then $\operatorname{pr}(k) \leqslant p$. The theorem is a consequence of (15).

Let us consider k. Let us note that primesFinS(k) is positive yielding and primesFinS(k) is increasing.

Let R be an extended real-valued binary relation. We say that R has values greater or equal one if and only if

(Def. 2) for every extended real r such that $r \in \operatorname{rng} R$ holds $r \geqslant 1$.

Observe that $\langle 1 \rangle$ has values greater or equal one and there exists a natural-valued finite sequence which has values greater or equal one.

Let f be an extended real-valued function. Let us observe that f has values greater or equal one if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every object x such that $x \in \operatorname{dom} f$ holds $f(x) \geqslant 1$.

Let f be an extended real-valued finite sequence. One can verify that f has values greater or equal one if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds $f(n) \geqslant 1$.

One can verify that every extended real-valued binary relation which is empty has also values greater or equal one and every extended real-valued binary relation which has values greater or equal one is also positive yielding.

Now we state the propositions:

(17) If $m \leqslant n$, then $\operatorname{primesFinS}(n) \restriction m = \operatorname{primesFinS}(m)$.

(18) Let us consider extended real-valued binary relations P, R. Suppose $\operatorname{rng} P \subseteq \operatorname{rng} R$ and R has values greater or equal one. Then P has values greater or equal one.

(19) Let us consider extended real-valued finite sequences f, g. Suppose $f \frown g$ has values greater or equal one. Then

(i) f has values greater or equal one, and

(ii) g has values greater or equal one.

(20) Let us consider an extended real r. If $\langle r \rangle$ has values greater or equal one, then $r \geqslant 1$.

Let us consider a real-valued finite sequence f with values greater or equal one. Now we state the propositions:

(21) $\prod f \geqslant 1$.

PROOF: Define $\mathcal{P}$[finite sequence of elements of $\mathbb{R}$] $\equiv$ for every real-valued finite sequence g with values greater or equal one such that $g = \$_1$ holds $\prod \$_1 \geqslant 1$. For every finite sequence p of elements of $\mathbb{R}$ and for every element x of $\mathbb{R}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$. For every finite sequence p of elements of $\mathbb{R}$, $\mathcal{P}[p]$. $\square$

(22) $\prod (f \restriction n) \leqslant \prod f$. The theorem is a consequence of (19) and (20).

Let us consider k. One can verify that $\operatorname{primesFinS}(k)$ has values greater or equal one.

Now we state the proposition:

(23) If $3 \leqslant k$, then $\operatorname{pr}(k) + \operatorname{pr}(k+1) \leqslant \prod \operatorname{primesFinS}(k)$. The theorem is a consequence of (8) and (16).

4. Problem 121

Let k, n be natural numbers. We say that n satisfies Sierpiński Problem 121 for k if and only if

(Def. 5) $k \cdot 2^{2^n} + 1$ is composite and for every positive natural number m such that $m < n$ holds $k \cdot 2^{2^m} + 1$ is not composite.

Now we state the propositions:

(24) 5 satisfies Sierpiński Problem 121 for 1. The theorem is a consequence of (3).

(25) 1 satisfies Sierpiński Problem 121 for 2.

(26) 2 satisfies Sierpiński Problem 121 for 3.

(27) 2 satisfies Sierpiński Problem 121 for 4.

(28) 1 satisfies Sierpiński Problem 121 for 5.

(29) 1 satisfies Sierpiński Problem 121 for 6.

(30) 3 satisfies Sierpiński Problem 121 for 7. The theorem is a consequence of (1).

(31) 1 satisfies Sierpiński Problem 121 for 8.

(32) 2 satisfies Sierpiński Problem 121 for 9.

(33) 2 satisfies Sierpiński Problem 121 for 10.

5. Problem 122

Let us consider a positive natural number n.

Now we state the propositions:

(34) $3 \mid (3 \cdot a + 2) \cdot 2^{2^n} + 1$.

(35) $2 \cdot 2^{2^n} + 1$ is composite.

(36) $5 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).

(37) $8 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).

(38) Let us consider a positive natural number k. Then $k \leqslant 10$ and for every positive natural number n, $k \cdot 2^{2^n} + 1$ is composite if and only if $k \in \{2, 5, 8\}$. The theorem is a consequence of (24), (26), (27), (30), (32), (33), (35), (36), and (37).

6. Problem 123

Now we state the propositions:

(39) $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 7$.

(40) If $n > 0$, then $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 21$.

(41) If $n > 1$, then $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 273$.

(42) If m is even or $m = 2 \cdot n$, then $2^m \bmod 3 = 1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2^{2\cdot \$_1} \bmod 3 = 1$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. □

(43) If m is odd or $m = 2 \cdot n + 1$, then $2^m \bmod 3 = 2$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2^{2\cdot \$_1+1} \bmod 3 = 2$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. □

(44) Let us consider a non zero natural number n. Then $3 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42).

(45) $7 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42) and (43).

Let n be a non zero natural number. Note that $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is natural. Now we state the proposition:

(46) Let us consider a non zero natural number n. If $n > 1$, then $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is composite. The theorem is a consequence of (39), (45), (44), and (41).

7. Problem 172

Now we state the proposition:

(47) Let us consider positive natural numbers n, x, y, z. Then $n^x + n^y = n^z$ if and only if $n = 2$ and $y = x$ and $z = x + 1$.

8. Problem 182

Now we state the proposition:

(48) Let us consider real numbers a, b, c. If $c > 1$ and $c^a = c^b$, then $a = b$.

Let us consider positive natural numbers n, x, y, z, t. Now we state the propositions:

(49) If $x \leqslant y \leqslant z$, then $n^x + n^y + n^z = n^t$ iff $n = 2$ and $y = x$ and $z = x + 1$ and $t = x + 2$ or $n = 3$ and $y = x$ and $z = x$ and $t = x + 1$.

(50) $n^x + n^y + n^z = n^t$ if and only if $n = 2$ and $y = x$ and $z = x + 1$ and $t = x + 2$ or $n = 2$ and $y = x + 1$ and $z = x$ and $t = x + 2$ or $n = 2$ and $z = y$ and $x = y + 1$ and $t = y + 2$ or $n = 3$ and $y = x$ and $z = x$ and $t = x + 1$. The theorem is a consequence of (49).

9. PROBLEM 183

Now we state the proposition:

(51) Let us consider positive natural numbers x, y, z, t. Then $4^x+4^y+4^z \neq 4^t$.

10. PROBLEM 191

Now we state the proposition:

(52) Let us consider non zero integers x, y, z, t. Then

(i) $x^{\mathbf{2}} + 5 \cdot y^{\mathbf{2}} \neq z^{\mathbf{2}}$, or

(ii) $5 \cdot x^{\mathbf{2}} + y^{\mathbf{2}} \neq t^{\mathbf{2}}$.

11. PROBLEM 192

Now we state the propositions:

(53) Let us consider non zero integers x, y, z, t. Then

(i) $x^{\mathbf{2}} + 6 \cdot y^{\mathbf{2}} \neq z^{\mathbf{2}}$, or

(ii) $6 \cdot x^{\mathbf{2}} + y^{\mathbf{2}} \neq t^{\mathbf{2}}$.

(54) (i) $3^{\mathbf{2}} + 7 \cdot 1^{\mathbf{2}} = 4^{\mathbf{2}}$, and

(ii) $7 \cdot 3^{\mathbf{2}} + 1^{\mathbf{2}} = 8^{\mathbf{2}}$.

