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Formal Proof of Transcendence of the Number e. Part II

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Summary. This article is continuation of [?] and we formalize the main part of Hurwitz's proof [10] using the Mizar formalism [3], [4]. For related proof developments in Coq or HOL Light, see [?] and [5], respectively. The following is a summary of the formalized proof:
In the first chapter we define a polynomial f_0 over $\mathbb{Z}$ and observe properties of f_0. It is defined by $f_0(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p$, where p is an odd prime number and $m+1$ is the number of component of the products. The f_0 is defined as `E_TRANS2:def 5`. The component $(x-j)_{(j=0,1,\ldots,m)}$ are represented by $\tau(j)$ in the article and obtain:

$$f_0 = \tau(0)^{p-1}\prod_{j=1}^{m}\tau(j)^p$$

The second chapter is about properties of f_0 and $F(f_0)$ where F is introduced [?], the transformation $F(f) = f + f' + f'' + \cdots + f^{(deg\ f)}$.
We observe k^{th} differentiation of the f_0 and evaluate by a number j. The following number-theoretical properties are obtained:

1. $\prod_{j=1}^{m}\tau(j)^p(0) = (((-1)|^m) * (m!))|^p$ (`E_TRANS2:17`),
2. $f_0^{(k)}(0) = 0$ if $0 \leqslant k \leqslant p-2$ (`E_TRANS2:18`),
3. $f_0^{(k)}(0) = k!(\prod_{j=1}^{m}\tau(j))(k-p+1)$ if $p \leqslant k$ (E_TRANS2 : 24),
4. $f_0^{(k)}(j) = 0$ if $k \leqslant p, 1 \leqslant j \leqslant m$ (E_TRANS2 : 26),
5. $f_0^{(k)} = \tau(j)u + p!v$ $(\exists u, v \in \mathbb{Z}[X])$ if $p \leqslant k, 1 \leqslant j \leqslant m$ (E_TRANS2 : 30),
6. $f_0^{(k)}(j) \in (p!)$ if $p \leqslant k, 1 \leqslant j \leqslant m$ (E_TRANS2 : 32).

We denote $\mathbf{F}$ for $F(f_0)$ for simplicity.

7. $\mathbf{F}(0) = (p-1)!(((-1)|^m) * (m!))|^p + p!u$ $(\exists u \in \mathbb{Z}[X])$ (E_TRANS2 : 33),
8. $\mathbf{F}(j) \in (p!)$ if $1 \leqslant j \leqslant m$ (E_TRANS2 : 34),

We then obtain an equation system shown as below: where C_i stands for coefficient of the i^{th} coefficient of g_0. This is based on the equation system (4) stated in Hurwitz's proof [10].

$$\begin{cases} \frac{1}{(p-1)!}C_0\mathbf{F}(0) & - & \frac{1}{(p-1)!}C_0e^0\mathbf{F}(0) & = \frac{1}{(p-1)!}C_0\varepsilon_0 \\ \frac{1}{(p-1)!}C_1\mathbf{F}(1) & - & \frac{1}{(p-1)!}C_1e^1\mathbf{F}(0) & = \frac{1}{(p-1)!}C_1\varepsilon_1 \\ \vdots & & \vdots & \vdots \\ \frac{1}{(p-1)!}C_m\mathbf{F}(m) & - & \frac{1}{(p-1)!}C_me^m\mathbf{F}(0) & = \frac{1}{(p-1)!}C_m\varepsilon_m \end{cases}$$

where each equation is a product of i^{th} coefficient of g_0 and $\mathbf{F}(i) - e^x\mathbf{F}(i)(= -ie^{(i-\vartheta)i}f_0(\vartheta i))$ which is from the result of the mean value theorem to $e^x\mathbf{F}(x)$. In actual coding the sequence $C_m\mathbf{F}(m)$ and $(p-1)!C_me^m\mathbf{F}(0)$ are defined as delta_1 and delta_2 respectively.
We have new equation by adding each term of the equation system vertically:

$$\frac{1}{(p-1)!}\sum_{i=1}^{m} C_i\mathbf{F}(i) - \frac{1}{(p-1)!}\sum_{i=1}^{m} C_ie^i\mathbf{F}(0) = \frac{1}{(p-1)!}\sum_{i=1}^{m} C_i\varepsilon_i$$

One can verify and formalize that the left hand side is not divided by p, because the first term of $p|\frac{1}{(p-1)!}\Sigma C_i\mathbf{F}(i)$ and $p \not| \frac{1}{(p-1)!}\Sigma e^iC_i\mathbf{F}(0)$. The right-hand side is a member of $\mathbb{Z}$ and bounded by 1/2 by choosing sufficiently large p, this means it is 0. This contradicts the left-hand side nature. Therefore e is transcendental number.

