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U-Small and U-Locally Small Categories[1]

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Summary. Category theory is developed from the beginning of Mizar Mathematical Library [4]. Gradually it was expanded, with several different definitions for a category: `Category` ([5]) and `category` (in [12] or with another definition in [15]) based on [14] and [10]. In the following, we will only use, among these 3 definitions, the first, as well as the notion $\mathcal{U}$ for Grothendieck's non-empty Universe.

The first part of this work is devoted to the definitions of $\mathcal{U}$`-set` and proper classes $\mathcal{U}$`-class`.

The second part is largely influenced by the number 0 *Universe* of the first presentation of SGA 4 [1], we define the notion of an $\mathcal{U}$`-small set` (and of $\mathcal{U}$`-small` group as well as of $\mathcal{U}$`-small` Category). This allows us to access the formalization of the definition of $\mathcal{U}$`-Category`.

Finally, we introduce the notions of $\mathcal{U}$`-small Category` and $\mathcal{U}$`-locally small Category` and some classic examples (adapted from "Example 1.1.4" by Emily Riehl in "Category theory in Context" [13]).

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a non empty set X. Then $\{\langle A, B\rangle$, where A, B are elements of X : if B $= \emptyset$, then $A = \emptyset\} = (X \times X) \setminus (X \setminus \{\emptyset\} \times \{\emptyset\})$.

(2) Let us consider a non empty set X. Suppose $\{\emptyset\}$ is an element of X. Then $\{\emptyset\} \notin \mathrm{Funcs}\, X$.

(3) $\mathbb{N}_{\mathrm{even}}$ is denumerable.

(4) $\mathbb{N}_{\mathrm{odd}}$ is denumerable.

(5) Let us consider non empty sets X, Y, and an element y of Y. Then $X \times \{y\} \subseteq \bigcup Y^X$.

(6) Let us consider a non empty set X, and a non zero natural number n. If X^n is finite, then X is finite.

Let us consider a non empty set X. Now we state the propositions:

(7) $\bigcup \mathrm{SmallestPartition}(X) = X$.

(8) $X \approx \mathrm{SmallestPartition}(X)$.

Now we state the proposition:

(9) Let us consider a strict object-category C. Then $(C^{\mathrm{op}})^{\mathrm{op}} = C$.

Let x_1, x_2, x_3, x_4, x_5 be objects. The functor $\langle x_1, x_2, x_3, x_4, x_5\rangle$ yielding an object is defined by the term

(Def. 1) $\langle\langle x_1, x_2, x_3, x_4\rangle, x_5\rangle$.

Let x be an object. We say that x is quintuple if and only if

(Def. 2) there exist objects x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5\rangle$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Let us note that $\langle x_1, x_2, x_3, x_4, x_5\rangle$ is quintuple.

Now we state the proposition:

(10) Let us consider objects x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5. Suppose $\langle x_1, x_2, x_3, x_4, x_5\rangle = \langle y_1, y_2, y_3, y_4, y_5\rangle$. Then

(i) $x_1 = y_1$, and

(ii) $x_2 = y_2$, and

(iii) $x_3 = y_3$, and

(iv) $x_4 = y_4$, and

(v) $x_5 = y_5$.

One can verify that there exists an object which is quintuple and there exists a set which is quintuple.

Let x be an object. Assume x is quintuple. The functor $(x)_{\mathbf{1}}$ yielding an object is defined by

(Def. 3) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_1$.

Assume x is quintuple. The functor $(x)_{\mathbf{2}}$ yielding an object is defined by

(Def. 4) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_2$.

Assume x is quintuple. The functor $(x)_{\mathbf{3}}$ yielding an object is defined by

(Def. 5) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_3$.

Assume x is quintuple. The functor $(x)_{\mathbf{4}}$ yielding an object is defined by

(Def. 6) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_4$.

Assume x is quintuple. The functor $(x)_{\mathbf{5}}$ yielding an object is defined by

(Def. 7) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_5$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Observe that $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{1}}$ reduces to x_1 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{2}}$ reduces to x_2 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{3}}$ reduces to x_3 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{4}}$ reduces to x_4 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{5}}$ reduces to x_5.

Let x be a quintuple object. Observe that $\langle (x)_{\mathbf{1}}, (x)_{\mathbf{2}}, (x)_{\mathbf{3}}, (x)_{\mathbf{4}}, (x)_{\mathbf{5}} \rangle$ reduces to x.