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Integral of Continuous Three Variable Functions[1]

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Summary. In this article we continue our proofs on integrals of continuous functions of three variables in Mizar. In fact, we use similar techniques as in the case of two variables: we deal with projections of continuous function, the continuity of three variable functions in general, aiming at pure real-valued functions (not necessarily extended real-valued functions), concluding with integrability and iterated integrals of continuous functions of three variables.

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Keywords: triple integral; iterated integral

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Introduction

In this article, following the previous article [9], we continue our proofs on integrals of continuous functions of three variables in Mizar [2], [3]; for a survey of formalizations of real analysis in another proof-assistants like ACL2 [11], Isabelle/HOL [10], Coq [4], see [5].

In the first section, continuity of functions of three variables is shown. These are used in the proofs of the later sections.

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The second section summarizes the basic properties of the projection of a continuous function in three variables, a result that is almost as obvious as in two variables, but is used to transform [8] Riemann and Lebesgue integrals for real-valued functions (not extended real-valued functions).

In the last section, we prove integrability and iterated integrals of continuous functions of three variables. Throughout the paper, the basic proof steps follow [1], [16], and [12].

1. Preliminaries

Now we state the propositions:

(1) Let us consider real normed spaces X, Y, Z, a point u of $X \times Y \times Z$, a point x of X, a point y of Y, and a point z of Z. Suppose $u = \langle x, y, z \rangle$. Then

  (i) $\|u\| \leqslant \|x\| + \|y\| + \|z\|$, and

  (ii) $\|x\| \leqslant \|u\|$, and

  (iii) $\|y\| \leqslant \|u\|$, and

  (iv) $\|z\| \leqslant \|u\|$.

(2) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, and a subset E of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$). If $E = (I \times J) \times K$, then E is compact.

(3) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a set E.

Suppose $f = g$ and $E \subseteq \operatorname{dom} f$. Then f is uniformly continuous on E if and only if for every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1 \rangle$, $\langle x_2, y_2, z_2 \rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$.

Proof: For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points p_1, p_2 of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) such that $p_1, p_2 \in E$ and $\|p_1 - p_2\| < r$ holds $\|f_{/p_1} - f_{/p_2}\| < e$. □

(4) Let us consider intervals I, J, K. Then

  (i) $(I \times J) \times K$ is a subset of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$), and

  (ii) $(I \times J) \times K \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$.

(5) Let us consider a point u of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and a real number r. Suppose $0 < r$. Then there exist real numbers s, x, y, z such that

(i) $0 < s < r$, and

(ii) $u = \langle x, y, z \rangle$, and

(iii) $]x - s, x + s[\times]y - s, y + s[\times]z - s, z + s[\subseteq \mathrm{Ball}(u, r)$.

Let us consider a subset A of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$). Now we state the propositions:

(6) Suppose for every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$. Then there exists a function F from A into $\mathbb{R}$ such that for every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist real numbers a, b, c and there exists a real-membered set R_{12} such that $\$_1 = \langle a, b, c \rangle$ and R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a - r, a + r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $\$_2 = \frac{\sup R_{12}}{2}$.

For every object x such that $x \in A$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$. Consider F being a function from A into $\mathbb{R}$ such that for every object x such that $x \in A$ holds $\mathcal{P}[x, F(x)]$. For every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a - r, a + r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$. $\square$

(7) If A is open, then $A \in \sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. The theorem is a consequence of (5), (6), and (1).

(8) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and

$|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$.

PROOF: Set $E = (I \times J) \times K$. f is uniformly continuous on E. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1\rangle$, $\langle x_2, y_2, z_2\rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$. For every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that x_1, $x_2 \in I$ and y_1, $y_2 \in J$ and z_1, $z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$. □

(9) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. If $f = g$, then $\|f\| = |g|$.

(10) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

  (i) $0 < r$, and

  (ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that x_1, $x_2 \in I$ and y_1, $y_2 \in J$ and z_1, $z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2\rangle) - |g|(\langle x_1, y_1, z_1\rangle)| < e$.

The theorem is a consequence of (9) and (8).

2. Properties on the Projective Function of a Three Variable Function

Now we state the propositions:

(11) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then ProjPMap1$(g, \langle x, y\rangle)$ is continuous.

PROOF: For every real number z_0 such that $z_0 \in$ dom(ProjPMap1$(g, \langle x, y\rangle)$) holds ProjPMap1$(g, \langle x, y\rangle)$ is continuous in z_0 by [13, (4)]. □

(12) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial

function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(g, z)$. Then p_2 is continuous on $\operatorname{dom} p_2$.

PROOF: For every point x_4 of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) such that $x_4 \in \operatorname{dom} p_2$ holds $p_2 \restriction \operatorname{dom} p_2$ is continuous in x_4 by [15, (18)], [14, (9)]. □

(13) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then $\mathrm{ProjPMap1}(|g|, \langle x, y \rangle)$ is continuous. The theorem is a consequence of (11).

(14) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(|g|, z)$. Then p_2 is continuous on $\operatorname{dom} p_2$. The theorem is a consequence of (12).

(15) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$. Then $\mathrm{ProjPMap1}(g, \langle x, y \rangle)$ is uniformly continuous.

PROOF: For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers z_1, z_2 such that $z_1, z_2 \in \operatorname{dom}(\mathrm{ProjPMap1}(g, \langle x, y \rangle))$ and $|z_1 - z_2| < s$ holds $|(\mathrm{ProjPMap1}(g, \langle x, y \rangle))(z_1) - (\mathrm{ProjPMap1}(g, \langle x, y \rangle))(z_2)| < r$. □

(16) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(g, z)$. Then p_2 is uniformly continuous on $\operatorname{dom} p_2$.

(17) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from

$(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $\mathrm{dom}\, f$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then P_8 is continuous. The theorem is a consequence of (11).

(18) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on $\mathrm{dom}\, f$ and $f = g$ and $P_7 = \mathrm{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then P_7 is continuous on $\mathrm{dom}\, P_7$. The theorem is a consequence of (12).

(19) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $\mathrm{dom}\, f$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then P_8 is continuous. The theorem is a consequence of (13).

(20) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on $\mathrm{dom}\, f$ and $f = g$ and $P_7 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then P_7 is continuous on $\mathrm{dom}\, P_7$. The theorem is a consequence of (14).

3. Integral of Continuous Three Variable Function

Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(21) Suppose $x \in I$ and $y \in J$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) $P_8 \restriction K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (17).

(22) Suppose $x \in I$ and $y \in J$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \,\mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle) \,\mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_K P_8(x)dx = (\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(\langle x, y \rangle)$.

The theorem is a consequence of (21).

(23) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then

(i) P_9 is integrable on ProdMeas(L-Meas, L-Meas), and

(ii) $\int P_9 \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$\int \text{ProjPMap2}(\overline{\mathbb{R}}(g), z) \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(iii) $\int P_9 \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$(\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)))(z)$.

The theorem is a consequence of (18).

(24) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then

(i) $P_8 \restriction K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (19).

(25) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose

$x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$ and $E = K$. Then P_8 is E-measurable. The theorem is a consequence of (24).

(26) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \, \mathrm{d}\, \text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle) \, \mathrm{d}\, \text{L-Meas}$, and

(iv) $\int\limits_K P_8(x)dx = (\mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(\langle x, y \rangle)$.

The theorem is a consequence of (24).

(27) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$ and $E = I \times J$. Then P_9 is E-measurable. The theorem is a consequence of (20).

