

# Integral of Continuous Three Variable Functions<sup>1</sup>

Noboru Endou  
National Institute of Technology, Gifu College  
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Yasunari Shidama  
Karuiizawa Hotch 244-1  
Nagano, Japan

**Summary.** In this article, following the previous article [11], we continue our proofs on integrals of continuous functions of three variables in Mizar [2], [3]. In the first section, continuity of functions of three variables is shown. These are used in the proofs of the later sections.

The second section summarizes the basic properties of the projection of a continuous function in three variables, a result that is almost as obvious as in two variables, but is used to transform [10] Riemann and Lebesgue integrals for real-valued functions (not extended real-valued functions).

In the last section, we prove integrability and iterated integrals of continuous functions of three variables. Throughout the paper, the basic operations follow [1], [19], and [13].

MSC: 26B15 97I50 68V20

Keywords: triple integral; iterated integral

MML identifier: MESFUN17, version: 8.1.14 5.79.1465

## 1. PRELIMINARIES

Now we state the propositions:

---

<sup>1</sup>This work was supported by JSPS KAKENHI 23K11242.

(1) Let us consider real normed spaces  $X, Y, Z$ , a point  $u$  of  $X \times Y \times Z$ , a point  $x$  of  $X$ , a point  $y$  of  $Y$ , and a point  $z$  of  $Z$ . Suppose  $u = \langle x, y, z \rangle$ . Then

(i)  $\|u\| \leq \|x\| + \|y\| + \|z\|$ , and

(ii)  $\|x\| \leq \|u\|$ , and

(iii)  $\|y\| \leq \|u\|$ , and

(iv)  $\|z\| \leq \|u\|$ .

(2) Let us consider closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , and a subset  $E$  of ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ). If  $E = (I \times J) \times K$ , then  $E$  is compact.

(3) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a set  $E$ . Suppose  $f = g$  and  $E \subseteq \text{dom } f$ . Then  $f$  is uniformly continuous on  $E$  if and only if for every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $\langle x_1, y_1, z_1 \rangle, \langle x_2, y_2, z_2 \rangle \in E$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$ .

PROOF: For every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every points  $p_1, p_2$  of ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) such that  $p_1, p_2 \in E$  and  $\|p_1 - p_2\| < r$  holds  $\|f/p_1 - f/p_2\| < e$  by [17, (9)], [14, (4)], [4, (60)], (1).  $\square$

(4) Let us consider intervals  $I, J, K$ . Then

(i)  $(I \times J) \times K$  is a subset of ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ), and

(ii)  $(I \times J) \times K \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$ .

(5) Let us consider a point  $u$  of (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ), and a real number  $r$ . Suppose  $0 < r$ . Then there exist real numbers  $s, x, y, z$  such that

(i)  $0 < s < r$ , and

(ii)  $u = \langle x, y, z \rangle$ , and

(iii)  $]x - s, x + s[ \times ]y - s, y + s[ \times ]z - s, z + s[ \subseteq \text{Ball}(u, r)$ .

Let us consider a subset  $A$  of (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ). Now we state the propositions:

- (6) Suppose for every real numbers  $a, b, c$  such that  $\langle a, b, c \rangle \in A$  there exists a real-membered set  $R_{12}$  such that  $R_{12}$  is non empty and upper bounded and  $R_{12} = \{r, \text{ where } r \text{ is a real number : } 0 < r \text{ and } ]a-r, a+r[ \times ]b-r, b+r[ \times ]c-r, c+r[ \subseteq A\}$ . Then there exists a function  $F$  from  $A$  into  $\mathbb{R}$  such that for every real numbers  $a, b, c$  such that  $\langle a, b, c \rangle \in A$  there exists a real-membered set  $R_{12}$  such that  $R_{12}$  is non empty and upper bounded and  $R_{12} = \{r, \text{ where } r \text{ is a real number : } 0 < r \text{ and } ]a-r, a+r[ \times ]b-r, b+r[ \times ]c-r, c+r[ \subseteq A\}$  and  $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$ .

PROOF: Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exist real numbers  $a, b, c$  and there exists a real-membered set  $R_{12}$  such that  $\$1 = \langle a, b, c \rangle$  and  $R_{12}$  is non empty and upper bounded and  $R_{12} = \{r, \text{ where } r \text{ is a real number : } 0 < r \text{ and } ]a-r, a+r[ \times ]b-r, b+r[ \times ]c-r, c+r[ \subseteq A\}$  and  $\$2 = \frac{\sup R_{12}}{2}$ . For every object  $x$  such that  $x \in A$  there exists an object  $y$  such that  $y \in \mathbb{R}$  and  $\mathcal{P}[x, y]$  by [17, (9)]. Consider  $F$  being a function from  $A$  into  $\mathbb{R}$  such that for every object  $x$  such that  $x \in A$  holds  $\mathcal{P}[x, F(x)]$  from [6, Sch. 1]. For every real numbers  $a, b, c$  such that  $\langle a, b, c \rangle \in A$  there exists a real-membered set  $R_{12}$  such that  $R_{12}$  is non empty and upper bounded and  $R_{12} = \{r, \text{ where } r \text{ is a real number : } 0 < r \text{ and } ]a-r, a+r[ \times ]b-r, b+r[ \times ]c-r, c+r[ \subseteq A\}$  and  $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$ .  $\square$

- (7) If  $A$  is open, then  $A \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$ . The theorem is a consequence of (5), (6), and (1).