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1. Preliminaries

From now on R denotes an integral domain, p denotes an odd, prime natural number, and m denotes a positive natural number.

Now we state the propositions:

(1) Let us consider a natural number i, and an element r of $\mathbb{R}_{\mathrm{F}}$. Then $\sum(i \mapsto r) = i \cdot r$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \sum(\$_1 \mapsto r) = \$_1 \cdot r$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [6, (60)], [24, (71)], [18, (13), (15)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. □

(2) Let us consider sequences p_1, q_1 of $\mathbb{Z}^{\mathrm{R}}$. Then $(p_1 * q_1)(0) = p_1(0) \cdot q_1(0)$.

2. On the Ring of Polynomials

Now we state the propositions:

(3) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and a natural number n. Then ${}^{@}f^{n} = ({}^{@}f)^{n}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv {}^{@}f^{\$_1} = ({}^{@}f)^{\$_1}$. $\mathcal{P}[0]$ by [18, (8)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [18, (10), (8)], [**?** , (27)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(4) Let us consider an element f of the carrier of Polynom-Ring R, and a natural number n. Then $\curvearrowleft f^{n} = (\curvearrowleft f)^{n}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv \curvearrowleft f^{\$_1} = (\curvearrowleft f)^{\$_1}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [18, (10), (8)], [14, (19)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(5) Let us consider a natural number n, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $n \cdot f = n(\in \mathbb{Z}^{\mathrm{R}}) \cdot f$.

Proof: Define $\mathcal{P}$[natural number] $\equiv \$_1 \cdot f = \$_1(\in \mathbb{Z}^{\mathrm{R}}) \cdot f$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [9, (9), (7)], [18, (13), (15)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(6) Let us consider an element M of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence F of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $|F(i)| \leqslant M$. Then $|\prod F| \leqslant M^{\operatorname{len} F}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence F of elements of $\mathbb{R}_{\mathrm{F}}$ such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $|F(i)| \leqslant M$ holds $|\prod F| \leqslant M^{\operatorname{len} F}$. $\mathcal{P}[0]$ by [24, (80)], [18, (8)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [7, (29)], [1, (11)], [2, (1)], [24, (78)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

Let p be a polynomial over $\mathbb{Z}^{\mathrm{R}}$. Observe that the functor $|p|$ yields a sequence of $\mathbb{Z}^{\mathrm{R}}$ and is defined by

(Def. 1) for every natural number n, $it(n) = |p(n)|$.

Note that $|p|$ is finite-Support as a (the carrier of $\mathbb{Z}^{\mathrm{R}}$)-valued function.

In the sequel g denotes a non zero polynomial over $\mathbb{Z}^{\mathrm{R}}$.

Let us consider g. One can verify that $\operatorname{rng}|g|$ is finite.

Now we state the proposition:

(7) Let us consider a non zero polynomial g over $\mathbb{Z}^{\mathrm{R}}$. Then there exists a natural number M such that for every natural number i, $|g(i)| \leqslant M$.

Proof: $\operatorname{rng}|g| \subseteq \mathbb{N}$. Reconsider $S = \operatorname{rng}|g|$ as a finite, non empty, natural-membered set. Reconsider $M = \max S$ as a natural number. For every natural number i, $|g(i)| \leqslant M$ by [8, (3)]. □

3. The Polynomial f_0 and Its Properties

Let i be a natural number. The functor $\tau(i)$ yielding an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 2) $\langle (-i)(\in \mathbb{Z}^{\mathrm{R}}), 1_{\mathbb{Z}^{\mathrm{R}}} \rangle$.

Let p be a non zero natural number and m be a natural number. The functor x.(m, p) yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by

(Def. 3) len $it = m$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = (\tau(i))^p$.