(28) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then

(i) P_9 is integrable on ProdMeas(L-Meas, L-Meas), and

(ii) $\int P_9 \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$\int \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z) \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(iii) $\int P_9 \,\mathrm{d}\,\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$(\mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|))(z)$.

The theorem is a consequence of (20).

(29) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $E = (I \times J) \times K$. Then g is E-measurable.

Proof: For every real number r, $E \cap \text{LE-dom}(g, r) \in \sigma(\mathrm{MeasRect}(\sigma(\mathrm{Meas}$ $\mathrm{Rect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. $\square$

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a real number e. Now we state the propositions:

(30) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in K$ holds $|(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_2))(z) - (\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_1))(z)| < e$.

Proof: For every element x of $\mathbb{R} \times \mathbb{R}$ and for every element y of $\mathbb{R}$ such that $x \in I \times J$ and $y \in K$ holds $(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x))(y) = |\overline{\mathbb{R}}(g)|(x, y)$ and $|\overline{\mathbb{R}}(g)|(x, y) = |g|(\langle x, y \rangle)$. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2 \rangle) - |g|(\langle x_1, y_1, z_1 \rangle)| < e$. $\square$

(31) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in$

K holds $|(\text{ProjPMap1}(\overline{\mathbb{R}}(g), u_2))(z) - (\text{ProjPMap1}(\overline{\mathbb{R}}(g), u_1))(z)| < e$.

The theorem is a consequence of (8).

(32) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$ is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and
- (ii) $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction (I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and
- (iii) $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and
- (iv) $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (26) and (22).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function F_4 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Now we state the propositions:

(33) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction (I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (30), (19), and (24).

(34) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (31), (17), (21), and (22).

(35) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|)$ is a function from $\mathbb{R}$ into $\mathbb{R}$, and
- (ii) $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|) \restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(iii) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(iv) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (20), (28), (18), and (23).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(36) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 =$ Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$)$\restriction K$. Then G_3 is continuous.

PROOF: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $||g|(\langle x, y, z_2\rangle) - |g|(\langle x, y, z_1\rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\text{ProjPMap2}(|R_{11}|, z))(x, y) = |R_{11}|(\langle x, y\rangle, z)$ and $|R_{11}|(\langle x, y\rangle, z) = |g(\langle x, y, z\rangle)|$ and $|R_{11}|(\langle x, y\rangle, z) = |g|(\langle x, y, z\rangle)$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_2), x))(y) - (\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_1), x))(y)| < e$. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$. $\square$

(37) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 =$ Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction K$. Then G_3 is continuous.

PROOF: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $|g(\langle x, y, z_2\rangle) - g(\langle x, y, z_1\rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\text{ProjPMap2}(R_{11}, z))(x, y) = R_{11}(\langle x, y\rangle, z)$

and $R_{11}(\langle x, y\rangle, z) = g(\langle x, y, z\rangle)$ and $R_{11}(\langle x, y\rangle, z) = g(\langle x, y, z\rangle)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(R_{11}, z_2), x))(y) - (\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(R_{11}, z_1), x))(y)| < e$. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$. □

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(38) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (24) and (25).

(39) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (20) and (27).

(40) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element u of $\mathbb{R} \times \mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$))$(u) < +\infty$. The theorem is a consequence of (32).

(41) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$))$(z) < +\infty$. The theorem is a consequence of (35).

(42) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$. Set $I_1 = I \times J$. Reconsider $G = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$.

Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. For every real number r, $H \cap \text{LE-dom}(F, r) \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. □

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(43) Suppose $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) g is integrable on ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas), and

(ii) for every element u of $\mathbb{R} \times \mathbb{R}$, $\text{ProjPMap1}(\overline{\mathbb{R}}(g), u)$ is integrable on L-Meas, and

(iii) for every element U of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$, $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is U-measurable, and

(iv) $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on ProdMeas(L-Meas, L-Meas), and

(v) $\int g \,\mathrm{d}\, \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \,\mathrm{d}\, \text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$.

PROOF: Set $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$. Set $I_1 = I \times J$. Reconsider $G = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $A_1 = I \times J$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. Reconsider $N_1 = (\mathbb{R} \times \mathbb{R}) \setminus A_1$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$.

F is H-measurable. Set $F_1 = F \restriction N_1$. For every object x such that $x \in \operatorname{dom} F_1$ holds $F_1(x) = 0$. Reconsider $K_1 = (I \times J) \times K$ as an element of σ(MeasRect(σ(MeasRect(L-Field, L-Field)), L-Field)). g is K_1-measurable. For every element x of $\mathbb{R} \times \mathbb{R}$, (Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$))$(x) < +\infty$. □

(44) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element z of $\mathbb{R}$, ProjPMap2($\overline{\mathbb{R}}(g), z$) is integrable on ProdMeas(L-Meas, L-Meas), and
- (ii) for every element V of L-Field, Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is V-measurable, and
- (iii) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is integrable on L-Meas, and
- (iv) $\int g \, \mathrm{d}$ ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) d L-Meas.

The theorem is a consequence of (43) and (41).

(45) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element x of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $x \in I$. Then ProjPMap1($|$Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$|, x$) is E-measurable.

PROOF: Set F_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$). Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set F_5 = ProjPMap1($|F_4|, x$). Set $L_0 = F_5 \restriction J$. For every element t of $\mathbb{R}$ such that $t \in J$ holds $0 \leqslant L_0(t)$. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap$ LE-dom$(F_5, r) \in$ L-Field. □

(46) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element x of $\mathbb{R}$, (Integral2(L-Meas, $|$Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$|$))$(x) < +\infty$, and

(ii) for every element x of $\mathbb{R}$, ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is integrable on L-Meas.

PROOF: Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element x of $\mathbb{R}$, (Integral2(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))(x) $< +\infty$ by [6, (5)], [7, (75)]. Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on ProdMeas(L-Meas, L-Meas). □

(47) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element y of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $y \in J$. Then ProjPMap2(| Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|, y) is E-measurable.

PROOF: Set F_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$). Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set F_6 = ProjPMap2($|F_4|$, y). Set $L_0 = F_6 \restriction I$. For every element t of $\mathbb{R}$ such that $t \in I$ holds $0 \leqslant L_0(t)$. For every element r of $\mathbb{R}$, $0_{\overline{\mathbb{R}}} \leqslant F_6(r)$. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap$ LE-dom$(F_6, r) \in$ L-Field. □

(48) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element y of $\mathbb{R}$, (Integral1(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))(y) $< +\infty$, and

(ii) for every element y of $\mathbb{R}$, ProjPMap2(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), y) is integrable on L-Meas.

PROOF: Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element y of $\mathbb{R}$, (Integral1(L-Meas, $|$ Integral2(L-Meas, $\overline{\mathbb{R}}(g))|$))$(y) < +\infty$. Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on ProdMeas(L-Meas, L-Meas). □

(49) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F \restriction (I \times J)$. Reconsider $G = F_0$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. Reconsider $R_2 = \mathbb{R} \times \mathbb{R}$ as an element of σ(MeasRect(L-Field, L-Field)). F is non-negative. For every real number r, $R_2 \cap$LE-dom$(F, r) \in \sigma$(MeasRect(L-Field, L-Field)). □

(50) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F =$ Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F \restriction K$. Reconsider $G = F_0$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. $G \restriction K$ is bounded and G is integrable on K. Reconsider $R = \mathbb{R}$ as an element of L-Field. F is non-negative. For every real number r, $R \cap$ LE-dom$(F, r) \in$ L-Field. □

(51) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element x of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) ProjPMap1($|$ Integral2(L-Meas, $\overline{\mathbb{R}}(g))|$, x) is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(52) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element y of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

  (i) ProjPMap2(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), y) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

  (ii) ProjPMap2(| Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|, y) is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(53) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then | Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)| is a function from $\mathbb{R}$ into $\mathbb{R}$. The theorem is a consequence of (35).