2. Some Elementary Properties

From now on $\mathcal{U}$ denotes a universal class and x denotes an element of $\mathcal{U}$.

Now we state the propositions:

(11) Let us consider objects x_1, x_2, x_3. Suppose $x = \langle x_1, x_2, x_3 \rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$.

(12) Let us consider objects x_1, x_2, x_3, x_4. Suppose $x = \langle x_1, x_2, x_3, x_4 \rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$, and

(iv) x_4 is an element of $\mathcal{U}$.

The theorem is a consequence of (11).

(13) Let us consider elements x_1, x_2, x_3, x_4, x_5 of $\mathcal{U}$. Then $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is an element of $\mathcal{U}$.

(14) Let us consider objects x_1, x_2, x_3, x_4, x_5. Suppose $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$, and

(iv) x_4 is an element of $\mathcal{U}$, and

(v) x_5 is an element of $\mathcal{U}$.

The theorem is a consequence of (12).

Let $\mathcal{U}$ be a universal class and u_1, u_2, u_3 be elements of $\mathcal{U}$. Observe that the functor $\langle u_1, u_2, u_3 \rangle$ yields an element of $\mathcal{U}$. Let u_4 be an element of $\mathcal{U}$. Let us observe that the functor $\langle u_1, u_2, u_3, u_4 \rangle$ yields an element of $\mathcal{U}$. Let u_5 be an element of $\mathcal{U}$. Observe that the functor $\langle u_1, u_2, u_3, u_4, u_5 \rangle$ yields an element of $\mathcal{U}$. Now we state the propositions:

(15) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is an element of $\mathbf{U}_0$.

(16) Let us consider a finite set X. If $X \subseteq \mathbf{U}_0$, then $X \in \mathbf{U}_0$.
PROOF: Consider p being a function such that $\operatorname{rng} p = X$ and $\operatorname{dom} p \in \omega$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 = \{p(\$_1)\}$. Consider g being a function such that $\operatorname{dom} g = \operatorname{dom} p$ and for every object x such that $x \in \operatorname{dom} p$ holds $\mathcal{P}[x, g(x)]$ from [2, Sch. 1]. $\operatorname{rng} g \subseteq \mathbf{U}_0$ by [11, (57)]. $\bigcup \operatorname{rng} g = X$. $\square$

(17) (i) $\bigcup\{\mathbb{N}\} \subseteq \mathbf{U}_0$, and

(ii) $\bigcup\{\mathbb{N}\} \notin \mathbf{U}_0$, and

(iii) $\{\mathbb{N}\} \not\subseteq \mathbf{U}_0$, and

(iv) $\{\mathbb{N}\} \notin \mathbf{U}_0$.

(18) Let us consider an object x. Then $x \in \mathcal{U}$ if and only if $\{x\} \in \mathcal{U}$.

Let us consider a set X and a non zero natural number n. Now we state the propositions:

(19) If $\{X\}^{\operatorname{Seg} n}$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (18).

(20) If $\{X\}^n$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (19).

Now we state the proposition:

(21) Let us consider a set X. If $\bigcup X \in \mathcal{U}$, then $X \in \mathcal{U}$.

3. Set and Class

Let X be a non empty set and x be an object. We say that x is X-Set if and only if

(Def. 8) $x \in X$.

A Set of X is a set defined by

(Def. 9) *it* is X-Set.

Now we state the propositions:

(22) Let us consider universal classes U_1, U_2. Suppose $U_1 \in U_2$. Let us consider an object x. If x is U_1-Set, then x is U_2-Set.

(23) Let us consider universal classes U_1, U_2. If $U_1 \in U_2$, then every Set of U_1 is a Set of U_2.

(24) Every Set of $\mathbf{U}_0$ is finite.

(25) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is a Set of $\mathbf{U}_0$. The theorem is a consequence of (15).

(26) Let us consider an object x. Then x is a Set of $\mathbf{U}_0$ if and only if x is a set of a finite rank.

Let $\mathcal{U}$ be a universal class and x be an object. We say that x is $\mathcal{U}$-Class if and only if

(Def. 10) $x \in 2^{\mathcal{U}}$ and $x \notin \mathcal{U}$.

Now we state the proposition:

(27) Let us consider a set x. If x is $\mathcal{U}$-Class, then x is not empty.

Let $\mathcal{U}$ be a universal class.