Let p be an odd, prime natural number and m be a positive natural number. The functor ff-0(m, p) yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 4) x.$(m, p) \frown \langle (\tau(0))^{p -' 1} \rangle$.

The functor f-0(m, p) yielding an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 5) $\prod$ ff-0(m, p).

Now we state the propositions:

(8) Let us consider natural numbers i, n. Then len$\frown(\tau(i))^n = n + 1$.

(9) Let us consider elements f, g of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\text{len}\frown f) \cdot (\text{len}\frown g) \neq 0$. Then len$\frown f \cdot g = \text{len}\frown f + \text{len}\frown g - 1$.

(10) Let us consider a non zero natural number k, and an odd, prime natural number p. Then

(i) x.$(k, p) \frown \langle (\tau(k + 1))^p \rangle = $ x.$(k + 1, p)$, and

(ii) $\prod$ x.$(k + 1, p) = (\prod$ x.$(k, p)) \cdot (\tau(k + 1))^p$.

Proof: x.$(k, p) \frown \langle (\tau(k + 1))^p \rangle = $ x.$(k + 1, p)$ by [6, (16)], [2, (9)], [1, (19)], [2, (5), (3)]. □

Let us consider an odd, prime natural number p and a positive natural number m. Now we state the propositions:

(11) len$\frown\prod$ x.$(m, p) = m \cdot p + 1$.
Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ len$\frown\prod$ x.$(\$_1, p) = \$_1 \cdot p + 1$. $\mathcal{P}[1]$ by [2, (40)], [22, (11)], (8). For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every non zero natural number k, $\mathcal{P}[k]$ from [1, Sch. 10]. □

(12) len$\frown$f-0$(m, p) = m \cdot p + p$. The theorem is a consequence of (11), (8), and (9).

Now we state the propositions:

(13) Let us consider a natural number i. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))(\tau(i)) = 1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$.

(14) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and a natural number i. Then

(i) $(\tau(0) * f)(i+1) = f(i)$, and

(ii) $(\tau(0) * f)(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

PROOF: For every natural number i, $(\tau(0) * f)(i+1) = f(i)$ and $(\tau(0) * f)(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$ by [14, (16)], [19, (12)], [23, (31)]. □

From now on f denotes an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$.

Now we state the propositions:

(15) Let us consider an odd, prime natural number p, and a positive natural number m. Then

(i) $\mathrm{len}\,\mathrm{x.}(m,p) = m$, and

(ii) $\mathrm{len}\,\mathrm{ff\text{-}0}(m,p) = m+1$, and

(iii) $(\mathrm{ff\text{-}0}(m,p))(\mathrm{len}\,\mathrm{x.}(m,p)+1) = (\tau(0))^{p-'1}$.

(16) Let us consider an odd, prime natural number p, a positive natural number m, and a natural number k. Suppose $0 \leqslant k \leqslant p-1$. Let us consider natural numbers i, j. Suppose $i \in \mathrm{Seg}(k+1)$. Then $\tau(j) \mid (\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(\mathrm{ff\text{-}0}(m,p))\restriction_j, (\tau(j))^p))_{/i}$.

PROOF: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. For every natural numbers i, j such that $i \in \mathrm{Seg}(k+1)$ holds $\tau(j) \mid (\mathrm{LBZ}(D, k, \prod(\mathrm{ff\text{-}0}(m,p))\restriction_j, (\tau(j))^p))_{/i}$ by (13), [15, (19)], [18, (8)], [2, (1)]. □

(17) Let us consider an odd, prime natural number p, and a positive natural number m. Then $(\curvearrowleft\prod \mathrm{x.}(m,p))(0) = ((-1)^m \cdot (m!))^p$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\curvearrowleft\prod \mathrm{x.}(\$_1,p))(0) = ((-1)^{\$_1} \cdot (\$_1!))^p$. $\mathcal{P}[1]$ by [2, (40)], [22, (11)], [13, (13)]. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (10), (2), [13, (7), (6), (15)]. For every non zero natural number k, $\mathcal{P}[k]$ from [1, Sch. 10]. □

Let us consider an odd, prime natural number p, a positive natural number m, and a natural number k. Now we state the propositions:

(18) If $0 \leqslant k \leqslant p -' 2$, then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(\mathrm{f\text{-}0}(m,p))(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(19) Suppose $0 \leqslant k \leqslant p-'2$. Then $\mathrm{eval}(\curvearrowleft(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(\mathrm{f\text{-}0}(m,p)), 0_{\mathbb{Z}^{\mathrm{R}}}) = 0_{\mathbb{Z}^{\mathrm{R}}}$. The theorem is a consequence of (18).