Now we state the propositions:

- (8) Let us consider closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Let us consider a real number  $e$ . Suppose  $0 < e$ . Then there exists a real number  $r$  such that

- (i)  $0 < r$ , and
- (ii) for every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $x_1, x_2 \in I$  and  $y_1, y_2 \in J$  and  $z_1, z_2 \in K$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$ .

PROOF: Set  $E = (I \times J) \times K$ .  $f$  is uniformly continuous on  $E$ . Consider  $r$  being a real number such that  $0 < r$  and for every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $\langle x_1, y_1, z_1 \rangle, \langle x_2, y_2, z_2 \rangle \in E$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$ . For every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $x_1, x_2 \in I$  and  $y_1, y_2 \in J$  and  $z_1, z_2 \in K$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$  by [7, (87)].  $\square$

- (9) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . If  $f = g$ , then  $\|f\| = |g|$ .
- (10) Let us consider closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Let us consider a real number  $e$ . Suppose  $0 < e$ . Then there exists a real number  $r$  such that
- (i)  $0 < r$ , and
  - (ii) for every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $x_1, x_2 \in I$  and  $y_1, y_2 \in J$  and  $z_1, z_2 \in K$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $\|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)\| < e$ .

The theorem is a consequence of (9) and (8).

## 2. PROPERTIES ON THE PROJECTIVE FUNCTION OF A THREE VARIABLE FUNCTION

Now we state the propositions:

- (11) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and elements  $x, y$  of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$ . Then  $\text{ProjPMap1}(g, \langle x, y \rangle)$  is continuous.  
 PROOF: For every real number  $z_0$  such that  $z_0 \in \text{dom}(\text{ProjPMap1}(g, \langle x, y \rangle))$  holds  $\text{ProjPMap1}(g, \langle x, y \rangle)$  is continuous in  $z_0$  by [15, (19)], [14, (4)], [17, (9)], [21, (15)].  $\square$
- (12) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , a partial function  $p_2$  from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and an element  $z$  of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $p_2 = \text{ProjPMap2}(g, z)$ . Then  $p_2$  is continuous on  $\text{dom } p_2$ .  
 PROOF: For every point  $x_4$  of (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) such that  $x_4 \in \text{dom } p_2$  holds  $p_2 \upharpoonright \text{dom } p_2$  is continuous in  $x_4$  by [18, (18)], [15, (19)], [17, (9)], [21, (15)].  $\square$

- (13) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and elements  $x, y$  of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$ . Then  $\text{ProjPMap1}(|g|, \langle x, y \rangle)$  is continuous. The theorem is a consequence of (11).
- (14) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , a partial function  $p_2$  from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and an element  $z$  of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $p_2 = \text{ProjPMap2}(|g|, z)$ . Then  $p_2$  is continuous on  $\text{dom } p_2$ . The theorem is a consequence of (12).
- (15) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and elements  $x, y$  of  $\mathbb{R}$ . Suppose  $f$  is uniformly continuous on  $\text{dom } f$  and  $f = g$ . Then  $\text{ProjPMap1}(g, \langle x, y \rangle)$  is uniformly continuous.
- PROOF: For every real number  $r$  such that  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every real numbers  $z_1, z_2$  such that  $z_1, z_2 \in \text{dom}(\text{ProjPMap1}(g, \langle x, y \rangle))$  and  $|z_1 - z_2| < s$  holds  $|(\text{ProjPMap1}(g, \langle x, y \rangle))(z_1) - (\text{ProjPMap1}(g, \langle x, y \rangle))(z_2)| < r$  by [14, (4)], [17, (9)], [21, (15)], [20, (22)].  $\square$
- (16) Let us consider a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , a partial function  $p_2$  from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and an element  $z$  of  $\mathbb{R}$ . Suppose  $f$  is uniformly continuous on  $\text{dom } f$  and  $f = g$  and  $p_2 = \text{ProjPMap2}(g, z)$ . Then  $p_2$  is uniformly continuous on  $\text{dom } p_2$ .
- (17) Let us consider elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(\mathbb{R}(g), \langle x, y \rangle)$ . Then  $P_8$  is continuous. The theorem is a consequence of (11).
- (18) Let us consider an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times$

$\mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_7$  from ((the real normed space of  $\mathbb{R}) \times$  (the real normed space of  $\mathbb{R})$  to the real normed space of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $P_7 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$ . Then  $P_7$  is continuous on  $\text{dom } P_7$ . The theorem is a consequence of (12).

