FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 139–145, 2024
DOI: 10.2478/forma-2024-0011
e-ISSN: 1898–9934

sciendo.com/journal/forma

Ascoli-Arzela's Theorem (Metric Space Version)

Keiichi Miyajima
Ibaraki University
Faculty of Engineering
Hitachi, Ibaraki, Japan

Hiroshi Yamazaki
Nagano Prefectural Institute of Technology
Nagano, Japan

Summary. In this article, the Ascoli-Arzela's theorem on metric space is formalized [12], [13], [16]. First, we gave definitions of equicontinuousness and equiboundedness of a set of continuous functions [19], [14], [9], [17], [18]. Next, we formalized the Ascoli-Arzela's theorem using those definitions, and proved this theorem. From this result, Ascoli-Arzela's theorem can be applied in a metric space that is easier to apply.

MSC: 46B50 68V20

Keywords: Ascoli-Arzela's theorem; equicontinuousness of continuous functions; equiboundedness of continuous functions

MML identifier: `ASCOLI2`, version: `8.1.14 5.84.1473`

1. Equicontinuousness and Equiboundedness of Continuous Functions

Now we state the propositions:

(1) Let us consider a non empty metric space T, and a subset A of T. Then $A \subseteq \overline{A}$.

(2) Let us consider a non empty topological space S, a non empty metric space T, a function f from S into $T_{\rm top}$, and a point x of S. Then f is continuous at x if and only if for every real number e such that $0 < e$ there exists a subset H of S such that H is open and $x \in H$ and for every point y of S such that $y \in H$ holds $\rho(f(x)(\in T), f(y)(\in T)) < e$.

Proof: For every subset G of T_{top} such that G is open and $f(x) \in G$ there exists a subset H of S such that H is open and $x \in H$ and $f^{\circ}H \subseteq G$ by [8, (15)], [10, (11)]. □

Let S, T be non empty metric spaces and F be a subset of (the carrier of $T)^{(\text{the carrier of } S)}$. We say that F is equibounded if and only if

(Def. 1) there exists a subset K of T such that K is bounded and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every element x of S, $f(x) \in K$.

Let x_0 be a point of S. We say that F is equicontinuous at x_0 if and only if

(Def. 2) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, x_0) < d$ holds $\rho(f(x), f(x_0)) < e$.

We say that F is equicontinuous if and only if

(Def. 3) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\rho(f(x_1), f(x_2)) < e$.

2. Ascoli-Arzela's Theorem

Now we state the proposition:

(3) Let us consider a non empty metric space Z, and a non empty subset F of Z. If Z is complete, then $Z \restriction \overline{F}$ is complete.

Proof: Set $N = Z \restriction \overline{F}$. Reconsider $S_1 = S2$ as a sequence of Z. For every real number r such that $r > 0$ there exists a natural number k such that for every natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\rho(S_1(n), S_1(m)) < r$. Consider H being a subset of Z_{top} such that $H = F$ and $\overline{F} = \overline{H}$. For every natural number n, $S_1(n) \in \overline{H}$ by [5, (4)]. Reconsider $L = \lim S_1$ as a point of N. For every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\rho(S2(n), L) < r$. □

Let us consider a non empty metric space Z and a non empty subset H of Z. Now we state the propositions:

(4) $Z \restriction H$ is totally bounded if and only if $Z \restriction \overline{H}$ is totally bounded.

Proof: Consider D being a subset of Z_{top} such that $D = H$ and $\overline{H} = \overline{D}$. $Z \restriction H$ is totally bounded by [10, (4)]. □

(5) If Z is complete and $Z\restriction H$ is totally bounded, then $\overline{H}$ is sequentially compact and $Z\restriction\overline{H}$ is compact. The theorem is a consequence of (3) and (4).

(6) Suppose Z is complete. Then

(i) $Z\restriction H$ is totally bounded iff $\overline{H}$ is sequentially compact, and

(ii) $Z\restriction H$ is totally bounded iff $Z\restriction\overline{H}$ is compact.

The theorem is a consequence of (3) and (4).

Let S be a non empty topological space and T be a non empty metric space. The continuous functions of S and T yielding a non empty set is defined by the term

(Def. 4) $\{f$, where f is a function from S into $T_{\rm top}$: f is continuous$\}$.

Now we state the propositions:

(7) Let us consider a metric space X, and elements x, y, v, w of X. Then $|\rho(x,y)-\rho(v,w)| \leqslant \rho(x,v)+\rho(y,w)$.