A Class of $\mathcal{U}$ is a non empty set defined by

(Def. 11) *it* is $\mathcal{U}$-Class.

Now we state the propositions:

(28) Let us consider a finite subset X of $\mathcal{U}$. Then $X \in \mathcal{U}$. The theorem is a consequence of (18) and (7).

(29) Every Class of $\mathcal{U}$ is not finite. The theorem is a consequence of (28).

(30) Let us consider a Set X of $\mathcal{U}$. Then $\mathcal{U} \setminus X$ is a Class of $\mathcal{U}$.

(31) Every non finite subset of $\mathbf{U}_0$ is a Class of $\mathbf{U}_0$.

(32) $\mathbb{N}$ is a Class of $\mathbf{U}_0$.

(33) $\mathbb{N}_{\mathrm{even}}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (3) and (31).

(34) $\mathbb{N}_{\mathrm{odd}}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (4) and (31).

(35) Let us consider an object x. Then

(i) x is not $\mathcal{U}$-Class, or

(ii) x is not $\mathcal{U}$-Set.

(36) Let us consider universal classes U_1, U_2. Suppose $U_1 \in U_2$. Let us consider an object x. If x is U_1-Class, then x is U_2-Set.

(37) (i) $\bigcup\{\mathbb{N}\}$ is $\mathbf{U}_0$-Class, and

(ii) $\{\mathbb{N}\}$ is not $\mathbf{U}_0$-Class, and

(iii) $\{\mathbb{N}\}$ is not $\mathbf{U}_0$-Set.

4. Categories of Groups and Universes

From now on U_1, U_2 denote universal classes.

Now we state the propositions:

(38) Let us consider an object x. Then there exists $\mathcal{U}$ such that x is $\mathcal{U}$-Set.

(39) Every set is (GrothendieckUniverse(x))-Set.

Let U_1, U_2 be universal classes. The functor $\sup(U_1, U_2)$ yielding a universal class is defined by the term

(Def. 12) $\begin{cases} U_1, & \textbf{if } U_2 \in U_1, \\ U_2, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(40) Let us consider universal classes U_1, U_2, a Set x of U_1, and a Set y of U_2. Then there exists a Set z of $\sup(U_1, U_2)$ such that for every object a, $a \in z$ iff $a = x$ or $a = y$.

(41) Let us consider a Set X of $\mathcal{U}$. Then $\bigcup X$ is a Set of $\mathcal{U}$.

Let us consider a set X. Now we state the propositions:

(42) If $\bigcup X$ is a Set of $\mathcal{U}$, then X is a Set of $\mathcal{U}$. The theorem is a consequence of (21).

(43) If $\bigcup X$ is empty, then X is $\mathcal{U}$-Set.

Now we state the propositions:

(44) Let us consider a Class X of $\mathcal{U}$. Then $\bigcup X$ is a Class of $\mathcal{U}$. The theorem is a consequence of (43) and (21).

(45) There exists a set X such that

(i) $\bigcup X$ is a Class of $\mathbf{U}_0$, and

(ii) X is not a Class of $\mathbf{U}_0$, and

(iii) X is not a Set of $\mathbf{U}_0$, and

(iv) X is a Set of $\mathbf{U}_1$.

The theorem is a consequence of (17).

(46) Let us consider a Set X of $\mathcal{U}$, and a set Y. If $Y \in X$, then Y is a Set of $\mathcal{U}$.

(47) Let us consider a Class X of $\mathcal{U}$, and a set Y. If $Y \in X$, then Y is a Set of $\mathcal{U}$.

5. U-petit

Let $\mathcal{U}$ be a universal class and x be a set. We say that x is $\mathcal{U}$-petit if and only if

(Def. 13) there exists an element u of $\mathcal{U}$ such that $u \approx x$.

Now we state the proposition:

(48) Every element of $\mathcal{U}$ is $\mathcal{U}$-petit.

Let us consider a set x. Now we state the propositions:

(49) x is $\mathcal{U}$-petit if and only if $\overline{\overline{x}} \in \overline{\overline{\mathcal{U}}}$.

(50) $\{x\}$ is $\mathcal{U}$-petit.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$-element if and only if

(Def. 14) the carrier of G is an element of $\mathcal{U}$.

Now we state the proposition:

(51) Let us consider a group G. Suppose G is $\mathcal{U}$-element. Then the multiplication of G is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$-petit if and only if

(Def. 15) there exists a group g such that g is $\mathcal{U}$-element and G and g are isomorphic.