- (19) Let us consider elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}) \times$  (the real normed space of  $\mathbb{R})$ )  $\times$  (the real normed space of  $\mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$ . Then  $P_8$  is continuous. The theorem is a consequence of (13).
- (20) Let us consider an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}) \times$  (the real normed space of  $\mathbb{R})$ )  $\times$  (the real normed space of  $\mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_7$  from (the real normed space of  $\mathbb{R}) \times$  (the real normed space of  $\mathbb{R})$  to the real normed space of  $\mathbb{R}$ . Suppose  $f$  is continuous on  $\text{dom } f$  and  $f = g$  and  $P_7 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$ . Then  $P_7$  is continuous on  $\text{dom } P_7$ . The theorem is a consequence of (14).

### 3. INTEGRAL OF CONTINUOUS THREE VARIABLE FUNCTIONS

Let us consider subsets  $I, J$  of  $\mathbb{R}$ , a non empty, closed interval subset  $K$  of  $\mathbb{R}$ , elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}) \times$  (the real normed space of  $\mathbb{R})$ )  $\times$  (the real normed space of  $\mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Now we state the propositions:

- (21) Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$ . Then
- (i)  $P_8|_K$  is bounded, and
  - (ii)  $P_8$  is integrable on  $K$ .

The theorem is a consequence of (17).

- (22) Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$ . Then
- (i)  $P_8$  is integrable on L-Meas, and
  - (ii)  $\int_K P_8(x)dx = \int P_8 \text{ dL-Meas}$ , and
  - (iii)  $\int_K P_8(x)dx = \int \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle) \text{ dL-Meas}$ , and

$$(iv) \int_K P_8(x)dx = (\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(\langle x, y \rangle).$$

The theorem is a consequence of (21).

Now we state the propositions:

- (23) Let us consider non empty, closed interval subsets  $I, J$  of  $\mathbb{R}$ , a subset  $K$  of  $\mathbb{R}$ , an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_9$  from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $z \in K$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_9 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$ . Then

(i)  $P_9$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$ , and

(ii)  $\int P_9 \text{ dProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{ProjPMap2}(\overline{\mathbb{R}}(g), z) \text{ dProdMeas}(\text{L-Meas}, \text{L-Meas})$   
and

(iii)  $\int P_9 \text{ dProdMeas}(\text{L-Meas}, \text{L-Meas}) = (\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)))(\langle x, y \rangle)$ .

The theorem is a consequence of (18).

- (24) Let us consider subsets  $I, J$  of  $\mathbb{R}$ , a non empty, closed interval subset  $K$  of  $\mathbb{R}$ , elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$ . Then

(i)  $P_8|_K$  is bounded, and

(ii)  $P_8$  is integrable on  $K$ .

The theorem is a consequence of (19).

- (25) Let us consider subsets  $I, J$  of  $\mathbb{R}$ , a non empty, closed interval subset  $K$  of  $\mathbb{R}$ , elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of L-Field. Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$  and  $E = K$ . Then  $P_8$  is  $E$ -measurable. The theorem is a consequence of (24).

- (26) Let us consider subsets  $I, J$  of  $\mathbb{R}$ , a non empty, closed interval subset  $K$  of  $\mathbb{R}$ , elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of L-Field. Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$  and  $E = K$ . Then  $P_8$  is  $E$ -measurable. The theorem is a consequence of (24).

$\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $y \in J$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$ . Then

- (i)  $P_8$  is integrable on L-Meas, and
- (ii)  $\int_K P_8(x) dx = \int P_8 d\text{L-Meas}$ , and
- (iii)  $\int_K P_8(x) dx = \int \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle) d\text{L-Meas}$ , and
- (iv)  $\int_K P_8(x) dx = (\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(\langle x, y \rangle)$ .

The theorem is a consequence of (24).

- (27) Let us consider non empty, closed interval subsets  $I, J$  of  $\mathbb{R}$ , a subset  $K$  of  $\mathbb{R}$ , an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , a partial function  $P_9$  from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ . Suppose  $z \in K$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_9 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$  and  $E = I \times J$ . Then  $P_9$  is  $E$ -measurable. The theorem is a consequence of (20).

- (28) Let us consider non empty, closed interval subsets  $I, J$  of  $\mathbb{R}$ , a subset  $K$  of  $\mathbb{R}$ , an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_9$  from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $z \in K$  and  $\text{dom } f = (I \times J) \times K$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_9 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$ . Then

- (i)  $P_9$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$ , and
- (ii)  $\int P_9 d\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z) d\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$   
and
- (iii)  $\int P_9 d\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = (\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|, z))$ .

The theorem is a consequence of (20).

- (29) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of



$\sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $E = (I \times J) \times K$ . Then  $g$  is  $E$ -measurable.