(8) Let us consider a non empty topological space S, a non empty metric space T, and functions f, g from S into $T_{\rm top}$. Suppose f is continuous and g is continuous. Let us consider a real map D_1 of S. Suppose for every point x of S, $D_1(x)=\rho(f(x)(\in T), g(x)(\in T))$. Then D_1 is continuous. The theorem is a consequence of (2) and (7).

(9) Let us consider a non empty, compact topological space S, a non empty metric space T, and functions f, g from S into $T_{\rm top}$. Suppose f is continuous and g is continuous. Let us consider a real map D_1 of S. Suppose for every point x of S, $D_1(x)=\rho(f(x)(\in T), g(x)(\in T))$. Then

(i) $\operatorname{rng} D_1 \neq \emptyset$, and

(ii) $\operatorname{rng} D_1$ is upper bounded and lower bounded.

The theorem is a consequence of (8).

(10) Let us consider a non empty topological space S, and a non empty metric space T. Then there exists a function F from (the continuous functions of S and T) $\times$ (the continuous functions of S and T) into $\mathbb{R}$ such that for every functions f, g from S into $T_{\rm top}$ such that f, $g \in$ the continuous functions of S and T there exists a real map D_1 of S such that for every point x of S, $D_1(x)=\rho(f(x)(\in T), g(x)(\in T))$ and $F(f,g)=\sup\operatorname{rng} D_1$.
PROOF: Set $F_1=$ the continuous functions of S and T. Define $\mathcal{P}$[object, object, object there exist functions f, g from S into $T_{\rm top}$ and there exists a real map D_1 of S such that $\$_1=f$ and $\$_2=g$ and for every point t of S, $D_1(t)=\rho(f(t)(\in T), g(t)(\in T))$ and $\$_3=\sup\operatorname{rng} D_1$. For every objects x, y such that x, $y\in F_1$ there exists an object z such that $z\in\mathbb{R}$ and

$\mathcal{P}[x, y, z]$. Consider F being a function from $F_1 \times F_1$ into $\mathbb{R}$ such that for every objects x, y such that x, $y \in F_1$ holds $\mathcal{P}[x, y, F(x, y)]$ from [3, Sch. 1]. □

Let S be a non empty topological space and T be a non empty metric space. The functor dist-Func(S, T) yielding a function from (the continuous functions of S and T) $\times$ (the continuous functions of S and T) into $\mathbb{R}$ is defined by

(Def. 5) for every functions f, g from S into $T_{\rm top}$ such that $f, g \in$ the continuous functions of S and T there exists a real map D_1 of S such that for every point x of S, $D_1(x) = \rho(f(x)(\in T), g(x)(\in T))$ and $it(f, g) = \sup \operatorname{rng} D_1$.

The functor MetricSpace-of-ContinuousFunctions(S, T) yielding a metric structure is defined by the term

(Def. 6) $\langle$the continuous functions of S and T, dist-Func$(S, T)\rangle$.

Let S be a non empty, compact topological space. Note that MetricSpace-of-Continuous is reflexive, discernible, symmetric, and triangle.

Let S be a non empty topological space. One can verify that MetricSpace-of-Continuous is non empty and strict and the continuous functions of S and T is non empty and functional.

Let S be a non empty, compact topological space. Note that MetricSpace-of-Continuous is constituted functions.

Let f be an element of MetricSpace-of-ContinuousFunctions(S, T) and v be a point of S. One can check that the functor $f(v)$ yields a point of $T_{\rm top}$. Now we state the propositions:

(11) Let us consider a non empty, compact topological space S, a non empty metric space T, points f, g of MetricSpace-of-ContinuousFunctions(S, T), and a point t of S. Then $\rho(f(t)(\in T), g(t)(\in T)) \leqslant \rho(f, g)$. The theorem is a consequence of (9).

(12) Let us consider a non empty, compact topological space S, a non empty metric space T, points f, g of MetricSpace-of-ContinuousFunctions(S, T), functions f_1, g_1 from S into T, and a real number e. Suppose $f = f_1$ and $g = g_1$ and for every point t of S, $\rho(f_1(t), g_1(t)) \leqslant e$. Then $\rho(f, g) \leqslant e$. The theorem is a consequence of (9).

(13) Let us consider a non empty, compact topological space S, and a non empty metric space T. Suppose T is complete. Then MetricSpace-of-ContinuousFunct is complete. The theorem is a consequence of (11), (2), and (12).

(14) Let us consider a non empty, compact topological space S, and a non empty metric space T. Suppose T is complete. Let us consider a non empty subset H of MetricSpace-of-ContinuousFunctions(S, T). Then $\overline{H}$ is sequentially compact if and only if MetricSpace-of-ContinuousFunctions$(S, T) \restriction H$

is totally bounded. The theorem is a consequence of (13), (3), and (4).

Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpace-of-ContinuousFunctions$(S, T$
Now we state the propositions:

(15) If $S = M_{\rm top}$, then if $G = H$ and MetricSpace-of-ContinuousFunctions$(S,T)\restriction H$ is totally bounded, then G is equicontinuous.

PROOF: Set Z = MetricSpace-of-ContinuousFunctions(S,T). Set $M_2 = Z\restriction H$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point w of M_2 such that $\$_2 = w$ and $\$_1 = \text{Ball}(w, 1)$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in G$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\rho(f(x_1), f(x_2)) < e$ by [6, (2)], [4, (3)], [11, (133)], [5, (35)]. □

(16) Suppose $S = M_{\rm top}$. Then suppose $G = H$ and MetricSpace-of-ContinuousFunctions
is totally bounded. Then

- (i) for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T\restriction H_1$ is totally bounded, and
- (ii) G is equicontinuous.

PROOF: For every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T\restriction H_1$ is totally bounded by [10, (11)], (11), [5, (35)]. □

(17) Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then MetricSpace-of-Continuous
is totally bounded if and only if G is equicontinuous and for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T\restriction\overline{H_1}$ is compact.

PROOF: Set Z = MetricSpace-of-ContinuousFunctions(S,T). Set $M_2 = Z\restriction H$. For every real number e such that $e > 0$ there exists a family L of subsets of M_2 such that L is finite and the carrier of $M_2 = \bigcup L$ and for every subset C of M_2 such that $C \in L$ there exists an element w of M_2 such that $C = \text{Ball}(w, e)$ by [2, (29)], [10, (1)], [7, (1)], [1, (93), (16)]. □

(18) Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then $\overline{H}$ is sequentially compact if and only if G is equicontinuous and for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T\restriction\overline{H_1}$ is compact. The theorem is a consequence of (14) and (17).

Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a non empty subset F of

MetricSpace-of-ContinuousFunctions(S, T), and a subset G of (the carrier of T)$^{(\text{the carrier of } M)}$. Now we state the propositions:

(19) Suppose $S = M_{\text{top}}$ and T is complete and $G = F$. Then MetricSpace-of-Continuous is compact if and only if G is equicontinuous and for every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T\restriction\overline{F_2}$ is compact. The theorem is a consequence of (14) and (17).

(20) Suppose $S = M_{\text{top}}$ and T is complete and $G = F$. Then MetricSpace-of-Continuous is compact if and only if for every point x of M, G is equicontinuous at x and for every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T\restriction\overline{F_2}$ is compact. The theorem is a consequence of (19).

Now we state the proposition:

(21) Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a compact subset U of T_{top}, a non empty subset F of MetricSpace-of-ContinuousFunctions(S, T), and a subset G of (the carrier of T)$^{\alpha}$. Suppose $S = M_{\text{top}}$ and T is complete and $G = F$ and for every function f such that $f \in F$ holds $\operatorname{rng} f \subseteq U$. Then MetricSpace-of-ContinuousFunctions(S, T)$\restriction\overline{F}$ is compact if and only if G is equicontinuous, where α is the carrier of M.

PROOF: Set $Z =$ MetricSpace-of-ContinuousFunctions(S, T). $\overline{F}$ is sequentially compact iff $Z\restriction F$ is totally bounded. For every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T\restriction\overline{F_2}$ is compact by [5, (4)], [2, (34)], [15, (19), (22)]. □

ACKNOWLEDGEMENT: The authors would like to express our gratitude to Prof. Yasunari Shidama for his support and encouragement.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[8] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[9] Bruce K. Driver. *Analysis Tools with Applications*. Springer, Berlin, 2003.
[10] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[11] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[12] Serge Lang. *Real and Functional Analysis (Texts in Mathematics)*. Springer-Verlag, 1993.
[13] Kazuo Matsuzaka. *Sets and Topology (Introduction to Mathematics)*. IwanamiShoten, 2000.
[14] Tohru Ozawa. Ascoli-Arzelà theorem. 2012.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Michael Read and Barry Simon. *Functional Analysis (Methods of Modern Mathematical Physics)*. Academic Press, 1980.
[17] Laurent Schwartz. *Théorie des ensembles et topologie, tome 1. Analyse*. Hermann, 1997.
[18] Laurent Schwartz. *Calcul différentiel, tome 2. Analyse*. Hermann, 1997.
[19] Kôsaku Yosida. *Functional Analysis*. Springer, 1980.

Accepted November 28, 2024