(54) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int$ ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x)$\restriction \mathbb{R} \setminus J$ d L-Meas $= 0$.

(55) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int$ ProjPMap2(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), y)$\restriction \mathbb{R} \setminus I$ d L-Meas $= 0$.

(56) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int$ Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction \mathbb{R} \setminus K$ d L-Meas $= 0$.

(57) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_1 =$ ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x)$\restriction J$. Then P_1 is continuous. The theorem is a consequence of (32) and (34).

(58) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2

from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction I$. Then P_2 is continuous. The theorem is a consequence of (32) and (34).

(59) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J$. Then

(i) $P_1 \restriction J$ is bounded, and

(ii) P_1 is integrable on J.

The theorem is a consequence of (32) and (34).

(60) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction I$. Then

(i) $P_2 \restriction I$ is bounded, and

(ii) P_2 is integrable on I.

The theorem is a consequence of (32) and (34).

(61) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \restriction K$. Then

(i) $G_3 \restriction K$ is bounded, and

(ii) G_3 is integrable on K.

The theorem is a consequence of (37).

(62) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on

$(I\times J)\times K$ and $f = g$ and $P_1 = \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J$. Then

(i) $\mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J$ is integrable on L-Meas, and

(ii) $\int_J P_1(x)dx = \int \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J \,\mathrm{d}\,\text{L-Meas}$, and

(iii) $\int_J P_1(x)dx = \int \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \,\mathrm{d}\,\text{L-Meas}$, and

(iv) $\int_J P_1(x)dx = (\mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(x)$.

The theorem is a consequence of (46), (59), and (54).

(63) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \mathrm{dom}\, f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$. Then

(i) $\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$ is integrable on L-Meas, and

(ii) $\int_I P_2(x)dx = \int \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I \,\mathrm{d}\,\text{L-Meas}$, and

(iii) $\int_I P_2(x)dx = \int \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \,\mathrm{d}\,\text{L-Meas}$, and

(iv) $\int_I P_2(x)dx = (\mathrm{Integral1}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(y)$.

The theorem is a consequence of (48), (60), and (55).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(64) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element U of L-Field, $\mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$ is U-measurable, and

(ii) Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(v) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ is integrable on ProdMeas(L-Meas, L-Meas), and

(vi) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$) d L-Meas.

The theorem is a consequence of (32), (43), (46), (40), and (34).

(65) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element V of L-Field, Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is V-measurable, and

(ii) Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(v) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$) d L-Meas.

The theorem is a consequence of (32), (43), (48), (40), and (34).

(66) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and P_1 = ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$, x). Then

(i) P_1 is continuous, and

(ii) dom(ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$, x)) = J, and

(iii) $P_1 \restriction J$ is bounded, and

(iv) P_1 is integrable on J, and

(v) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), x) \, \text{d}\,\text{L-Meas}$, and

(vi) $\int\limits_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)))(x)$, and

(vii) $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), x)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(67) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), y)$. Then

(i) P_2 is continuous, and

(ii) $\text{dom}(\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), y)) = I$, and

(iii) $P_2 \restriction I$ is bounded, and

(iv) P_2 is integrable on I, and

(v) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), y) \, \text{d}\,\text{L-Meas}$, and

(vi) $\int\limits_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)))(y)$, and

(vii) $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J), y)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(68) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_8 = \text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I$. Then

(i) $\text{dom}\, G_8 = I$, and

(ii) G_8 is continuous, and

(iii) $G_8 \restriction I$ is bounded, and

(iv) G_8 is integrable on I, and

(v) Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I$ is integrable on L-Meas, and

(vi) $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I \, \mathrm{d}$ L-Meas $= \int\limits_I G_8(x)dx$, and

(vii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}$ ProdMeas(L-Meas, L-Meas) $= \int\limits_I G_8(x)dx$.

The theorem is a consequence of (32) and (34).

(69) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_7 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_7 =$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$. Then

(i) $\operatorname{dom} G_7 = J$, and

(ii) G_7 is continuous, and

(iii) $G_7 \restriction J$ is bounded, and

(iv) G_7 is integrable on J, and

(v) Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$ is integrable on L-Meas, and

(vi) $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J \, \mathrm{d}$ L-Meas $= \int\limits_J G_7(x)dx$, and

(vii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}$ ProdMeas(L-Meas, L-Meas) $= \int\limits_J G_7(x)dx$.

The theorem is a consequence of (32) and (34).

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Separable Polynomials and Separable Extensions

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Summary. We continue the formalization of field theory in Mizar [2], [3], [4]. We introduce separability of polynomials and field extensions: a polynomial is separable, if it has no multiple roots in its splitting field; an algebraic extension E of F is separable, if the minimal polynomial of each $a \in E$ is separable. We prove among others that a polynomial $q(X)$ is separable if and only if the gcd of $q(X)$ and its (formal) derivation equals 1 – and that a irreducible polynomial $q(X)$ is separable if and only if its derivation is not 0 – and that $q(X)$ is separable if and only if the number of $q(X)$'s roots in some field extension equals the degree of $q(X)$.

A field F is called perfect if all irreducible polynomials over F are separable, and as a consequence every algebraic extension of F is separable. Every field with characteristic 0 is perfect [13]. To also consider separability in fields with prime characteristic p we define the rings $R^p = \{ a^p \mid a \in R\}$ and the polynomials $X^n - a$ for $a \in R$. Then we show that a field F with prime characteristic p is separable if and only if $F = F^p$ and that finite fields are perfect. Finally we prove that for fields $F \subseteq K \subseteq E$ where E is a separable extension of F both E is separable over K and K is separable over F.

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Introduction

In this paper we formalize separability [7] using the Mizar formalism [2], [3], [6]. A polynomial is separable, if it has no multiple roots in its splitting field; an algebraic extension E of F is separable, if the minimal polynomial of each $a \in E$ is separable [8], [10], [5].

In the first two sections we provide some technical lemmas necessary later. They concern for example divisibility and gcds of integers, in particular we show that a prime p divides $\binom{p}{m}$ for $1 \leqslant m < p$. We also need a number of results on powers of polynomials among them that a polynomial $q(X)$ divides $(X - a)^n$ if and only if $q(X) = (X - a)^{`l}$ for some $0 \leqslant l \leqslant n$ or that a is an n-fold root of $(X - a)^n$.

In the third section we define the ring $R^p = \{\ a^p \mid a \in R\}$ for a given ring R with prime characteristic p. In order to do so we proved that $(a+b)^p = a^p + b^p$, also called freshman's dream.

Then we define the polynomial $q(X) = X^n - a$ necessary to describe separability in fields with characteristic $p \neq 0$. Note that the roots of $q(X)$ are the elements b with $b^p = a$, so that $q(X) = (X - b)^p$ if there exists such a b and is irreducible otherwise.