PROOF: For every real number  $r$ ,  $E \cap \text{LE-dom}(g, r) \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$  by [11, (17), (24)], (7).  $\square$

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , elements  $x, y$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a real number  $e$ . Now we state the propositions:

- (30) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then suppose  $0 < e$ . Then there exists a real number  $r$  such that
- (i)  $0 < r$ , and
  - (ii) for every elements  $u_1, u_2$  of  $\mathbb{R} \times \mathbb{R}$  and for every real numbers  $x_1, y_1, x_2, y_2$  such that  $u_1 = \langle x_1, y_1 \rangle$  and  $u_2 = \langle x_2, y_2 \rangle$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $u_1, u_2 \in I \times J$  for every element  $z$  of  $\mathbb{R}$  such that  $z \in K$  holds  $|(\text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_2))(z) - (\text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_1))(z)| < e$ .

PROOF: For every element  $x$  of  $\mathbb{R} \times \mathbb{R}$  and for every element  $y$  of  $\mathbb{R}$  such that  $x \in I \times J$  and  $y \in K$  holds  $(\text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x))(y) = |\overline{\mathbb{R}}(g)|(x, y)$  and  $|\overline{\mathbb{R}}(g)|(x, y) = |g|(\langle x, y \rangle)$  by [7, (87)], [12, (12)]. Consider  $r$  being a real number such that  $0 < r$  and for every real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$  such that  $x_1, x_2 \in I$  and  $y_1, y_2 \in J$  and  $z_1, z_2 \in K$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $|z_2 - z_1| < r$  holds  $||g|(\langle x_2, y_2, z_2 \rangle) - |g|(\langle x_1, y_1, z_1 \rangle)| < e$ .  $\square$

- (31) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then suppose  $0 < e$ . Then there exists a real number  $r$  such that
- (i)  $0 < r$ , and
  - (ii) for every elements  $u_1, u_2$  of  $\mathbb{R} \times \mathbb{R}$  and for every real numbers  $x_1, y_1, x_2, y_2$  such that  $u_1 = \langle x_1, y_1 \rangle$  and  $u_2 = \langle x_2, y_2 \rangle$  and  $|x_2 - x_1| < r$  and  $|y_2 - y_1| < r$  and  $u_1, u_2 \in I \times J$  for every element  $z$  of  $\mathbb{R}$  such that  $z \in K$  holds  $|(\text{ProjPMap1}(\overline{\mathbb{R}}(g), u_2))(z) - (\text{ProjPMap1}(\overline{\mathbb{R}}(g), u_1))(z)| < e$ .

The theorem is a consequence of (8).

Now we state the proposition:

- (32) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and

a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i)  $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$  is a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ , and
- (ii)  $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \upharpoonright (I \times J)$  is a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ , and
- (iii)  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  is a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ , and
- (iv)  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)$  is a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ .

The theorem is a consequence of (26) and (22).

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $F_4$  from  $(\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ . Now we state the propositions:

- (33) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $F_4 = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \upharpoonright (I \times J)$ . Then  $F_4$  is uniformly continuous on  $I \times J$ . The theorem is a consequence of (30), (19), and (24).
- (34) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $F_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)$ . Then  $F_4$  is uniformly continuous on  $I \times J$ . The theorem is a consequence of (31), (17), (21), and (22).

Now we state the proposition:

- (35) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then
  - (i)  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|)$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ , and
  - (ii)  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|) \upharpoonright K$  is a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ , and
  - (iii)  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ , and
  - (iv)  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \upharpoonright K$  is a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ .

The theorem is a consequence of (20), (28), (18), and (23).

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $G_3$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Now we state the propositions:

- (36) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|) \upharpoonright K$ . Then  $G_3$  is continuous.

PROOF: Consider  $a, b$  being real numbers such that  $I = [a, b]$ . Consider  $c, d$  being real numbers such that  $J = [c, d]$ . For every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every real numbers  $z_1, z_2$  such that  $|z_2 - z_1| < r$  and  $z_1, z_2 \in K$  for every real numbers  $x, y$  such that  $x \in I$  and  $y \in J$  holds  $||g(\langle x, y, z_2 \rangle) - |g(\langle x, y, z_1 \rangle)| < e$ . Set  $R_{11} = \overline{\mathbb{R}}(g)$ . For every elements  $x, y, z$  of  $\mathbb{R}$  such that  $x \in I$  and  $y \in J$  and  $z \in K$  holds  $(\text{ProjPMap2}(|R_{11}|, z))(x, y) = |R_{11}|(\langle x, y \rangle, z)$  and  $|R_{11}|(\langle x, y \rangle, z) = |g(\langle x, y, z \rangle)|$  and  $|R_{11}|(\langle x, y \rangle, z) = |g(\langle x, y, z \rangle)|$  by [7, (87)], [12, (12)]. For every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every elements  $z_1, z_2$  of  $\mathbb{R}$  such that  $|z_2 - z_1| < r$  and  $z_1, z_2 \in K$  for every elements  $x, y$  of  $\mathbb{R}$  such that  $x \in I$  and  $y \in J$  holds  $|(\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_2), x))(y) - (\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_1), x))(y)| < e$  by [11, (28)], [7, (87)], [12, (12)]. For every real numbers  $z_0, r$  such that  $z_0 \in K$  and  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every real number  $z_1$  such that  $z_1 \in K$  and  $|z_1 - z_0| < s$  holds  $|G_3(z_1) - G_3(z_0)| < r$  by [11, (30), (28)], (20), [11, (51), (53)].  $\square$

- (37) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \upharpoonright K$ . Then  $G_3$  is continuous.