Let C be an object-category. We say that C is $\mathcal{U}$-element if and only if

(Def. 16) the carrier of C is an element of $\mathcal{U}$ and the carrier' of C is an element of $\mathcal{U}$.

Now we state the propositions:

(52) Let us consider an object-category C. Suppose C is $\mathcal{U}$-element. Then

  (i) the source of C is an element of $\mathcal{U}$, and

  (ii) the target of C is an element of $\mathcal{U}$, and

  (iii) the composition of C is an element of $\mathcal{U}$.

(53) Let us consider elements o, m of $\mathcal{U}$. Then $\dot{\circlearrowright}(o, m)$ is $\mathcal{U}$-element.

Let $\mathcal{U}$ be a universal class. Observe that there exists an object-category which is $\mathcal{U}$-element.

Let C be an object-category. We say that C is $\mathcal{U}$-petit if and only if

(Def. 17) there exists a strict object-category c such that c is $\mathcal{U}$-element and $C \cong c$.

Now we state the propositions:

(54) Let us consider an object-category A, and an object a of A. Then $\langle\langle \mathrm{id}_a, \mathrm{id}_a\rangle, \mathrm{id}_a\rangle \in$ the composition of A.

(55) Let us consider objects o, m. Then the composition of $\dot{\circlearrowleft}(o, m) = \{\langle\langle m, m\rangle, m\rangle\}$.

PROOF: Set $A = \dot{\circlearrowleft}(o, m)$. The composition of $A \subseteq \{\langle\langle m, m\rangle, m\rangle\}$ by [5, (16)]. $\square$

(56) Let us consider objects o, m, and an object c of $\dot{\circlearrowleft}(o, m)$. Then $c = o$.

(57) Let us consider objects o, m, and an element c of $\dot{\circlearrowleft}(o, m)$. Then

(i) c is an object of $\dot{\circlearrowleft}(o, m)$, and

(ii) $c = o$, and

(iii) $\mathrm{id}_c = m$.

(58) Let us consider objects o_1, o_2, m_1, m_2. Then $\dot{\circlearrowleft}(o_1, m_1) \cong \dot{\circlearrowleft}(o_2, m_2)$. The theorem is a consequence of (57).

(59) Let us consider objects o, m. Then $\dot{\circlearrowleft}(o, m)$ is $\mathcal{U}$-petit. The theorem is a consequence of (53) and (58).

Let $\mathcal{U}$ be a universal class. Let us observe that there exists an object-category which is $\mathcal{U}$-petit.

Now we state the propositions:

(60) There exists a $\mathcal{U}$-petit object-category C such that

(i) the carrier of C is not an element of $\mathcal{U}$, and

(ii) the carrier' of C is an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(61) There exists a $\mathcal{U}$-petit object-category C such that

(i) the carrier of C is not an element of $\mathcal{U}$, and

(ii) the carrier' of C is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(62) There exists a $\mathcal{U}$-petit object-category C such that

(i) the carrier of C is an element of $\mathcal{U}$, and

(ii) the carrier' of C is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(63) There exists a $\mathcal{U}$-petit object-category C such that C is not $\mathcal{U}$-element. The theorem is a consequence of (62).

(64) Let us consider a strict object-category C. If C is $\mathcal{U}$-element, then C is $\mathcal{U}$-petit.

Let $\mathcal{U}$ be a universal class and C be an object-category. We say that C is $\mathcal{U}$-Category if and only if

(Def. 18) for every objects x, y of C, $\hom(x, y)$ is $\mathcal{U}$-petit.

Let us observe that there exists an object-category which is $\mathcal{U}$-Category.

Now we state the proposition:

(65) Let us consider object-categories C, D, and a functor F from C to D. Then $F \subseteq$ (the carrier' of C) $\times$ (the carrier' of D).

Let us consider object-categories C, D. Now we state the propositions:

(66) $\mathrm{Funct}(C, D) \subseteq 2^{\alpha \times \beta}$, where α is the carrier' of C and β is the carrier' of D.

(67) $\mathrm{NatTrans}(C, D) \subseteq (2^{\alpha \times \beta} \times 2^{\alpha \times \beta}) \times 2^{\gamma \times \beta}$, where α is the carrier' of C, β is the carrier' of D, and γ is the carrier of C.

