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On the Properties of Curves and Parametrization-Independent Isoperimetric Inequality[1]

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Summary. In this article we formalize in Mizar [1], [2] several properties of curves and establishes a parametrization-independent isoperimetric inequality. The paper is structured into three main sections:

1. Preliminaries and Basic Theorems: Introduces fundamental definitions, notations, and initial theorems, including the definition of the ArcLenP function.

2. Arc Length Parametrization: Constructs arc length parametrization and explores its properties, including differentiability and characteristics of its inverse function.

3. Parametrization-Independent Isoperimetric Inequality: Proves an isoperimetric inequality that holds regardless of the curve's parametrization.

This formalization provides a rigorous foundation for further work in differential geometry and analysis. We referred to [12], [11] and [9] in this formalization.

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1. Preliminaries and Basic Theorems

From now on a, b, r denote real numbers, A denotes a non empty set, X, x denote sets, f, g, F, G denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n denotes an element of $\mathbb{N}$.

Let a, b be real numbers and x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor ArcLenP(x, y, a, b) yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by

(Def. 1) dom $it = [a, b]$ and for every real number t such that $t \in [a, b]$ holds
$$it(t) = \int_a^t (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx.$$

Now we state the propositions:

(1) Let us consider real numbers a, b, d, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq \mathrm{dom}\, f$ and $f{\restriction}[a, b]$ is continuous and $f(a) < d < f(b)$. Then there exists a real number c such that

(i) $a < c < b$, and

(ii) $d = f(c)$.

Proof: Reconsider $g = f{\restriction}[a, b]$ as a function from $[a,\ b]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. Set $T = [a,\ b]_{\mathrm{T}}$. For every point p of T and for every positive real number r, there exists an open subset W of T such that $p \in W$ and $g^{\circ}W \subseteq]g(p) - r, g(p) + r[$ by [14, (14)], [8, (39)], [5, (17), (18)]. Consider c being a real number such that $g(c) = d$ and $a < c < b$. $\square$

(2) Let us consider real numbers a, b, and an open subset Z of $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq Z$. Then there exist real numbers a_1, b_1 such that

(i) $a_1 < a$, and

(ii) $b < b_1$, and

(iii) $a_1 < b_1$, and

(iv) $[a_1, b_1] \subseteq Z$, and

(v) $[a, b] \subseteq]a_1, b_1[$.

2. Arc Length Parametrization

Let us consider real numbers a, b and partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(3) Suppose $a < b$ and x is differentiable and y is differentiable and $[a, b] \subseteq \mathrm{dom}\, x$ and $[a, b] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then there exist real numbers a_1, b_1 and there exists a partial function l from $\mathbb{R}$ to $\mathbb{R}$ and there exists an open subset Z of $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and $Z = \mathrm{dom}\, x \cap \mathrm{dom}\, y$ and $[a, b] \subseteq]a_1, b_1[$ and $[a_1, b_1] \subseteq Z$ and $\mathrm{dom}\, l = Z$ and for every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx$ and l is differentiable on $]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[} = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y}) \restriction]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[}$ is continuous and for every real number t such that $t \in]a_1, b_1[$ holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and for every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$.

PROOF: Reconsider $Z_1 = \mathrm{dom}\, x$, $Z_2 = \mathrm{dom}\, y$ as an open subset of $\mathbb{R}$. Reconsider $Z = Z_1 \cap Z_2$ as an open subset of $\mathbb{R}$. Consider d_1 being a real number such that $0 < d_1$ and $]a - d_1, a + d_1[\subseteq Z$. Consider d_2 being a real number such that $0 < d_2$ and $]b - d_2, b + d_2[\subseteq Z$. Reconsider $d = \min(d_1, d_2)$ as a real number. Set $a_1 = a - \frac{d}{2}$. Set $b_1 = b + \frac{d}{2}$. $[a_1, b_1] \subseteq Z$. Define $\mathcal{F}(\text{real number}) = (\int\limits_{a_1}^{\$_1} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx)(\in \mathbb{R})$. Consider l_0 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element t of $\mathbb{R}$, $l_0(t) = \mathcal{F}(t)$ from [4, Sch. 4]. For every real number t, $l_0(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx$. Set $l = l_0 \restriction Z$. Set $X_2 = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})$. For every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} X_2(x) dx$ by [3, (49)]. For every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$ by [6, (10), (11)], [7, (17)]. For every real number t such that $t \in]a_1, b_1[$ holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ by [3, (12)], [7, (28)]. □

(4) Suppose $a < b$ and x is differentiable and y is differentiable and $[a, b] \subseteq \mathrm{dom}\, x$ and $[a, b] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then there exist real numbers a_1, b_1 and there exists a one-to-one partial function L from $\mathbb{R}$ to $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and