PROOF: Consider  $a, b$  being real numbers such that  $I = [a, b]$ . Consider  $c, d$  being real numbers such that  $J = [c, d]$ . For every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every real numbers  $z_1, z_2$  such that  $|z_2 - z_1| < r$  and  $z_1, z_2 \in K$  for every real numbers  $x, y$  such that  $x \in I$  and  $y \in J$  holds  $|g(\langle x, y, z_2 \rangle) - g(\langle x, y, z_1 \rangle)| < e$ . Set  $R_{11} = \overline{\mathbb{R}}(g)$ . For every elements  $x, y, z$  of  $\mathbb{R}$  such that  $x \in I$  and  $y \in J$  and  $z \in K$  holds  $(\text{ProjPMap2}(R_{11}, z))(x, y) = R_{11}(\langle x, y \rangle, z)$  and  $R_{11}(\langle x, y \rangle, z) = g(\langle x, y, z \rangle)$  and  $R_{11}(\langle x, y \rangle, z) = g(\langle x, y, z \rangle)$  by [7, (87)]. For every real number  $e$  such that  $0 < e$  there exists a real number  $r$  such that  $0 < r$  and for every elements  $z_1, z_2$  of  $\mathbb{R}$  such that  $|z_2 - z_1| < r$  and  $z_1, z_2 \in K$  for every elements  $x, y$  of  $\mathbb{R}$  such that  $x \in I$  and  $y \in J$  holds  $|(\text{ProjPMap1}(\text{ProjPMap2}(R_{11}, z_2), x))(y) -$

$(\text{ProjPMap1}(\text{ProjPMap2}(R_{11}, z_1), x))(y) < e$  by [11, (28)], [7, (87)], [12, (12)]. For every real numbers  $z_0, r$  such that  $z_0 \in K$  and  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every real number  $z_1$  such that  $z_1 \in K$  and  $|z_1 - z_0| < s$  holds  $|G_3(z_1) - G_3(z_0)| < r$  by [11, (30), (28)], (18), [11, (51), (53)].  $\square$

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Now we state the propositions:

(38) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$  is non-negative. The theorem is a consequence of (24) and (25).

(39) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|)$  is non-negative. The theorem is a consequence of (20) and (27).

Now we state the propositions:

(40) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , an element  $u$  of  $\mathbb{R} \times \mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $(\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(u) < +\infty$ . The theorem is a consequence of (32).

(41) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , an element  $z$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $(\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|))(z) < +\infty$ . The theorem is a consequence of (35).

(42) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$  is  $E$ -measurable.

PROOF: Set  $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$ . Set  $I_1 = I \times J$ . Reconsider  $G = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)|_{I_1}$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Re-

consider  $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright I_1$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $G_1 = G$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ . Reconsider  $R_6 = R_4$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ .  $G_1$  is uniformly continuous on  $I \times J$ .  $R_6$  is uniformly continuous on  $I \times J$ .  $F$  is non-negative. Reconsider  $H = \mathbb{R} \times \mathbb{R}$  as an element of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ . For every real number  $r$ ,  $H \cap \text{LE-dom}(F, r) \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$  by [16, (4)], [5, (49)], [16, (3)].  $\square$

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Now we state the propositions:

(43) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i)  $g$  is integrable on  $\text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas})$ , and
- (ii) for every element  $u$  of  $\mathbb{R} \times \mathbb{R}$ ,  $\text{ProjPMap1}(\overline{\mathbb{R}}(g), u)$  is integrable on  $\text{L-Meas}$ , and
- (iii) for every element  $U$  of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ ,  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  is  $U$ -measurable, and
- (iv)  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$ , and
- (v)  $\int g \, d \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$

PROOF: Set  $F = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright I_1$ . Set  $I_1 = I \times J$ . Reconsider  $G = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright I_1$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright I_1$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $A_1 = I \times J$  as an element of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ . Reconsider  $G_1 = G$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ . Reconsider  $R_6 = R_4$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ .  $G_1$  is uniformly continuous on  $I \times J$ .  $R_6$  is uniformly continuous on  $I \times J$ . Reconsider  $N_1 = (\mathbb{R} \times \mathbb{R}) \setminus A_1$  as an element of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ .  $F$  is non-negative. Reconsider  $H = \mathbb{R} \times \mathbb{R}$  as an element of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ .  $F$  is  $H$ -measurable. Set  $F_1 = F \upharpoonright N_1$ . For every object  $x$  such that  $x \in \text{dom } F_1$  holds  $F_1(x) = 0$  by [5, (49)]. Reconsider  $K_1 = (I \times J) \times K$  as an element of  $\sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$ .  $g$  is  $K_1$ -

measurable. For every element  $x$  of  $\mathbb{R} \times \mathbb{R}$ ,  $(\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x) < +\infty$ .  $\square$

(44) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i) for every element  $z$  of  $\mathbb{R}$ ,  $\text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$  and
- (ii) for every element  $V$  of  $\text{L-Field}$ ,  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))$  is  $V$ -measurable, and
- (iii)  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))$  is integrable on  $\text{L-Meas}$ , and
- (iv)  $\int g \, d \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \, d \text{L-Meas}$ .