Now we state the propositions:

(20) Let us consider an odd, prime natural number p, and a positive natural number m. Then $\mathrm{eval}(\curvearrowleft(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{p-'1}(\mathrm{f\text{-}0}(m,p)), 0_{\mathbb{Z}^{\mathrm{R}}}) = (p -' 1)! \cdot (((-1)^m \cdot (m!))^p (\in \mathbb{Z}^{\mathrm{R}}))$. The theorem is a consequence of (17).

(21) Let us consider an odd, prime natural number p, a positive natural number m, and a non zero natural number k. Suppose $p \leqslant k$. Then $\mathrm{eval}(\curvearrowleft(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{k}(\text{f-0}(m,p)), 0_{\mathbb{Z}^{\mathrm{R}}}) = k! \cdot (\curvearrowleft\prod \mathrm{x}.(m,p))(k -' (p -' 1))$.

(22) Let us consider a natural number j, and an element u of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $\mathrm{eval}(\curvearrowleft(\tau(j)) \cdot u, j(\in \mathbb{Z}^{\mathrm{R}})) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(23) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $k < p$ and $j \in \mathrm{Seg}\, m$. Then $\mathrm{eval}(\curvearrowleft(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{k}(\text{f-0}(m,p)), j(\in \mathbb{Z}^{\mathrm{R}})) = 0_{\mathbb{Z}^{\mathrm{R}}}$. The theorem is a consequence of (16) and (22).

(24) Let us consider a natural number i. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))(\tau(i)) = 1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$.

(25) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers j, k. Suppose $j \in \mathrm{Seg}\, m$ and $p \leqslant k$. Let us consider a natural number i. Suppose $i \in \mathrm{Seg}\, p$. Then $\tau(j) \mid (\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(\text{ff-0}(m,p))_{\restriction j}, (\tau(j))^{p}))_{/i}$.

PROOF: For every natural number i such that $i \in \mathrm{Seg}\, p$ holds $\tau(j) \mid (\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(\text{ff-0}(m,p))_{\restriction j}, (\tau(j))^{p}))_{/i}$ by [2, (1)], (24), [15, (19)], [18, (8)]. □

(26) Let us consider an odd, prime natural number p, a positive natural number m, natural numbers k, j, and a natural number i. Suppose $p+1 < i$ and $i \in \mathrm{dom}(\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(\text{ff-0}(m,p))_{\restriction j}, (\tau(j))^{p}))$. Then $(\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(\text{ff-0}$
$0_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$.

PROOF: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. Set $P_1 = \text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}$. Set $x_1 = \tau(j)$. Set $y_1 = \prod(\text{ff-0}(m,p))_{\restriction j}$. $1_{P_1} = D(x_1)$. For every natural number i such that $p+1 < i$ and $i \in \mathrm{dom}(\mathrm{LBZ}(D, k, y_1, {x_1}^{p}))$ holds $(\mathrm{LBZ}(D, k, y_1, {x_1}^{p}))_{/i} = 0_{P_1}$ by [2, (1)], [**?** , (21)]. □

(27) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $j \in \mathrm{Seg}\, m$ and $p \leqslant k$. Then there exist elements u, v of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{k}(\text{f-0}(m,p)) = (\tau(j)) \cdot u + p! \cdot v$.