In section five we deal with multiplicity of polynomials. We show among others that a polynomial $q(X)$ has a multiple root (in a field extension where $q(X)$ splits) if and only if the gcd of $q(X)$ and its (formal) derivation is not 1. For irreducible $q(X)$ this can be sharpened to $q(X)$'s derivation being 0. We also prove that in fields with characteristic $p \neq 0$ the derivation of a polynomial $q(X)$ is 0 if and only if there exists a polynomial $r(X)$ such that $q(X) = r(X^p)$.

The next two sections are devoted to separability of polynomials. We define a polynomial $q(X)$ to be separable, if it has no multiple roots in its splitting field. Note that the splitting field of $q(X)$ is unique only up to isomorphism, so that we had to prove that the definition indeed is independent of a particular splitting field. We prove a number of characterizations of separability found in the literature, for example that $q(X)$ is separable if and only if the number of $q(X)$'s roots equals the degree of $q(X)$ in some field extension if and only if $q(X)$ is square free in every field extension in which q splits. Then we introduce perfect fields, e.g. fields in which every irreducible polynomial is separable. Fields with characteristic 0 are perfect (see [13]). Fields F with characteristic $p \neq 0$ are perfect if and only if $F = F^p$. This is shown using the polynomial $X^p - a$, which is inseparable and irreducible if there is no b with $b^p = a$. Because in finite fields the multiplicative group is cyclic in finite fields such a b always exists and so finite field are perfect.

In the last section we define separable extensions: an algebraic extension is separable if the minimal polynomial of every $a \in E$ is separable. As an easy consequence we get that for $p(X) \in F[X] \backslash F$, where F is perfect, the splitting field of $p(X)$ is both normal and separable. We also show that for fields $F \subseteq K \subseteq E$ where E is a separable extension of F both E is a separable extension of K and K is a separable extension of F.

1. Preliminaries

Let R be a ring and k be a non zero natural number. One can check that $(0_R)^k$ reduces to 0_R.

Let k be a natural number. Note that $(1_R)^k$ reduces to 1_R.

Let p be a prime number. Observe that there exists a field which is finite and has characteristic p.

Let F be a finite field. Let us observe that $\mathrm{char}(F)$ is prime.

Let R be a non degenerated ring. One can verify that every element of the carrier of Polynom-Ring R which is monic is also non zero.

Let F be a field, p be a non constant element of the carrier of Polynom-Ring F, and a be a non zero element of F. One can verify that the functor $a \cdot p$ yields a non constant element of the carrier of Polynom-Ring F. Now we state the propositions:

(1) Let us consider a natural number n, and a non zero natural number m. Then $\frac{n}{m}$ is a natural number if and only if $m \mid n$.

(2) Let us consider a prime number p, and natural numbers n, a, b. If $p \mid a$ and $p \nmid b$ and $n = \frac{a}{b}$, then $p \mid n$. The theorem is a consequence of (1).

(3) Let us consider a prime number p, and a non zero natural number n. If $n < p$, then $\gcd(n, p) = 1$.

(4) Let us consider a non zero natural number n, and a prime number p. Then there exist natural numbers k, m such that

 (i) $n = m \cdot p^k$, and

 (ii) $p \nmid m$.

 The theorem is a consequence of (1).

Let R be an integral domain, a be a non zero element of R, and n be a natural number. One can check that a^n is non zero.

Now we state the propositions:

(5) Let us consider a ring R, an element a of R, and an even natural number n. Then $(-a)^n = a^n$.

(6) Let us consider a ring R, an element a of R, and an odd natural number n. Then $(-a)^n = -a^n$.

(7) Let us consider a ring R with characteristic 2, and an element a of R. Then $-a = a$.

(8) Let us consider an add-associative, right zeroed, right complementable, Abelian, non empty double loop structure R, and an integer i. Then $i \star 0_R = 0_R$.

PROOF: Define $\mathcal{P}[\text{integer}] \equiv \$_1 \star 0_R = 0_R$. For every integer u such that $\mathcal{P}[u]$ holds $\mathcal{P}[u-1]$ and $\mathcal{P}[u+1]$ by [12, (64), (60), (62)]. For every integer i, $\mathcal{P}[i]$. □

Let F be a finite field. Let us observe that MultGroup(F) is cyclic.

Now we state the propositions:

(9) Let us consider a field F, and an extension E of F. Then MultGroup(F) is a subgroup of MultGroup(E).

(10) Let us consider a skew field R, a natural number n, an element a of R, and an element b of MultGroup(R). If $a = b$, then $a^n = b^n$ by [1, (17)], [11, (8)].

Let us consider a ring R, a polynomial p over R, and elements a, b of R. Now we state the propositions:

(11) $(a + b) \cdot p = a \cdot p + b \cdot p$.

(12) $(a \cdot b) \cdot p = a \cdot (b \cdot p)$.

(13) Let us consider a ring R, an element q of the carrier of Polynom-Ring R, a polynomial p over R, and a natural number n. If $p = q$, then $n \cdot (1_R) \cdot p = n \cdot q$ by [9, (26)].

(14) Let us consider a ring R, an element q of the carrier of Polynom-Ring R, a polynomial p over R, and natural numbers n, j. If $p = n \cdot q$, then $p(j) = n \cdot q(j)$.

(15) Let us consider a field F, an element a of F, a polynomial p over F, an extension E of F, an element b of E, and a polynomial q over E. If $a = b$ and $p = q$, then $a \cdot p = b \cdot q$.

(16) Let us consider a field F, an irreducible element p of the carrier of Polynom-Ring F, and an element q of the carrier of Polynom-Ring F. If $q \mid p$, then q is unital or associated to p.

(17) Let us consider a field F, an irreducible element p of the carrier of Polynom-Ring F, and a monic element q of the carrier of Polynom-Ring F. If $q \mid p$, then $q = \mathbf{1}.F$ or $q = \text{NormPoly}\, p$.

Let us consider a field F and a non zero element p of the carrier of Polynom-Ring F. Now we state the propositions:

(18) p is reducible if and only if p is a unit of Polynom-Ring F or there exists a monic element q of the carrier of Polynom-Ring F such that $q \mid p$ and $1 \leqslant \deg(q) < \deg(p)$.

(19) p is reducible if and only if there exists a monic element q of the carrier of Polynom-Ring F such that $q \mid p$ and $1 \leqslant \deg(q) < \deg(p)$.

2. On Powers of Polynomials

Let R be an integral domain, p be a non zero polynomial over R, and n be a natural number. Observe that p^n is non zero. Let F be a field, p be a non constant polynomial over F, and n be a non zero natural number. One can verify that p^n is non constant.

Let p be a non constant element of the carrier of Polynom-Ring F. Let us note that p^n is non constant. Let p be a constant element of the carrier of Polynom-Ring F. One can check that p^n is constant and p^n is constant. Now we state the propositions:

(20) Let us consider an integral domain R, a polynomial p over R, and a natural number n. Then $\mathrm{LC}\, p^n = (\mathrm{LC}\, p)^n$.

(21) Let us consider an integral domain R, a non zero polynomial p over R, and a natural number n. Then $\deg(p^n) = n \cdot (\deg(p))$.

(22) Let us consider a commutative ring R, a polynomial p over R, and a non zero natural number n. Then $(p^n)(0) = p(0)^n$.

(23) Let us consider an integral domain R, a non zero element a of R, and a natural number n. Then $\langle 0_R, a\rangle^n = a^n \cdot (\langle 0_R, 1_R\rangle^n)$.

(24) Let us consider a field F, an element a of F, and a natural number n. Then $(a \restriction F)^n = a^n \restriction F$.

