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Formalization of Orthogonal Decomposition for Hilbert Spaces

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Summary. In this article, we formalize the theorems about orthogonal decomposition of Hilbert spaces, using the Mizar system [1], [2]. For any subspace S of a Hilbert space H, any vector can be represented by the sum of a vector in S and a vector orthogonal to S. The formalization of orthogonal complements of Hilbert spaces has been stored in the Mizar Mathematical Library [4]. We referred to [5] and [6] in the formalization.

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1. Preliminaries

From now on X denotes a real unitary space and x, y, y_1, y_2 denote points of X. Now we state the proposition:

(1) Let us consider a real unitary space X, points x, y of X, and points z, t of MetricSpaceNorm(the real normed space of X). If $x = z$ and $y = t$, then $\|x - y\| = \rho(z, t)$.

Let us consider a real unitary space X, an element z of MetricSpaceNorm(the real normed space of X), and a real number r. Now we state the propositions:

(2) There exists a point x of X such that

(i) $x = z$, and

(ii) $\mathrm{Ball}(z, r) = \{y, \text{ where } y \text{ is a point of } X : \|x - y\| < r\}$.

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The theorem is a consequence of (1).

(3) There exists a point x of X such that

(i) $x = z$, and

(ii) $\overline{\mathrm{Ball}}(z, r) = \{y$, where y is a point of $X : \|x - y\| \leqslant r\}$.

The theorem is a consequence of (1).

(4) Let us consider a real unitary space X, a sequence S of X, a sequence S_1 of MetricSpaceNorm(the real normed space of X), a point x of X, and a point x_2 of MetricSpaceNorm(the real normed space of X). Suppose $S = S_1$ and $x = x_2$. Then S_1 is convergent to x_2 if and only if for every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (1).

Let us consider a real unitary space X, a sequence S of X, and a sequence S_1 of MetricSpaceNorm(the real normed space of X). Now we state the propositions:

(5) If $S = S_1$, then S_1 is convergent iff S is convergent. The theorem is a consequence of (4).

(6) If $S = S_1$ and S_1 is convergent, then $\lim S_1 = \lim S$. The theorem is a consequence of (5) and (4).

2. Topological Space Generated from Real Unitary Space

Now we state the proposition:

(7) Let us consider a real unitary space X, and a subset V of TopSpaceNorm (the real normed space of X). Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\{y$, where y is a point of $X : \|x - y\| < r\} \subseteq V$. The theorem is a consequence of (2).

Let us consider a real unitary space X, a point x of X, and a real number r. Now we state the propositions:

(8) $\{y$, where y is a point of $X : \|x - y\| < r\}$ is an open subset of TopSpaceNorm(the real normed space of X). The theorem is a consequence of (2).

(9) $\{y$, where y is a point of $X : \|x - y\| \leqslant r\}$ is a closed subset of TopSpaceNorm(the real normed space of X). The theorem is a consequence of (3).

(10) Let us consider a real unitary space M, a subset X of TopSpaceNorm(the real normed space of M), and an object x. Then $x \in \overline{X}$ if and only if there exists a sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent and $\lim S = x$. The theorem is a consequence of (5) and (6).

(11) Let us consider a real unitary space M, and a subset X of TopSpaceNorm (the real normed space of M). Then X is closed if and only if for every sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent holds $\lim S \in X$. The theorem is a consequence of (5) and (6).

(12) Let us consider a real unitary space S, and a subset X of S. Then X is a closed subset of TopSpaceNorm(the real normed space of S) if and only if for every sequence s_1 of S such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$. The theorem is a consequence of (11).

(13) Let us consider a real unitary space S, a point x of S, a point y of MetricSpaceNorm(the real normed space of S), and a real number r. If $x = y$, then $\mathrm{Ball}(x, r) = \mathrm{Ball}(y, r)$. The theorem is a consequence of (1).

(14) Let us consider a real unitary space S. Then TopSpaceNorm(the real normed space of S) = TopUnitSpace S. The theorem is a consequence of (13).

Let us consider a real unitary space S, a subset U of S, and a subset V of TopSpaceNorm(the real normed space of S). Now we state the propositions:

(15) If $U = V$, then U is closed iff V is closed.

(16) If $U = V$, then U is open iff V is open.

(17) Let us consider a real unitary space X, a subspace M of X, and points x, m_0 of X. Suppose $m_0 \in M$. Then for every point m of X such that $m \in M$ holds $\|x - m_0\| \leqslant \|x - m\|$ if and only if for every point m of X such that $m \in M$ holds $((x - m_0)|m) = 0$.