Now we state the propositions:

(68) Let us consider sets X, Y, Z. Suppose $X, Y, Z \in \mathcal{U}$. Then $2^{(2^{X \times Y} \times 2^{X \times Y}) \times 2^{Z \times Y}} \in \mathcal{U}$.

(69) Let us consider non empty sets X, Y. Suppose Y^X is an element of $\mathcal{U}$. Then X is an element of $\mathcal{U}$. The theorem is a consequence of (5).

Now we state the propositions:

(70) PROP 1.1.1 A) SGA4:
Let us consider object-categories C, D. Suppose C is $\mathcal{U}$-element and D is $\mathcal{U}$-element. Then $\mathrm{Functors}(D, C)$ is $\mathcal{U}$-element. The theorem is a consequence of (66) and (67).

(71) Let us consider a set c. Suppose $c \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^c}} \in \mathcal{U}$.

(72) Let us consider cardinal numbers c_1, c_2. Suppose $c_1, c_2 \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^{c_1 \times c_2}}} \in \mathcal{U}$.

6. Category GroupCat

Let x be an object. The functor $\mathrm{op0}(x)$ yielding an element of $\{x\}$ is defined by the term

(Def. 19) x.

The functor $\mathrm{op1}(x)$ yielding a unary operation on $\{x\}$ is defined by the term

(Def. 20) $x \longmapsto x$.

The functor op2(x) yielding a binary operation on $\{x\}$ is defined by the term

(Def. 21) $[\langle x, x\rangle \mapsto x]$.

Now we state the proposition:

(73) (i) op0(0) = op_0, and

(ii) op1(0) = op_1, and

(iii) op2(0) = op_2.

Let x be an object. The functor TrivialAddLoopStr(x) yielding a non empty additive loop structure is defined by the term

(Def. 22) $\langle\{x\}, \mathrm{op2}(x), \mathrm{op0}(x)\rangle$.

Now we state the propositions:

(74) Trivial-addLoopStr = TrivialAddLoopStr(0).

(75) Let us consider an object x. Then TrivialAddLoopStr(x) is a strict group.

(76) (i) op0(x) is an element of $\mathcal{U}$, and

(ii) op1(x) is an element of $\mathcal{U}$, and

(iii) op2(x) is an element of $\mathcal{U}$.

(77) comp TrivialAddLoopStr(x) is an element of $\mathcal{U}$.

(78) There exists an element y of $\mathcal{U}$ such that $\mathrm{P_{ob}}\, y$, TrivialAddLoopStr(x). The theorem is a consequence of (76) and (77).

(79) $\bigcup$the set of all the carrier of TrivialAddLoopStr(x) where x is an element of $\mathcal{U} = \mathcal{U}$.

(80) TrivialAddLoopStr(x) $\in$ GroupObj($\mathcal{U}$). The theorem is a consequence of (78).

(81) GroupObj($\mathcal{U}$) $\approx \mathcal{U}$.

PROOF: Set G_1 = GroupObj($\mathcal{U}$). Reconsider $G_2 = G_1$ as a non empty set. Define $\mathcal{P}$[object, object] $\equiv$ $\$_2 \in \mathcal{U}$ and $\mathrm{P_{ob}}\, \$_2, \$_1$. For every element x of G_2, there exists an element y of $\mathcal{U}$ such that $\mathcal{P}[x, y]$. Consider f being a function from G_2 into $\mathcal{U}$ such that for every element x of G_2, $\mathcal{P}[x, f(x)]$ from [7, Sch. 3]. Define $\mathcal{Q}$(object) = TrivialAddLoopStr($\$_1$). For every object x such that $x \in \mathcal{U}$ holds $\mathcal{Q}(x) \in$ GroupObj($\mathcal{U}$). Consider g being a function from $\mathcal{U}$ into GroupObj($\mathcal{U}$) such that for every object x such that $x \in \mathcal{U}$ holds $g(x) = \mathcal{Q}(x)$ from [7, Sch. 2]. □

(82) GroupObj($\mathcal{U}$) is not $\mathcal{U}$-petit. The theorem is a consequence of (81).

7. Object-category Represented by a Set

Let C be an object-category. The functor CatToSet(C) yielding a set is defined by the term

(Def. 23) ⟨the carrier of C, the carrier' of C, the source of C, the target of C, the composition of C⟩.