(25) Let us consider a field F, a non zero element a of F, and natural numbers n, m. Then $(\mathrm{anpoly}(a, m))^n = \mathrm{anpoly}(a^n, n \cdot m)$.

(26) Let us consider a field F, an element a of F, and a natural number n. Then $\deg((\mathrm{X}- a)^n) = n$.

(27) Let us consider a field F, an element a of F, and a non zero natural number n. Then $\mathrm{Roots}((\mathrm{X}- a)^n) = \{a\}$.

Let us consider a field F, an element a of F, and a natural number n. Now we state the propositions:

(28) $\mathrm{multiplicity}((\mathrm{X}- a)^n, a) = n$. The theorem is a consequence of (26).

(29) $\overline{\overline{\mathrm{BRoots}((\mathrm{X}- a)^n)}} = n$.

(30) Let us consider a non degenerated commutative ring R, a commutative ring extension S of R, an element a of R, an element b of S, and an element n of $\mathbb{N}$. If $a = b$, then $(\mathrm{X}- b)^n = (\mathrm{X}- a)^n$.

(31) Let us consider a field F, a monic polynomial p over F, an element a of F, and a natural number n. Then $p \mid (\mathrm{X}- a)^n$ if and only if there exists a natural number l such that $l \leqslant n$ and $p = (\mathrm{X}- a)^l$. The theorem is a consequence of (27), (28), and (26).

(32) Let us consider a non degenerated commutative ring R, elements a, b of R, and a natural number n. Then $\mathrm{eval}((\mathrm{X}+a)^n, b) = (a+b)^n$.

(33) Let us consider a field F, an element a of F, and a non zero natural number n. Then $(\mathrm{X}-a)^n$ splits in F.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{X}-a)^{\$_1}$ splits in F. For every natural number k such that $k \geqslant 1$ holds $\mathcal{P}[k]$. □

(34) Let us consider a field F_1, an F_1-homomorphic field F_2, a homomorphism h from F_1 to F_2, an element a of F_1, and a natural number n. Then $(\mathrm{PolyHom}(h))((\mathrm{X}-a)^n) = (\mathrm{X}-h(a))^n$.

3. The Rings R^p for Primes p

Let p be a prime number. One can verify that every commutative ring with characteristic p is non degenerated. Now we state the propositions:

(35) Let us consider a prime number p, a commutative ring R with characteristic p, and an element a of R. Then $p \cdot a = 0_R$.

(36) Let us consider a prime number p, a commutative ring R with characteristic p, a non zero element a of R, and a non zero natural number n. If $n < p$, then $n \cdot a \neq 0_R$.

Let us consider a prime number p, a commutative ring R with characteristic p, an element a of R, and a natural number n. Now we state the propositions:

(37) $n \cdot p \cdot a = 0_R$.

(38) If $p \mid n$, then $n \cdot a = 0_R$. The theorem is a consequence of (37).

(39) Let us consider a prime number p, a commutative ring R with characteristic p, a non zero element a of R, and a natural number n. Then $p \mid n$ if and only if $n \cdot a = 0_R$. The theorem is a consequence of (37) and (36).

(40) Let us consider a prime number p, a commutative ring R with characteristic p, and elements a, b of R. Then $(a+b)^p = a^p + b^p$.

PROOF: Set $F = \langle \binom{p}{0} a^0 b^p, \ldots, \binom{p}{p} a^p b^0 \rangle$. Consider f_1 being a sequence of the carrier of R such that $\sum F = f_1(\mathrm{len}\, F)$ and $f_1(0) = 0_R$ and for every natural number j and for every element v of R such that $j < \mathrm{len}\, F$ and $v = F(j+1)$ holds $f_1(j+1) = f_1(j) + v$. Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv \$_1 = 0$ and $f_1(\$_1) = 0_R$ or $0 < \$_1 < \mathrm{len}\, F$ and $f_1(\$_1) = a^p$ or $\$_1 = \mathrm{len}\, F$ and $f_1(\$_1) = a^p + b^p$. For every element j of $\mathbb{N}$ such that $0 \leqslant j \leqslant \mathrm{len}\, F$ holds $\mathcal{P}[j]$. □

(41) Let us consider a prime number p, a commutative ring R with characteristic p, elements a, b of R, and a natural number i. Then $(a+b)^{p^i} = a^{p^i} + b^{p^i}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (a+b)^{p^{\$_1}} = a^{p^{\$_1}} + b^{p^{\$_1}}$. For every natural number k, $\mathcal{P}[k]$. $\square$

(42) Let us consider a prime number p, a commutative ring R with characteristic p, and an element a of R. Then $-a^p = (-a)^p$. The theorem is a consequence of (40).

Let p be a prime number and R be a commutative ring with characteristic p. The functor R^p yielding a strict double loop structure is defined by

(Def. 1) the carrier of *it* = the set of all a^p where a is an element of R and the addition of *it* = (the addition of R) $\upharpoonright$ (the carrier of *it*) and the multiplication of *it* = (the multiplication of R) $\upharpoonright$ (the carrier of *it*) and $1_{it} = 1_R$ and $0_{it} = 0_R$.

Let us observe that R^p is non degenerated.

Let us consider a prime number p, a commutative ring R with characteristic p, elements a, b of R, and elements x, y of R^p. Now we state the propositions:

(43) If $a = x$ and $b = y$, then $a + b = x + y$.

(44) If $a = x$ and $b = y$, then $a \cdot b = x \cdot y$.

Let p be a prime number and R be a commutative ring with characteristic p. Note that R^p is Abelian, add-associative, right zeroed, and right complementable and R^p is commutative, associative, well unital, and distributive.

Let F be a field with characteristic p. One can verify that F^p is almost left invertible. Let R be a commutative ring with characteristic p. Observe that R^p has characteristic p. Let F be a field with characteristic p. One can verify that the functor F^p yields a strict subfield of F.

4. The Polynomials $X^n - a$

Let R be a unital, non empty double loop structure, a be an element of R, and n be a non zero natural number. The functor $X^n - a$ yielding a sequence of R is defined by the term

(Def. 2) $\mathbf{0}.R{+\cdot}[0 \longmapsto -a, n \longmapsto 1_R]$.

Let us observe that $X^n - a$ is finite-Support.

Let R be a unital, non degenerated double loop structure. One can verify that $X^n - a$ is non constant and monic.

Let R be a non degenerated ring. One can verify that the functor $X^n - a$ yields a non constant, monic element of the carrier of Polynom-Ring R. Now we state the proposition:

(45) Let us consider a unital, non degenerated double loop structure L, an element a of L, and a non zero natural number n. Then

(i) $(X^n - a)(0) = -a$, and
(ii) $(X^n - a)(n) = 1_L$, and
(iii) for every natural number m such that $m \neq 0$ and $m \neq n$ holds $(X^n - a)(m) = 0_L$.

Let us consider a unital, non degenerated double loop structure R, a non zero natural number n, and an element a of R. Now we state the propositions:

(46) $\deg(X^n - a) = n$.

(47) $\mathrm{LC}\, X^n - a = 1_R$.