(18) Let us consider a real unitary space X, a subspace M of X, and points x, m_1, m_2 of X. Suppose $m_1, m_2 \in M$ and for every point m of X such that $m \in M$ holds $\|x - m_1\| \leqslant \|x - m\|$ and for every point m of X such that $m \in M$ holds $\|x - m_2\| \leqslant \|x - m\|$. Then $m_1 = m_2$.

(19) Let us consider a real Hilbert space of X, a subspace M of X, and a point x of X. Suppose the carrier of M is a closed subset of TopSpaceNorm(the real normed space of X). Then there exists a point m_0 of X such that

  (i) $m_0 \in M$, and

  (ii) for every point m of X such that $m \in M$ holds $\|x - m_0\| \leqslant \|x - m\|$.

  The theorem is a consequence of (12).

Let X be a real unitary space and M be a subset of X. The functor OrtCompSet(M) yielding a non empty subset of X is defined by

(Def. 1) for every point x of X, $x \in it$ iff for every point y of X such that $y \in M$ holds $(y|x) = 0$.

Now we state the propositions:

(20) Let us consider a real unitary space X, and a subset M of X. Then OrtCompSet(M) is linearly closed.
PROOF: For every vectors v, u of X such that v, $u \in$ OrtCompSet(M) holds $v+u \in$ OrtCompSet(M). For every real number a and for every vector v of X such that $v \in$ OrtCompSet(M) holds $a \cdot v \in$ OrtCompSet(M). □

(21) Let us consider a real unitary space X, a non empty subset M of X, and a sequence s_2 of X. Suppose rng $s_2 \subseteq$ the carrier of OrtComp(M) and s_2 is convergent. Then $\lim s_2 \in$ the carrier of OrtComp(M).

(22) Let us consider a real unitary space S, a non empty subset M of S, and a subset L of S. Suppose $L =$ the carrier of OrtComp(M). Then L is a closed subset of TopSpaceNorm(the real normed space of S). The theorem is a consequence of (21) and (12).

(23) Let us consider a real unitary space X. Then every non empty subset of X is a subset of OrtComp(OrtComp(M)).

(24) Let us consider a real unitary space X, and non empty subsets S, T of X. Suppose $S \subseteq T$. Then OrtComp(T) is a subspace of OrtComp(S).

(25) Let us consider a real Hilbert space of X, and a subspace M of X. Suppose X is strict and the carrier of M is a closed subset of TopSpaceNorm(the real normed space of X). Then X is the direct sum of M and OrtComp(M).
PROOF: For every object z, $z \in$ the carrier of M + OrtComp(M) iff $z \in$ the carrier of X. For every object z, $z \in$ the carrier of $M \cap$ OrtComp(M) iff $z \in \{0_X\}$. □

(26) Let us consider a real Hilbert space of X, and a strict subspace M of X. Suppose X is strict and the carrier of M is a closed subset of TopSpaceNorm(the real normed space of X).
Then $M =$ OrtComp(OrtComp(M)).
PROOF: Reconsider $N =$ the carrier of M as a subset of X. N is a subset of OrtComp(OrtComp(N)). The carrier of OrtComp(OrtComp(M)) $\subseteq N$. □

(27) Let us consider a real unitary space X, a subspace M of X, a subset K of X, and a subset L of TopSpaceNorm(the real normed space of X). Suppose the carrier of $M = L$ and $K = \overline{L}$. Then K is linearly closed.

Proof: For every vectors v, u of X such that $v, u \in K$ holds $v+u \in K$. For every real number a and for every vector v of X such that $v \in K$ holds $a \cdot v \in K$ by (10), [3, (15)]. □

(28) Let us consider a real Hilbert space of X, and a non empty subset M of X. Suppose X is strict. Then

(i) the carrier of OrtComp(OrtComp(M)) is a closed subset of TopSpaceNorm(the real normed space of X), and

(ii) there exists a subset L of TopSpaceNorm(the real normed space of X) such that $L =$ the carrier of Lin(M) and the carrier of OrtComp(OrtComp(M)) $= \overline{L}$, and

(iii) Lin(M) is a subspace of OrtComp(OrtComp(M)).

(29) Let us consider a real Hilbert space of X, a strict subspace K of X, and a non empty subset M of X. Suppose X is strict and the carrier of K is a closed subset of TopSpaceNorm(the real normed space of X) and Lin(M) is a subspace of K. Then OrtComp(OrtComp(M)) is a subspace of K.

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