PROOF: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. Set $P_1 = \text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}$. Set $t_1 = \tau(j)$. Set $\mathrm{j} = \prod(\text{ff-0}(m,p))_{\restriction j}$. $1_{P_1} = D(t_1)$. Reconsider $l_3 = \mathrm{LBZ}(D, k, \mathrm{j}, {t_1}^{p})$ as a non empty finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Set $l_4 = l_3 \restriction p$. For every natural number i such that $i \in \mathrm{Seg}\, p$ holds $\tau(j) \mid {l_4}_{/i}$ by [2, (1)], [8, (49)], (25). Consider u being an element of P_1 such that $\sum l_4 = (\tau(j)) \cdot u$. Set $k_2 = k + 1 -' (p+1)$. For every natural number i_1 such that $i_1 \in \mathrm{dom}({l_3}_{\lfloor p+1})$ holds $({l_3}_{\lfloor p+1})_{/i_1} = 0_{P_1}$ by [2, (1)], [7, (27)], (26). ${l_3}_{\lfloor p+1} = k_2 \mapsto 0_{P_1}$ by [6, (57)]. □

(28) Let us consider an element u of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and elements a, b of $\mathbb{Z}^{\mathrm{R}}$. Then $\mathrm{eval}(a \cdot (\curvearrowleft u), b) \in \{a\}$–ideal.

(29) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $j \in \operatorname{Seg} m$ and $p \leqslant k$. Then $\operatorname{eval}(\curvearrowleft(\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}))^{k}(\text{f-0}(m,p)), j(\in \mathbb{Z}^{\mathrm{R}})) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. The theorem is a consequence of (27), (22), (5), and (28).

Now we state the propositions:

(30) NOW WE APPLY THE POLYNOMIAL TRANSFORMATION 'F' TO F_0.: Let us consider an odd, prime natural number p, and a positive natural number m. Then there exists an element u of $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathcal{F}\,\text{f-0}(m,p))(0) = (p -' 1)! \cdot (((-1)^{m} \cdot (m!))^{p}(\in \mathbb{Z}^{\mathrm{R}})) + p!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$.

PROOF: Set $G_3 = \mathcal{G}\,\text{f-0}(m,p)$. Set $p_1 = p -' 1$. $\operatorname{eval}(G_3{\restriction}(p -' 1), 0_{\mathbb{Z}^{\mathrm{R}}}) = p_1 \mapsto 0_{\mathbb{Z}^{\mathrm{R}}}$ by [2, (1)], [21, (25)], [8, (49)], (19). For every natural number j such that $j \in \operatorname{dom}(\operatorname{eval}(G_{3\,\lfloor p}, 0_{\mathbb{Z}^{\mathrm{R}}}))$ holds $(\operatorname{eval}(G_{3\,\lfloor p}, 0_{\mathbb{Z}^{\mathrm{R}}}))(j) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal by [2, (1)], [11, (6)], (21), [12, (18), (19)]. Consider u being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $(\operatorname{Eval}(\curvearrowleft^{@}\sum G_{3\,\lfloor p}))(0) = p!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$. $\square$

(31) Let us consider an odd, prime natural number p, a positive natural number m, and a natural number j. Suppose $j \in \operatorname{Seg} m$. Then $(\mathcal{F}\,\text{f-0}(m,p))(j(\in \mathbb{R}_{\mathrm{F}})) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal.

PROOF: Set $G_3 = \mathcal{G}\,\text{f-0}(m,p)$. $\operatorname{eval}(G_3{\restriction}p, j(\in \mathbb{Z}^{\mathrm{R}})) = p \mapsto 0_{\mathbb{Z}^{\mathrm{R}}}$ by [2, (1)], [21, (25)], [8, (49)], (23). For every natural number k such that $k \in \operatorname{dom}(\operatorname{eval}(G_{3\,\lfloor p}, j(\in \mathbb{Z}^{\mathrm{R}})))$ holds $(\operatorname{eval}(G_{3\,\lfloor p}, j(\in \mathbb{Z}^{\mathrm{R}})))(k) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal by [2, (1)], (29). $\square$

4. THE MAIN PART OF THE PROOF

Now we state the proposition:

(32) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Then $(\operatorname{Eval}(\curvearrowleft^{@}\text{f-0}(m,p)))(x) = (\operatorname{eval}(\curvearrowleft^{@}\prod \mathrm{x}.(m,p), x)) \cdot (\operatorname{eval}(\curvearrowleft^{@}(\tau(0))^{p-'1}, x))$.

Let us consider m, p, and g. The functor delta-1(m,p,g) yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 6) $\operatorname{len} it = m$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = g(i) \cdot (\mathcal{F}\,\text{f-0}(m,p))(i(\in \mathbb{R}_{\mathrm{F}}))$.