The theorem is a consequence of (43) and (41).

Now we state the propositions:

(45) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , an element  $x$  of  $\mathbb{R}$ , and an element  $E$  of  $\text{L-Field}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $x \in I$ . Then  $\text{ProjPMap1}(|\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, x)$  is  $E$ -measurable.

PROOF: Set  $F_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ . Reconsider  $G_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  as a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ . Reconsider  $G = G_4 \upharpoonright (I \times J)$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $F = G$  as a partial function from  $(\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ .  $F$  is uniformly continuous on  $I \times J$ . Set  $F_5 = \text{ProjPMap1}(|F_4|, x)$ . Set  $L_0 = F_5 \upharpoonright J$ . For every element  $t$  of  $\mathbb{R}$  such that  $t \in J$  holds  $0 \leq L_0(t)$  by [5, (49)], [12, (14)]. Reconsider  $H = \mathbb{R}$  as an element of  $\text{L-Field}$ . For every real number  $r$ ,  $H \cap \text{LE-dom}(F_5, r) \in \text{L-Field}$  by [5, (49)], [16, (4), (3)].  $\square$

(46) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i) for every element  $x$  of  $\mathbb{R}$ ,  $(\text{Integral2}(\text{L-Meas}, |\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(x) < +\infty$ , and
- (ii) for every element  $x$  of  $\mathbb{R}$ ,  $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)$  is integrable on  $\text{L-Meas}$ .

PROOF: Reconsider  $G_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  as a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ . Reconsider  $G = G_4 \upharpoonright (I \times J)$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $F = G$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ .  $F$  is uniformly continuous on  $I \times J$ . For every element  $x$  of  $\mathbb{R}$ ,  $(\text{Integral2}(\text{L-Meas}, |\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(x) < +\infty$  by [11, (25)], [8, (5)], [9, (75)], [16, (5), (6)].  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ .  $\square$

- (47) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , an element  $y$  of  $\mathbb{R}$ , and an element  $E$  of L-Field. Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $y \in J$ . Then  $\text{ProjPMap2}(|\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, y)$  is  $E$ -measurable.

PROOF: Set  $F_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ . Reconsider  $G_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  as a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ . Reconsider  $G = G_4 \upharpoonright (I \times J)$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $F = G$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ .  $F$  is uniformly continuous on  $I \times J$ . Set  $F_6 = \text{ProjPMap2}(|F_4|, y)$ . Set  $L_0 = F_6 \upharpoonright I$ . For every element  $t$  of  $\mathbb{R}$  such that  $t \in I$  holds  $0 \leq L_0(t)$  by [5, (49)], [12, (14)]. For every element  $r$  of  $\mathbb{R}$ ,  $0_{\overline{\mathbb{R}}} \leq F_6(r)$  by [5, (49)]. Reconsider  $H = \mathbb{R}$  as an element of L-Field. For every real number  $r$ ,  $H \cap \text{LE-dom}(F_6, r) \in \text{L-Field}$  by [16, (4)], [5, (49)], [16, (3)].  $\square$

- (48) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from ((the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ))  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i) for every element  $y$  of  $\mathbb{R}$ ,  $(\text{Integral1}(\text{L-Meas}, |\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(y) < +\infty$ , and
- (ii) for every element  $y$  of  $\mathbb{R}$ ,  $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)$  is integrable on L-Meas.

PROOF: Reconsider  $G_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  as a function from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ . Reconsider  $G = G_4 \upharpoonright (I \times J)$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $F = G$  as a partial function from (the real normed space of  $\mathbb{R}$ )  $\times$  (the real normed space of  $\mathbb{R}$ ) to the real normed space of  $\mathbb{R}$ .  $F$  is uniformly continuous on  $I \times J$ . For every element  $y$  of  $\mathbb{R}$ ,

$(\text{Integral1}(\text{L-Meas}, |\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(y) < +\infty$  by [11, (26)], [8, (5)], [9, (75)], [16, (5), (6)].  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$  is integrable on  $\text{ProdMeas}(\text{L-Meas})$ .  $\square$

- (49) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$  is  $E$ -measurable.