Let C be a quintuple set. We say that C is StrCategory-like if and only if

(Def. 24) there exist sets y_1, y_2, y_3, y_4, y_5 such that $y_1 = (C)_{\mathbf{1}}$ and $y_2 = (C)_{\mathbf{2}}$ and $y_3 = (C)_{\mathbf{3}}$ and $y_4 = (C)_{\mathbf{4}}$ and $y_5 = (C)_{\mathbf{5}}$ and y_3 is a function from y_2 into y_1 and y_4 is a function from y_2 into y_1 and y_5 is a partial function from $y_2 \times y_2$ to y_2.

Observe that there exists a quintuple set which is StrCategory-like.

Let C be a StrCategory-like, quintuple set. The functor SetToCat(C) yielding a strict category structure is defined by

(Def. 25) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (C)_{\mathbf{1}}$ and $y_2 = (C)_{\mathbf{2}}$ and $y_3 = (C)_{\mathbf{3}}$ and $y_4 = (C)_{\mathbf{4}}$ and $y_5 = (C)_{\mathbf{5}}$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

We say that C is category-like if and only if

(Def. 26) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (C)_{\mathbf{1}}$ and $y_2 = (C)_{\mathbf{2}}$ and $y_3 = (C)_{\mathbf{3}}$ and $y_4 = (C)_{\mathbf{4}}$ and $y_5 = (C)_{\mathbf{5}}$ and $\langle y_1, y_2, y_3, y_4, y_5 \rangle$ is an object-category.

Let us observe that there exists a StrCategory-like, quintuple set which is category-like and there exists a StrCategory-like, quintuple set which is non empty and category-like.

Let C be a category-like, StrCategory-like, quintuple set. The functor Obj C yielding a set is defined by the term

(Def. 27) $(C)_{\mathbf{1}}$.

The functor Mor C yielding a set is defined by the term

(Def. 28) $(C)_{\mathbf{2}}$.

We say that C is non-empty if and only if

(Def. 29) Obj C is not empty.

Observe that there exists a category-like, StrCategory-like, quintuple set which is non-empty.

A CategorySet is a non-empty, category-like, StrCategory-like, quintuple set. Now we state the proposition:

(83) Every CategorySet is not empty.

Observe that every CategorySet is non empty.

Let C be a CategorySet. The functor SetToCat(C) yielding a strict object-category is defined by

(Def. 30) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (C)_{\mathbf{1}}$ and $y_2 = (C)_{\mathbf{2}}$ and $y_3 = (C)_{\mathbf{3}}$ and $y_4 = (C)_{\mathbf{4}}$ and $y_5 = (C)_{\mathbf{5}}$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

Let C be a strict object-category. One can check that the functor CatToSet(C) yields a CategorySet. Now we state the propositions:

(84) Let us consider a CategorySet C. Then CatToSet(SetToCat(C)) $= C$.

(85) Let us consider a strict object-category C. Then SetToCat(CatToSet(C)) $= C$.

(86) Let us consider an object-category C. Then C is $\mathcal{U}$-element if and only if CatToSet(C) is $\mathcal{U}$-Set. The theorem is a consequence of (52), (13), and (14).

(87) Let us consider a CategorySet C. Then C is $\mathcal{U}$-Set if and only if SetToCat(C) is $\mathcal{U}$-element. The theorem is a consequence of (14) and (84).

Let C, D be CategorySets. We say that $C \cong D$ if and only if

(Def. 31) SetToCat(C) $\cong$ SetToCat(D).

Now we state the proposition:

(88) Let us consider strict object-categories C, D. If $C \cong D$, then CatToSet(C) $\cong$ CatToSet(D). The theorem is a consequence of (85).

Let $\mathcal{U}$ be a universal class and C be a CategorySet. We say that C is $\mathcal{U}$-petit if and only if

(Def. 32) there exists a CategorySet c such that c is $\mathcal{U}$-Set and $C \cong c$.

Now we state the proposition:

(89) Let us consider a strict object-category C. Then C is $\mathcal{U}$-petit if and only if CatToSet(C) is $\mathcal{U}$-petit. The theorem is a consequence of (86), (88), (85), and (87).

Let C, D be CategorySets. The functor Funct(C, D) yielding a set is defined by the term

(Def. 33) Funct(SetToCat(C), SetToCat(D)).

The functor Functors(D, C) yielding a CategorySet is defined by the term

(Def. 34) CatToSet(Functors(SetToCat(D), SetToCat(C))).

Now we state the proposition:

(90) Let us consider CategorySets C, D. Then

(i) Obj Functors(D, C) $=$ Funct(SetToCat(C), SetToCat(D)), and

(ii) Mor Functors(D, C) = NatTrans(SetToCat(C), SetToCat(D)).