(48) Let us consider a non degenerated ring R, a non zero natural number n, and elements a, x of R. Then $\mathrm{eval}(X^n - a, x) = x^n - a$.
PROOF: Set $q = X^n - a$. Consider F being a finite sequence of elements of R such that $\mathrm{eval}(q, x) = \sum F$ and $\mathrm{len}\, F = \mathrm{len}\, q$ and for every element j of $\mathbb{N}$ such that $j \in \mathrm{dom}\, F$ holds $F(j) = q(j -' 1) \cdot \mathrm{power}_R(x, j -' 1)$. $n = \deg(q)$. Consider f_1 being a sequence of the carrier of R such that $\sum F = f_1(\mathrm{len}\, F)$ and $f_1(0) = 0_R$ and for every natural number j and for every element v of R such that $j < \mathrm{len}\, F$ and $v = F(j+1)$ holds $f_1(j+1) = f_1(j) + v$. Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv \$_1 = 0$ and $f_1(\$_1) = 0_R$ or $0 < \$_1 < \mathrm{len}\, F$ and $f_1(\$_1) = -a$ or $\$_1 = \mathrm{len}\, F$ and $f_1(\$_1) = x^n - a$. For every element j of $\mathbb{N}$ such that $0 \leqslant j \leqslant \mathrm{len}\, F$ holds $\mathcal{P}[j]$. □

(49) Let us consider a field F, a non zero natural number n, and elements a, b of F. Then b is a root of $X^n - a$ if and only if $b^n = a$. The theorem is a consequence of (48).

(50) Let us consider a field F, an extension E of F, a non zero natural number n, an element a of F, and an element b of E. If $b = a$, then $X^n - a = X^n - b$. The theorem is a consequence of (43).

(51) Let us consider a non degenerated, commutative ring R, a non trivial natural number n, and an element a of R. Then $(\mathrm{Deriv}(R))(X^n - a) = n \cdot (X^{(n-1)} - (0_R))$. The theorem is a consequence of (43) and (14).

(52) Let us consider a prime number p, a commutative ring R with characteristic p, and an element a of R. Then $(\mathrm{Deriv}(R))(X^p - a) = \mathbf{0}.R$. The theorem is a consequence of (43) and (38).

(53) Let us consider a prime number p, a field F with characteristic p, and elements a, b of F. If $b^p = a$, then $X^p - a = (\mathrm{X}- b)^p$. The theorem is a consequence of (7), (43), (40), (22), and (6).

(54) Let us consider a prime number p, a field F with characteristic p, and an element a of F. Suppose there exists no element b of F such that $b^p = a$. Then $X^p - a$ is irreducible. The theorem is a consequence of (50), (49), (53), (18), (31), (22), (5), (6), (3), (9), and (10).

5. More on Multiplicity of Roots

Now we state the propositions:

(55) Let us consider a field F, a non zero polynomial p over F, and an element a of F. Then $\deg(p) \geqslant \text{multiplicity}(p, a)$.

(56) Let us consider a field F, a non zero polynomial p over F, an element a of F, and an element n of $\mathbb{N}$. Then $(\text{X}- a)^n \mid p$ if and only if $\text{multiplicity}(p, a) \geqslant n$.

(57) Let us consider a field F, an extension E of F, a non zero element p of the carrier of Polynom-Ring F, and an element a of E. Then a is a root of p in E if and only if $\text{multiplicity}(p, a) \geqslant 1$. The theorem is a consequence of (56).

(58) Let us consider a field F, a non zero polynomial p over F, an extension E of F, and a non zero polynomial q over E. Suppose $q = p$. Let us consider an E-extending extension K of F, and an element a of K. Then $\text{multiplicity}(q, a) = \text{multiplicity}(p, a)$.

(59) Let us consider a field F, a non zero polynomial p over F, an extension E of F, and a non zero polynomial q over E. Suppose $q = p$. Let us consider an element a of E. Then $\text{multiplicity}(q, a) = \text{multiplicity}(p, a)$. The theorem is a consequence of (58).

(60) Let us consider a field F, a non zero polynomial p over F, a non zero element c of F, and an element a of F. Then $\text{multiplicity}(c \cdot p, a) = \text{multiplicity}(p, a)$.

(61) Let us consider a field F, an extension E of F, a non zero polynomial p over F, a non zero element c of F, and an element a of E. Then $\text{multiplicity}(c \cdot p, a) = \text{multiplicity}(p, a)$. The theorem is a consequence of (15) and (59).

(62) Let us consider a field F, an extension E of F, non zero polynomials p, q over F, and an element a of E. Then $\text{multiplicity}(p*q, a) = \text{multiplicity}(p, a) + \text{multiplicity}(q, a)$. The theorem is a consequence of (59).

(63) Let us consider a field F, a non zero polynomial p over F, extensions E_1, E_2 of F, and a function i from E_1 into E_2. Suppose i is F-fixing and isomorphism. Let us consider an element a of E_1. Then $\text{multiplicity}(p, a) = \text{multiplicity}(p, i(a))$.

PROOF: Set $n = \text{multiplicity}(p, a)$. Reconsider $E_3 = E_2$ as an E_1-homomorphic field. Reconsider $h = i$ as an additive function from E_1 into E_3. Reconsider $X_1 = (\text{X}- a)^n$ as an element of the carrier of Polynom-Ring E_1. Reconsider $X_2 = (\text{X}- a)^{n+1}$ as an element of the carrier of Polynom-Ring E_1.

(PolyHom(h))(X_1) = (X$-$ $h(a))^n$ and (PolyHom(h))(X_2) = (X$-$ $h(a))^{n+1}$. (PolyHom(h))(p) = p. □

(64) Let us consider a field F, a non zero polynomial p over F, an extension E of F, and an element a of F. Then multiplicity(p, ${}^{@}(a, E)$) = multiplicity(p, a).

(65) Let us consider a field F, a non zero polynomial p over F, an extension E of F, an E-extending extension K of F, and an element a of E. Then multiplicity(p, ${}^{@}(a, K)$) = multiplicity(p, a).

(66) Let us consider a field F, a non zero polynomial p over F, a polynomial q over F, and an element a of F. Suppose p = (X$-$ $a)^{\text{multiplicity}(p,a)} * q$. Then eval($q, a$) $\neq 0_F$.

(67) Let us consider a field F, and a non zero polynomial p over F. Then $\overline{\overline{\text{Roots}(p)}} < \overline{\overline{\text{BRoots}(p)}}$ if and only if there exists an element a of F such that multiplicity(p, a) > 1.

(68) Let us consider a field F, a non zero polynomial p over F, and an element a of F. Then multiplicity(NormPoly p, a) = multiplicity(p, a).

(69) Let us consider a field F, and a non constant polynomial p over F. Then $\deg(p) = \overline{\overline{\text{Roots}(p)}}$ if and only if p splits in F and for every element a of F, multiplicity(p, a) $\leqslant 1$. The theorem is a consequence of (67) and (68).

(70) Let us consider a field F, a non zero element p of the carrier of Polynom-Ring F, and an element a of F. Suppose a is a root of p. Then

   (i) multiplicity(p, a) = 1 iff eval((Deriv(F))(p), a) $\neq 0_F$, and

   (ii) multiplicity(p, a) > 1 iff eval((Deriv(F))(p), a) $= 0_F$.

   The theorem is a consequence of (66).

(71) Let us consider a field F, and a non zero element p of the carrier of Polynom-Ring F. Then there exists an element a of F such that multiplicity (p, a) > 1 if and only if gcd(p, (Deriv(F))(p)) has roots. The theorem is a consequence of (70).

(72) Let us consider a field F, a non zero element p of the carrier of Polynom-Ring F, and an extension E of F. Suppose p splits in E. Then there exists an element a of E such that multiplicity(p, a) > 1 if and only if gcd(p, (Deriv(F))(p)) $\neq$ **1**.F. The theorem is a consequence of (70).