PROOF: Set  $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$ . Set  $F_0 = F \upharpoonright (I \times J)$ . Reconsider  $G = F_0$  as a partial function from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . Reconsider  $G_1 = G$  as a partial function from  $(\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ .  $G_1$  is uniformly continuous on  $I \times J$ . Reconsider  $R_2 = \mathbb{R} \times \mathbb{R}$  as an element of  $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$ .  $F$  is non-negative. For every real number  $r$ ,  $R_2 \cap \text{LE-dom}(F, r) \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$  by [16, (4)], [5, (49)], [16, (3)].  $\square$

- (50) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $E$  of  $\text{L-Field}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|)$  is  $E$ -measurable.

PROOF: Set  $F = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|)$ . Set  $F_0 = F \upharpoonright K$ . Reconsider  $G = F_0$  as a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ .  $G \upharpoonright K$  is bounded and  $G$  is integrable on  $K$ . Reconsider  $R = \mathbb{R}$  as an element of  $\text{L-Field}$ .  $F$  is non-negative. For every real number  $r$ ,  $R \cap \text{LE-dom}(F, r) \in \text{L-Field}$  by [16, (4)], [5, (49)], [16, (3)].  $\square$

- (51) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $x$  of  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i)  $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ , and
- (ii)  $\text{ProjPMap1}(|\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, x)$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ .

The theorem is a consequence of (32).



- (52) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and an element  $y$  of  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then
- (i)  $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ , and
  - (ii)  $\text{ProjPMap2}(|\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, y)$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ .

The theorem is a consequence of (32).

- (53) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then  $|\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))|$  is a function from  $\mathbb{R}$  into  $\mathbb{R}$ . The theorem is a consequence of (35).
- (54) Let us consider an element  $x$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } g$ . Then  $\int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright \mathbb{R} \setminus J \, d\text{L-Meas} = 0$ .
- (55) Let us consider an element  $y$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } g$ . Then  $\int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright \mathbb{R} \setminus I \, d\text{L-Meas} = 0$ .
- (56) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } g$ . Then  $\int \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \upharpoonright \mathbb{R} \setminus K \, d\text{L-Meas} = 0$ .
- (57) Let us consider an element  $x$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_1$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright J$ . Then  $P_1$  is continuous. The theorem is a consequence of (32) and (34).
- (58) Let us consider an element  $y$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_2$

from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $y \in J$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright I$ . Then  $P_2$  is continuous. The theorem is a consequence of (32) and (34).

- (59) Let us consider an element  $x$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_1$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright J$ . Then

- (i)  $P_1 \upharpoonright J$  is bounded, and
- (ii)  $P_1$  is integrable on  $J$ .

The theorem is a consequence of (32) and (34).

- (60) Let us consider an element  $y$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_2$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $y \in J$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright I$ . Then

- (i)  $P_2 \upharpoonright I$  is bounded, and
- (ii)  $P_2$  is integrable on  $I$ .

The theorem is a consequence of (32) and (34).

- (61) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $G_3$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \upharpoonright K$ . Then

- (i)  $G_3 \upharpoonright K$  is bounded, and
- (ii)  $G_3$  is integrable on  $K$ .

The theorem is a consequence of (37).

- (62) Let us consider an element  $x$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_1$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on

$(I \times J) \times K$  and  $f = g$  and  $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright J$ .  
Then

- (i)  $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright J$  is integrable on L-Meas, and
- (ii)  $\int_J P_1(x) dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \upharpoonright J d \text{L-Meas}$ , and
- (iii)  $\int_J P_1(x) dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) d \text{L-Meas}$ , and
- (iv)  $\int_J P_1(x) dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(x)$ .

The theorem is a consequence of (46), (59), and (54).

(63) Let us consider an element  $y$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_2$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $y \in J$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright I$ .  
Then

- (i)  $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright I$  is integrable on L-Meas, and
- (ii)  $\int_I P_2(x) dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \upharpoonright I d \text{L-Meas}$ , and
- (iii)  $\int_I P_2(x) dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) d \text{L-Meas}$ , and
- (iv)  $\int_I P_2(x) dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(y)$ .

The theorem is a consequence of (48), (60), and (55).

Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , and a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ . Now we state the propositions:

(64) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i) for every element  $U$  of L-Field,  $\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$  is  $U$ -measurable, and

- (ii)  $\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$  is integrable on L-Meas,  
and
- (iii)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, d \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, d \text{L-Meas}$ ,  
and
- (iv)  $\int g \, d \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, d \text{L-Meas}$ ,  
and
- (v)  $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)$  is integrable on  $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$ ,  
and
- (vi)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J) \, d \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, d \text{L-Meas}$ .

The theorem is a consequence of (32), (43), (46), (40), and (34).

- (65) Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$ . Then

- (i) for every element  $V$  of L-Field,  $\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$  is  $V$ -measurable, and
- (ii)  $\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$  is integrable on L-Meas,  
and
- (iii)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, d \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))) \, d \text{L-Meas}$ ,  
and
- (iv)  $\int g \, d \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))) \, d \text{L-Meas}$ ,  
and
- (v)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J) \, d \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))) \upharpoonright (I \times J) \, d \text{L-Meas}$ .