In the sequel z_0 denotes a non zero element of $\mathbb{R}_{\mathrm{F}}$.

Let us consider m, p, g, and z_0. The functor delta-2(m,p,g,z_0) yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 7) $\operatorname{len} it = m$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = -g(i) \cdot (\operatorname{power}_{\mathbb{R}_{\mathrm{F}}}(z_0, i) \cdot (\mathcal{F}\,\text{f-0}(m,p))(0))$.

The functor delta(m,p,g,z_0) yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by the term

(Def. 8) $\text{delta-1}(m,p,g)+\text{delta-2}(m,p,g,z_0)$.

The functor $\hat{\text{delta}}(m,p,g)$ yielding a finite sequence of elements of $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 9) $\text{delta-1}(m,p,g)$.

Now we state the propositions:

(33) $\sum \text{delta-1}(m,p,g) \in \mathbb{Z}^{\mathrm{R}}$.
PROOF: For every natural number i such that $i \in \text{dom}(\text{delta-1}(m,p,g))$ holds $(\text{delta-1}(m,p,g))(i) \in \mathbb{Z}$ by [**?** , (30)]. $\square$

(34) Let us consider a non zero polynomial g over $\mathbb{Z}^{\mathrm{R}}$. Suppose $\deg(g)=m$. Let us consider a non zero element x of $\mathbb{R}_{\mathrm{F}}$. Then $\sum \text{delta-2}(m,p,g,x) = g(0)\cdot(\mathcal{F}\,\text{f-0}(m,p))(0) - (\text{ExtEval}(g,x))\cdot(\mathcal{F}\,\text{f-0}(m,p))(0)$.
PROOF: For every non zero element x of $\mathbb{R}_{\mathrm{F}}$, $\sum \text{delta-2}(m,p,g,x) = g(0)\cdot(\mathcal{F}\,\text{f-0}(m,p))(0) - (\text{ExtEval}(g,x))\cdot(\mathcal{F}\,\text{f-0}(m,p))(0)$ by [18, (8)], [24, (72)], (30), [2, (39), (22), (1)]. $\square$

(35) $\sum \text{delta-1}(m,p,g) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. The theorem is a consequence of (31).

(36) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Suppose $0 < x \leqslant m$. Let us consider a natural number i. Suppose $i \in \operatorname{Seg} m$. Then $|\,\text{eval}(\curvearrowleft^{@}(\text{x}.(m,p))_{/i}, x)| \leqslant m^p$.
PROOF: Set $F_1 = \mathbb{R}_{\mathrm{F}}$. Reconsider $z_0 = -i$ as an element of F_1. $|(z_0+x)^p| \leqslant m^p$ by [17, (9)]. $\square$

(37) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Then $\text{eval}(\curvearrowleft^{@}(\tau(0))^{p-'1}, x) = x^{p-'1}$. The theorem is a consequence of (3) and (4).

(38) (i) $m^{m+1}\,\text{ExpSeq}_{\mathbb{R}}$ is convergent, and

(ii) $\lim m^{m+1}\,\text{ExpSeq}_{\mathbb{R}} = 0$.

(39) Let us consider a non zero natural number M, and a non zero element z_0 of $\mathbb{R}_{\mathrm{F}}$. Suppose $z_0 = e$. Then there exists a natural number n_1 such that for every natural number n such that $n_1 \leqslant n$ holds $|\frac{(m^{m+1})^n}{n!} - 0| < \frac{1}{2\cdot(M\cdot(z_0{}^m))}$. The theorem is a consequence of (38).

(40) Every $\mathbb{Z}$-valued polynomial over $\mathbb{F}_{\mathbb{Q}}$ is a polynomial over $\mathbb{Z}^{\mathrm{R}}$.

The following theorem corresponds to the equation (3) in [**?**].

Now we state the proposition:

(41) Suppose e is algebraic. Then there exists a $\mathbb{Z}$-valued polynomial g over $\mathbb{F}_{\mathbb{Q}}$ such that

(i) $\hat{g}$ is irreducible, and

(ii) $\text{ExtEval}(g, e(\in \mathbb{R}_{\mathrm{F}})) = 0$, and

(iii) $\deg(g) \geqslant 2$, and

(iv) $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$.