Now we state the proposition:

(91) PROP 1.1.1 A) SGA4:
Let us consider CategorySets C, D. Suppose C is $\mathcal{U}$-Set and D is $\mathcal{U}$-Set. Then Functors(D, C) is $\mathcal{U}$-Set. The theorem is a consequence of (87), (70), and (86).

8. Small and Locally-small Categories

Let $\mathcal{U}$ be a universal class and C be an object-category. We introduce the notation C is $\mathcal{U}$-small as a synonym of C is $\mathcal{U}$-element.

Observe that there exists an object-category which is $\mathcal{U}$-small.

Now we state the propositions:

(92) Let us consider sets o, m. Suppose m is not $\mathcal{U}$-Set or o is not $\mathcal{U}$-Set. Then $\dot{\circlearrowleft}(o, m)$ is not $\mathcal{U}$-small. The theorem is a consequence of (18).

(93) Let us consider objects o, m. Suppose $\dot{\circlearrowleft}(o, m)$ is $\mathcal{U}$-small. Then

(i) m is $\mathcal{U}$-Set, and

(ii) o is $\mathcal{U}$-Set.

The theorem is a consequence of (92).

Let $\mathcal{U}$ be a universal class. One can verify that there exists an object-category which is non $\mathcal{U}$-small.

Let C be an object-category. We say that C is $\mathcal{U}$-locally small if and only if

(Def. 35) for every objects x, y of C, $\hom(x, y)$ is $\mathcal{U}$-Set.

Note that there exists an object-category which is $\mathcal{U}$-locally small and there exists a non void, non empty object-category which is $\mathcal{U}$-locally small.

Now we state the propositions:

(94) Every $\mathcal{U}$-small object-category is $\mathcal{U}$-locally small.

(95) Let us consider an object o. Then $\dot{\circlearrowleft}(o, \mathcal{U})$ is not $\mathcal{U}$-locally small.
PROOF: Set $C = \dot{\circlearrowleft}(o', \mathcal{U})$. C is not $\mathcal{U}$-locally small by [3, (3)], (18). $\square$

Let $\mathcal{U}$ be a universal class. Let us observe that there exists an object-category which is non $\mathcal{U}$-locally small.

Let us consider a $\mathcal{U}$-locally small object-category C. Now we state the propositions:

(96) Suppose the carrier of C is $\mathcal{U}$-Set. Then $\bigcup$ the set of all $\hom(a, b)$ where a, b are objects of C is an element of $\mathcal{U}$.

PROOF: Define $\mathcal{P}$[object of C, element of $\mathcal{U}$] $\equiv \bigcup$ the set of all hom($\$_1, b$) where b is an object of $C = \$_2$. Consider f being a function from the carrier of C into $\mathcal{U}$ such that for every element x of the carrier of C, $\mathcal{P}[x, f(x)]$ from [7, Sch. 3]. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) \in \mathcal{U}$. □

(97) If the carrier of C is $\mathcal{U}$-Set, then C is $\mathcal{U}$-small. The theorem is a consequence of (96).

Now we state the propositions:

(98) Let us consider $\mathcal{U}$-small object-categories C, D. Then

(i) Functors(D, C) is $\mathcal{U}$-small, and

(ii) NatTrans(C, D) is $\mathcal{U}$-Set.

The theorem is a consequence of (70).

(99) Let us consider a $\mathcal{U}$-small object-category C. Then C^{op} is a $\mathcal{U}$-small object-category.

(100) Let us consider a $\mathcal{U}$-locally small object-category C. Then C^{op} is a $\mathcal{U}$-locally small object-category.

9. Examples

Let X be a set. One can verify that the functor id_X yields an element of X^X. Now we state the propositions:

(101) Funcs $\mathcal{U} \subset \mathcal{U}$.

PROOF: Funcs $\mathcal{U} \subseteq \mathcal{U}$ by [8, (77)], [9, (81)]. □

(102) Funcs $\mathcal{U}$ is $\mathcal{U}$-Class. The theorem is a consequence of (101).

(103) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\})$ is not an element of $\mathcal{U}$.

(104) (i) $\pi_1(\mathrm{Maps}\,\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$, and

(ii) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\}) \subseteq \pi_1(\mathrm{Maps}\,\mathcal{U})$, and

(iii) $\pi_2(\mathrm{Maps}\,\mathcal{U}) = \mathrm{Funcs}\,\mathcal{U}$.