$[a_1, b_1] \subseteq \operatorname{dom} x \cap \operatorname{dom} y$ and $\operatorname{dom} L =]a_1, b_1[$ and for every real number t such that $t \in]a_1, b_1[$ holds $L(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$ and for every real number t such that $t \in [a, b]$ holds $(\operatorname{ArcLenP}(x, y, a, b))(t) = L(t) - L(a)$ and L is increasing and $L \restriction [a, b]$ is continuous and $L^{\circ}[a, b] = [L(a), L(b)]$ and for every real number t such that $t \in]a_1, b_1[$ holds L is differentiable in t and L is differentiable on $]a_1, b_1[$ and for every real number t such that $t \in]a_1, b_1[$ holds $L'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and L^{-1} is differentiable on $\operatorname{dom}(L^{-1})$ and for every real number t such that $t \in \operatorname{dom}(L^{-1})$ holds $(L^{-1})'(t) = \frac{1}{L'((L^{-1})(t))}$ and L^{-1} is continuous and for every real number s such that $s \in \operatorname{rng} L$ holds $x \cdot (L^{-1})$ is differentiable in s and $y \cdot (L^{-1})$ is differentiable in s and $(x \cdot (L^{-1}))'(s) = x'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(y \cdot (L^{-1}))'(s) = y'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ and $(x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))} = x'_{\restriction \operatorname{dom} x} \cdot (L^{-1}) \cdot (L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ and $(y \cdot (L^{-1}))'_{\restriction \operatorname{dom}(y \cdot (L^{-1}))} = y'_{\restriction \operatorname{dom} y} \cdot (L^{-1}) \cdot (L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ and $(L^{-1})'_{\restriction \operatorname{dom}(L^{-1})} = \frac{1}{L'_{\restriction \operatorname{dom} L} \cdot (L^{-1})}$ and $(L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ is continuous and $[L(a), L(b)] \subseteq \operatorname{dom}(L^{-1})$ and $[L(a), L(b)] \subseteq \operatorname{dom}(x \cdot (L^{-1}))$ and $[L(a), L(b)] \subseteq \operatorname{dom}(y \cdot (L^{-1}))$ and $[L(a), L(b)] \subseteq \operatorname{rng} L$ and $\operatorname{dom}(x \cdot (L^{-1})) = \operatorname{dom}(L^{-1})$ and $\operatorname{dom}(y \cdot (L^{-1})) = \operatorname{dom}(L^{-1})$ and $x \cdot (L^{-1})$ is differentiable and $y \cdot (L^{-1})$ is differentiable and $(x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))}$ is continuous and $(y \cdot (L^{-1}))'_{\restriction \operatorname{dom}(y \cdot (L^{-1}))}$ is continuous and for every real number s such that $s \in \operatorname{dom}(x \cdot (L^{-1})) \cap \operatorname{dom}(y \cdot (L^{-1}))$ holds $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ and $\int\limits_{a}^{b} (y \cdot x'_{\restriction \operatorname{dom} x})(x)dx = \int\limits_{L(a)}^{L(b)} (y \cdot (L^{-1}) \cdot (x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))})(x)dx.$

PROOF: Consider a_1, b_1 being real numbers, l being a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z being an open subset of $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and $Z = \operatorname{dom} x \cap \operatorname{dom} y$ and $[a, b] \subseteq]a_1, b_1[$ and $[a_1, b_1] \subseteq Z$ and $\operatorname{dom} l = Z$ and for every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$ and l is differentiable on $]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[} = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y}) \restriction]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[}$ is continuous and for every real number t such that $t \in]a_1, b_1[$ holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and for every real number t such that $t \in [a, b]$ holds $(\operatorname{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$.