PROOF: Consider x being an element of $\mathbb{C}_{\mathrm{F}}$ such that $x = e$ and x is integral over $\mathbb{F}_{\mathbb{Q}}$. Consider f_0 being an element of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$ such that $f_0 \neq \mathbf{0}.\mathbb{F}_{\mathbb{Q}}$ and $\{f_0\}$–ideal $=$ AnnPoly$(x, \mathbb{F}_{\mathbb{Q}})$ and $f_0 =$ NormPoly f_0. Consider f being a polynomial over $\mathbb{F}_{\mathbb{Q}}$ such that $f_0 = f$ and ExtEval$(f, x) = 0_{\mathbb{C}_{\mathrm{F}}}$. Reconsider $m = \prod$ denomi-seq(f_0) as a non zero natural number. Reconsider $\mho_0 = m \cdot f_0$ as an element of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. rng $\mho_0 \subseteq \mathbb{Z}$ by [23, (27)], [**?** , (10)]. □

Now we state the proposition:

(42) e is transcendental.

PROOF: Consider g being a $\mathbb{Z}$-valued polynomial over $\mathbb{F}_{\mathbb{Q}}$ such that $\hat{g}$ is irreducible and ExtEval$(g, e(\in \mathbb{R}_{\mathrm{F}})) = 0$ and $\deg(g) \geqslant 2$ and $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$. Reconsider $g_0 = g$ as a polynomial over $\mathbb{Z}^{\mathrm{R}}$. Reconsider $g_0 = g$ as a non zero polynomial over $\mathbb{Z}^{\mathrm{R}}$. Reconsider $m_0 = \deg(g_0)$ as a positive natural number. Reconsider $z_0 = e$ as a non zero element of $\mathbb{R}_{\mathrm{F}}$. Consider M_0 being a natural number such that for every natural number i, $|g_0(i)| \leqslant M_0$. Consider n_1 being a natural number such that for every natural number n such that $n_1 \leqslant n$ holds $|\frac{(m_0{}^{m_0+1})^n}{n!} - 0| < \frac{1}{2 \cdot (m_0 \cdot M_0 \cdot m_0{}^{m_0+1} \cdot (z_0{}^{m_0}))}$. Consider p_1 being a prime number such that $n_1 + m_0 + M_0 < p_1$. $\sum$ delta$(m_0, p_1, g_0, z_0) = \sum$ delta-1$(m_0, p_1, g_0) + \sum$ delta-2(m_0, p_1, g_0, z_0) by [18, (7)]. $\sum$ delta-1$(m_0, p_1, g_0) \in \mathbb{Z}^{\mathrm{R}}$. Consider u being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathcal{F}\,$f-0$(m_0, p_1))(0) = (p_1 -' 1)! \cdot (((-1)^{m_0} \cdot (m_0!))^{p_1} (\in \mathbb{Z}^{\mathrm{R}})) + p_1!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$. $\frac{\sum \text{delta-2}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!}$ is an element of $\mathbb{Z}^{\mathrm{R}}$ and $\frac{\sum \text{delta-2}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!} = (((-1)^{m_0} \cdot (m_0!))^{p_1} (\in \mathbb{Z}^{\mathrm{R}}) + p_1 \cdot u) \cdot g_0(0)$ by (34), [**?** , (1)], [23, (1)], [18, (19)]. $\sum$ delta-1$(m_0, p_1, g_0) \in \{p_1!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. Consider v being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $\sum$ delta-1(m_0, p_1, g_0) $p_1!(\in \mathbb{Z}^{\mathrm{R}}) \cdot v$. $\frac{\sum \text{delta-1}(m_0, p_1, g_0)}{(p_1 -' 1)!} = p_1 \cdot v$. $\frac{\sum \text{delta}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!} \in \mathbb{Z}^{\mathrm{R}}$ and $\frac{\sum \text{delta}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!} = \frac{\sum \text{delta-1}(m_0, p_1, g_0)}{(p_1 -' 1)!} + \frac{\sum \text{delta-2}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!}$. $|\frac{\sum \text{delta}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!}| \leqslant \frac{1}{2}$ by [20, (11)], [16, (5)]. $\frac{\sum \text{delta}(m_0, p_1, g_0, z_0)}{(p_1 -' 1)!} = 0$ by [1, (14)]. □

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