PROOF: $\pi_1(\mathrm{Maps}\,\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$. $(\mathcal{U} \times \mathcal{U}) \cap \{\langle A, B \rangle$, where A, B are elements of $\mathcal{U}$: if B $= \emptyset$, then $A = \emptyset\} \subseteq \pi_1(\mathrm{Maps}\,\mathcal{U})$ by [6, (1)]. $\pi_2(\mathrm{Maps}\,\mathcal{U}) = \mathrm{Funcs}\,\mathcal{U}$ by [6, (1)]. □

(105) Maps $\mathcal{U} \subseteq \mathcal{U}$.

(106) The carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class. The theorem is a consequence of (102), (104), and (105).

(107) (i) $\mathbf{Ens}_{\mathcal{U}}$ is a non $\mathcal{U}$-small object-category, and

(ii) the carrier of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class, and

(iii) the carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class.

The theorem is a consequence of (106).

(108) Let us consider universal classes $\mathcal{U}$, V. Suppose $\mathcal{U} \in V$. Then $\mathbf{Ens}_{\mathcal{U}}$ is a V-small object-category. The theorem is a consequence of (107).

Let A_1 be an Abelian group. The functor $\# A_1$ yielding a function from (the carrier of A_1) $\times$ (the carrier of A_1) into the carrier of A_1 is defined by the term

(Def. 36) the addition of A_1.

Let K be a field, o be an object, and n be a natural number. The functor nMatrixFieldCat(K, o, n) yielding a non empty, non void, strict category structure is defined by the term

(Def. 37) $\langle \{o\}$, the carrier of $K_{\mathrm{G}}^{n \times n}$, ((the carrier of $K_{\mathrm{G}}^{n \times n}) \longmapsto o)$, ((the carrier of $K_{\mathrm{G}}^{n \times n}) \longmapsto o), \# K_{\mathrm{G}}^{n \times n} \rangle$.

One can verify that nMatrixFieldCat(K, o, n) is category-like and nMatrixFieldCat$(K,$ is transitive and nMatrixFieldCat(K, o, n) is associative and nMatrixFieldCat(K, o, n) is reflexive and nMatrixFieldCat(K, o, n) has identities.

Now we state the proposition:

(109) Let us consider a field K, an element o of $\mathcal{U}$, and a non zero natural number n. Suppose the carrier of K is an element of $\mathcal{U}$. Then

(i) the carrier of nMatrixFieldCat(K, o, n) is trivial, and

(ii) nMatrixFieldCat(K, o, n) is $\mathcal{U}$-small object-category and $\mathcal{U}$-locally small object-category.

The theorem is a consequence of (18) and (94).

Let us consider an element o of $\mathbf{U}_0$ and a non zero natural number n. Now we state the propositions:

(110) (i) the carrier of nMatrixFieldCat$(\mathbb{R}_{\mathrm{F}}, o, n)$ is trivial and $\mathbf{U}_0$-Set, and

(ii) nMatrixFieldCat$(\mathbb{R}_{\mathrm{F}}, o, n)$ is not a $\mathbf{U}_0$-small object-category, and

(iii) nMatrixFieldCat$(\mathbb{R}_{\mathrm{F}}, o, n)$ is not a $\mathbf{U}_0$-locally small object-category, and

(iv) nMatrixFieldCat$(\mathbb{R}_{\mathrm{F}}, o, n)$ is $\mathbf{U}_1$-small object-category and $\mathbf{U}_1$-locally small object-category.

The theorem is a consequence of (18), (6), and (109).

(111) (i) the carrier of nMatrixFieldCat$(\mathbb{C}_{\mathrm{F}}, o, n)$ is trivial and $\mathbf{U}_0$-Set, and

(ii) nMatrixFieldCat$(\mathbb{C}_{\mathrm{F}}, o, n)$ is not a $\mathbf{U}_0$-small object-category, and

(iii) nMatrixFieldCat$(\mathbb{C}_{\mathrm{F}}, o, n)$ is not a $\mathbf{U}_0$-locally small object-category, and

(iv) nMatrixFieldCat($\mathbb{C}_{\mathrm{F}}, o, n$) is $\mathbf{U}_1$-small object-category and $\mathbf{U}_1$-locally small object-category.

The theorem is a consequence of (18), (6), and (109).

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