(73) Let us consider a field F, an irreducible element p of the carrier of Polynom-Ring F, and an extension E of F. Suppose p splits in E. Then there exists an element a of E such that multiplicity(p, a) > 1 if and only if (Deriv(F))(p) = **0**.F. The theorem is a consequence of (17) and (72).

(74) Let us consider a prime number p, a commutative ring R with characteristic p, and an element f of the carrier of Polynom-Ring R. Then

(Deriv(R))(f) $= \mathbf{0}.R$ if and only if for every natural number i such that $i \in$ Support f holds $p \mid i$. The theorem is a consequence of (38) and (39).

6. Separable Polynomials

Let F be a field and p be a non constant element of the carrier of Polynom-Ring F. We say that p is separable if and only if

(Def. 3) for every element a of the splitting field of p such that a is a root of p in the splitting field of p holds multiplicity$(p, a) = 1$.

We introduce the notation p is inseparable as an antonym for p is separable.

Let us observe that there exists a non constant, monic element of the carrier of Polynom-Ring F which is separable and there exists a non constant, monic element of the carrier of Polynom-Ring F which is inseparable.

Let us consider a field F and a non constant element p of the carrier of Polynom-Ring F. Now we state the propositions:

(75) p is separable if and only if for every extension E of F such that p splits in E for every element a of E such that a is a root of p in E holds multiplicity$(p, a) = 1$. The theorem is a consequence of (63).

(76) p is separable if and only if there exists an extension E of F such that p splits in E and for every element a of E such that a is a root of p in E holds multiplicity$(p, a) = 1$. The theorem is a consequence of (63).

(77) p is separable if and only if for every extension E of F and for every element a of E, multiplicity$(p, a) \leqslant 1$. The theorem is a consequence of (58), (57), (75), and (76).

(78) p is separable if and only if there exists an extension E of F such that p splits in E and for every element a of E, multiplicity$(p, a) \leqslant 1$. The theorem is a consequence of (57) and (76).

(79) Let us consider a field F, and a separable, non constant element p of the carrier of Polynom-Ring F. Then $\deg(p) = \overline{\overline{\mathrm{Roots}(p)}}$ if and only if p splits in F. The theorem is a consequence of (75), (60), and (69).

(80) Let us consider a field F, and a non constant element p of the carrier of Polynom-Ring F. Then p is separable if and only if gcd$(p, (\mathrm{Deriv}(F))(p)) = \mathbf{1}.F$. The theorem is a consequence of (77) and (72).

(81) Let us consider a field F, and a non constant, irreducible element p of the carrier of Polynom-Ring F. Then p is separable if and only if (Deriv(F)) $(p) \neq \mathbf{0}.F$. The theorem is a consequence of (77) and (73).

(82) Let us consider a field F, and a non constant element p of the carrier of Polynom-Ring F. Then p is separable if and only if for every splitting field

E of p, there exists an element a of E and there exists a product of linear polynomials q of E and $\mathrm{Roots}(E, p)$ such that $p = a \cdot q$. The theorem is a consequence of (75), (59), and (60).

(83) Let us consider a field F, and a non constant, monic element p of the carrier of Polynom-Ring F. Then p is separable if and only if for every splitting field E of p, p is a product of linear polynomials of E and $\mathrm{Roots}(E, p)$. The theorem is a consequence of (82).

Let us consider a field F and a non constant element p of the carrier of Polynom-Ring F. Now we state the propositions:

(84) p is separable if and only if for every extension E of F such that p splits in E holds p is square-free over E. The theorem is a consequence of (60), (75), and (56).

(85) p is separable if and only if there exists an extension E of F such that $\overline{\overline{\mathrm{Roots}(E, p)}} = \deg(p)$. The theorem is a consequence of (77), (58), (79), (69), and (78).

(86) Let us consider a field F, a non constant element p of the carrier of Polynom-Ring F, and a non zero element a of F. Then $a \cdot p$ is separable if and only if p is separable. The theorem is a consequence of (15), (75), and (61).

(87) Let us consider a field F, non constant elements p, q of the carrier of Polynom-Ring F, and an element r of the carrier of Polynom-Ring F. If $p = q * r$, then if p is separable, then q is separable. The theorem is a consequence of (77) and (62).

(88) Let us consider a field F, an extension E of F, a non constant element p of the carrier of Polynom-Ring F, and a non constant element q of the carrier of Polynom-Ring E. If $p = q$, then p is separable iff q is separable. The theorem is a consequence of (80).

Let F be a field and a be an element of F. One can verify that $\mathrm{X}- a$ is separable and irreducible. Let n be a non trivial natural number. Note that $(\mathrm{X}- a)^n$ is inseparable and reducible. Let F be a field with characteristic 0. One can check that every irreducible element of the carrier of Polynom-Ring F is separable. Now we state the proposition:

(89) Let us consider a prime number p, a field F with characteristic p, and an element a of F. If $a \notin F^p$, then $X^p - a$ is irreducible and inseparable. The theorem is a consequence of (54), (50), (49), (53), (28), and (77).

7. Perfect Fields

Let F be a field. We say that F is perfect if and only if

(Def. 4) every irreducible element of the carrier of Polynom-Ring F is separable.

Let us note that every field with characteristic 0 is perfect. Now we state the propositions:

(90) Let us consider a prime number p, a field F with characteristic p, and an element q of the carrier of Polynom-Ring F. Suppose for every natural number i such that $i \in \text{Support}\, q$ holds $p \mid i$ and there exists an element a of F such that $a^p = q(i)$. Then there exists an element r of the carrier of Polynom-Ring F such that $r^p = q$. The theorem is a consequence of (25) and (40).

(91) Let us consider a prime number p, and a field F with characteristic p. Then F is perfect if and only if $F \approx F^p$. The theorem is a consequence of (89), (75), (57), (73), (74), and (90).

(92) Let us consider a field F. Then F is finite if and only if there exists a non zero natural number n such that $\overline{\overline{F}} = (\text{char}(F))^n$. The theorem is a consequence of (39) and (4).

(93) Let us consider a prime number p, a finite field F with characteristic p, and an element a of F. Then there exists an element b of F such that $b^p = a$. The theorem is a consequence of (92) and (10).

Observe that every finite field is perfect and every algebraic closed field is perfect.

8. Separable Extensions

Let F be a field, E be an extension of F, and a be an element of E. We say that a is F-separable if and only if

(Def. 5) there exists an F-algebraic element b of E such that $b = a$ and MinPoly(b, F) is separable.

One can verify that there exists an element of E which is non zero and F-separable and every element of E which is F-separable is also F-algebraic. Let a be an F-separable element of E. Observe that MinPoly(a, F) is separable. We say that E is F-separable if and only if

(Def. 6) E is F-algebraic and every element of E is F-separable.

We introduce the notation E is F-inseparable as an antonym for E is F-separable. Let us observe that there exists an extension of F which is F-finite and F-separable and every extension of F which is F-separable is also F-algebraic. Let E be an F-separable extension of F. Note that every element of E is F-separable. Now we state the proposition:

(94) Let us consider a field F, an extension K of F, and a K-extending extension E of F. Suppose E is F-separable. Then

(i) E is K-separable, and

(ii) K is F-separable.

The theorem is a consequence of (88) and (87).

Let F be a perfect field. One can verify that every F-algebraic extension of F is F-separable and there exists an extension of F which is F-normal and F-separable. Let p be a non constant element of the carrier of Polynom-Ring F. Let us note that every splitting field of p is F-normal and F-separable.

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