The theorem is a consequence of (32), (43), (48), (40), and (34).

Now we state the propositions:

- (66) Let us consider an element  $x$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_1$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $x \in I$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), x)$ . Then
- (i)  $P_1$  is continuous, and
  - (ii)  $\text{dom}(\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), x)) = J$ , and
  - (iii)  $P_1 \upharpoonright J$  is bounded, and
  - (iv)  $P_1$  is integrable on  $J$ , and

$$(v) \int_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), x) d \text{L-Meas},$$

and

$$(vi) \int_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)))(x),$$

and

(vii)  $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), x)$  is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(67) Let us consider an element  $y$  of  $\mathbb{R}$ , non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $P_2$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $y \in J$  and  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), y)$ . Then

(i)  $P_2$  is continuous, and

(ii)  $\text{dom}(\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), y)) = I$ , and

(iii)  $P_2 \upharpoonright I$  is bounded, and

(iv)  $P_2$  is integrable on  $I$ , and

$$(v) \int_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), y) d \text{L-Meas},$$

and

$$(vi) \int_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)))(y),$$

and

(vii)  $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J), y)$  is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(68) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $G_8$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $G_8 = \text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright I$ . Then

(i)  $\text{dom } G_8 = I$ , and

(ii)  $G_8$  is continuous, and

(iii)  $G_8 \upharpoonright I$  is bounded, and

(iv)  $G_8$  is integrable on  $I$ , and

(v)  $\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright I$  is integrable on L-Meas, and

(vi)  $\int \text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright I \, d\text{L-Meas} = \int_I G_8(x) dx,$   
and

(vii)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J) \, d\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int_I G_8(x) dx.$

The theorem is a consequence of (32) and (34).

(69) Let us consider non empty, closed interval subsets  $I, J, K$  of  $\mathbb{R}$ , a partial function  $f$  from  $((\text{the real normed space of } \mathbb{R}) \times (\text{the real normed space of } \mathbb{R})) \times (\text{the real normed space of } \mathbb{R})$  to the real normed space of  $\mathbb{R}$ , a partial function  $g$  from  $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$  to  $\mathbb{R}$ , and a partial function  $G_7$  from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $(I \times J) \times K = \text{dom } f$  and  $f$  is continuous on  $(I \times J) \times K$  and  $f = g$  and  $G_7 = \text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright J$ . Then

(i)  $\text{dom } G_7 = J$ , and

(ii)  $G_7$  is continuous, and

(iii)  $G_7 \upharpoonright J$  is bounded, and

(iv)  $G_7$  is integrable on  $J$ , and

(v)  $\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright J$  is integrable on L-Meas, and

(vi)  $\int \text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J)) \upharpoonright J \, d\text{L-Meas} = \int_J G_7(x) dx,$  and

(vii)  $\int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \upharpoonright (I \times J) \, d\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int_J G_7(x) dx.$

The theorem is a consequence of (32) and (34).

## REFERENCES

- [1] Tom M. Apostol. *Calculus*, volume II. Wiley, second edition, 1969.
- [2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8\_17.
- [3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

- [4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Noboru Endou. Product pre-measure. *Formalized Mathematics*, 24(1):69–79, 2016. doi:10.1515/forma-2016-0006.
- [9] Noboru Endou. Reconstruction of the one-dimensional Lebesgue measure. *Formalized Mathematics*, 28(1):93–104, 2020. doi:10.2478/forma-2020-0008.
- [10] Noboru Endou. Relationship between the Riemann and Lebesgue integrals. *Formalized Mathematics*, 29(4):185–199, 2021. doi:10.2478/forma-2021-0018.
- [11] Noboru Endou and Yasunari Shidama. Integral of continuous functions of two variables. *Formalized Mathematics*, 31(1):309–324, 2023. doi:10.2478/forma-2023-0025.
- [12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(3):491–494, 2001.
- [13] Serge Lang. *Calculus of Several Variables*. Springer, third edition, 2012.
- [14] Keiko Narita, Noboru Endou, and Yasunari Shidama. Weak convergence and weak\* convergence. *Formalized Mathematics*, 23(3):231–241, 2015. doi:10.1515/forma-2015-0019.
- [15] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(3):269–275, 2004.
- [16] Andrzej Nędzusiak.  $\sigma$ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [17] Hiroyuki Okazaki and Kazuhisa Nakasho. The 3-fold product space of real normed spaces and its properties. *Formalized Mathematics*, 29(4):241–248, 2021. doi:10.2478/forma-2021-0022.
- [18] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. Cartesian products of family of real linear spaces. *Formalized Mathematics*, 19(1):51–59, 2011. doi:10.2478/v10037-011-0009-2.
- [19] Edward Staples Smith, Salkover Meyer, and Howard K. Justice. *Calculus*. John Wiley and Sons, second edition, 1958.
- [20] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.

*Accepted June 18, 2024*

---