Set $L = l{\restriction}]a_1, b_1[$. For every real number t such that $t \in]a_1, b_1[$ holds $L(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x)dx$ by [3, (49)]. For every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = L(t) - L(a)$ by [3, (49)]. For every real number t such that $t \in]a_1, b_1[$ holds $0 < l'(t)$ by [13, (81)]. For every real number t such that $t \in]a_1, b_1[$ holds $L'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ by [? , (11)]. For every real number t such that $t \in]a_1, b_1[$ holds $0 < L'(t)$ by [? , (11)]. For every real number s such that $s \in \mathrm{rng}\, L$ holds $x \cdot (L^{-1})$ is differentiable in s and $y \cdot (L^{-1})$ is differentiable in s and $(x \cdot (L^{-1}))'(s) = x'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(y \cdot (L^{-1}))'(s) = y'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ by [3, (3), (33)], [10, (13), (48)]. Set $L_1 = (L^{-1})'_{\restriction \mathrm{dom}(L^{-1})}$. $L'_{\restriction \mathrm{dom}\, L} \cdot (L^{-1})^{-1}(\{0\}) = \emptyset$ by [3, (3), (33), (13)], [13, (81)]. For every real number t such that $t \in \mathrm{dom}\, L_1$ holds L_1 is continuous in t by [3, (3), (33), (13)], [13, (81)]. For every object s, $s \in L^{\circ}[a, b]$ iff $s \in [L(a), L(b)]$ by [15, (25)], (1). Set $e_1 = \frac{a - a_1}{2}$. Set $e_2 = \frac{b_1 - b}{2}$. Set $a_2 = a_1 + e_1$. Set $b_2 = b_1 - e_2$. $a_2 < a$ and $a_2 < b < b_2$ and $a_2 < b_2$ and $[a, b] \subseteq]a_2, b_2[$ and $[a_2, b_2] \subseteq]a_1, b_1[$. Define $\mathcal{FX}(\text{real number}) = (\int\limits_{a_2}^{\$_1} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx)(\in \mathbb{R})$. Consider F_0 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element t of $\mathbb{R}$, $F_0(t) = \mathcal{FX}(t)$ from [4, Sch. 4]. For every real number t, $F_0(t) = \int\limits_{a_2}^{t} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx$. Set $F = F_0{\restriction}[a_2, b_2]$. For every real number t such that $t \in]a_2, b_2[$ holds $F(t) = \int\limits_{a_2}^{t} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx$ by [3, (49)]. $[a_2, b_2] \subseteq \mathrm{dom}(y \cdot x'_{\restriction \mathrm{dom}\, x})$. $[a_2, b] \subseteq \mathrm{dom}(y \cdot x'_{\restriction \mathrm{dom}\, x})$. For every real number t such that $t \in]a_2, b_2[$ holds F is differentiable in t and $F'(t) = (y \cdot x'_{\restriction \mathrm{dom}\, x})(t)$ by [7, (28)]. $[L(a), L(b)] \subseteq]L(a_2), L(b_2)[$. Set $G = F \cdot (L^{-1}{\restriction}]L(a_2), L(b_2)[)$. For every object s, $s \in L^{\circ}[a_2, b_2]$ iff $s \in [L(a_2), L(b_2)]$ by [15, (25)], (1). $]L(a_2), L(b_2)[\subseteq \mathrm{rng}\, L$ by [3, (3)]. $\mathrm{rng}(L^{-1}{\restriction}]L(a_2), L(b_2)[) \subseteq \mathrm{dom}\, F$ by [3, (49), (34)]. For every real number t such that $t \in]L(a_2), L(b_2)[$ holds G is differentiable in t and $(L^{-1}{\restriction}]L(a_2), L(b_2)[)(t) \in]a_2, b_2[$ and $G'(t) = F'((L^{-1}{\restriction}]L(a_2), L(b_2)[)(t)) \cdot (L^{-1}{\restriction}]L(a_2), L(b_2)[)'(t)$ by [? , (10)], [3, (49), (34), (35)]. For every object s such that $s \in \mathrm{dom}\, G'_{\restriction]L(a_2), L(b_2)[}$ holds $G'_{\restriction]L(a_2), L(b_2)[}(s) = ((y \cdot (L^{-1}) \cdot (x \cdot (L^{-1}))'_{\restriction \mathrm{dom}(x \cdot (L^{-1}))}){\restriction}]L(a_2), L(b_2)[)(s)$ by [3, (3), (33), (49)]. For every real number s such that $s \in \mathrm{dom}(x \cdot (L^{-1})) \cap \mathrm{dom}(y \cdot (L^{-1}))$ holds $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ by [3, (33)]. $\square$

3. Parametrization-Independent Isoperimetric Inequality

Now we state the proposition:

(5) Let us consider real numbers a, b, l, and partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $(\mathrm{ArcLenP}(x, y, a, b))(b) = l$ and $y(a) = 0$ and $y(b) = 0$ and x is differentiable and y is differentiable and $[a, b] \subseteq \mathrm{dom}\, x$ and $[a, b] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then

(i) $\int\limits_a^b (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx \leqslant \frac{\frac{1}{2} \cdot l^{\mathbf{2}}}{\pi}$, and

(ii) $\int\limits_a^b (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx = \frac{\frac{1}{2} \cdot l^{\mathbf{2}}}{\pi}$ iff for every real number s such that $s \in [a, b]$ holds $y(s) = \frac{l}{\pi} \cdot (\text{the function sin})(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l})$ and $x(s) = \frac{l}{\pi} \cdot (-(\text{the function cos})(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l}) + (\text{the function cos})(0) + \frac{\pi}{l} \cdot x(a))$ or for every real number s such that $s \in [a, b]$ holds $y(s) = -\frac{l}{\pi} \cdot (\text{the function sin})(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l})$ and $x(s) = \frac{l}{\pi} \cdot ((\text{the function cos})(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l}) - (\text{the function cos})(0) + \frac{\pi}{l} \cdot x(a))$.

The theorem is a consequence of (4).

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