

Elementary Number Theory Problems. Part XIII

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Summary. This paper formalizes problems 41, 92, 121–123, 172, 182, 183, 191, 192 and 192a from "250 Problems in Elementary Number Theory" by Wacław Sierpiński [8].

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INTRODUCTION

In this paper, Problems 41 from Section I, 92, 121, 122, 123 from Section IV, 172, 182, 183, 191, 192, and 192a from Section V of [8] are formalized, using the Mizar formalism [2], [1]. The paper is a part of the project *Formalization of Elementary Number Theory in Mizar* [7], [4], [5], [6], [3].

In the preliminary section, we proved some trivial but useful facts about numbers.

In problem 92 the inequality $p_{k+1} + p_{k+2} \leq p_1 \cdot p_2 \cdot \ldots \cdot p_k$ should be justified for any integer $k \geq 3$, where p_k denotes the k-th prime. Because we count primes starting from the index 0, we formulated the fact as:

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3 <= k implies
primenumber(k) + primenumber(k+1) <= Product primesFinS(k);</pre>
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where primesFinS(k) denotes the finite sequence of primes of the length k, and elements of finite sequences are indexed from 1.

Problem 121 about finding the least positive integer n for which $k \cdot 2^{2^n} + 1$ is composite is represented as separated theorems for every positive $k \leq 10$.

Problem 122 requires finding all positive integers $k \leq 10$ such that every number $k \cdot 2^{2^n} + 1$ (n = 1, 2, ...) is composite. The proof lies in the fact that numbers $(3 \cdot t + 2) \cdot 2^{2^n} + 1$ are all divisible by 3 and greater than 3, for every natural t, and every positive natural n. In the book, there are minor misprints in the proof, where $2 \cdot 2^{2^2} + 1$ should be $2 \cdot 2^{2^n} + 1$ and $5 \cdot 2^{2^2} + 1$ should be $5 \cdot 2^{2^n} + 1$.

Problems 191 and 192 are generalized from positive integers to non-zero integers.

Problem 192a is formulated incorrectly in the book. It asks to prove that the system of two equations $x^2 + 7y^2 = z^2$ and $7x^2 + y^2 = t^2$ has no solutions in positive integers x, y, z, and t. However, it has solutions, for instance, x = 3, y = 1, z = 4, and t = 8. The example is provided in the book.

Proofs of other problems are straightforward formalizations of solutions given in the book.

1. Preliminaries

From now on a, b, c, k, m, n denote natural numbers, i, j denote integers, and p denotes a prime number.

Now we state the propositions:

(1) If n < 3, then n = 0 or n = 1 or n = 2.

(2) If n < 4, then n = 0 or n = 1 or n = 2 or n = 3.

(3) If n < 5, then n = 0 or n = 1 or n = 2 or n = 3 or n = 4.

Let us note that $\frac{1}{2}$ is non integer and there exists a rational number which is non natural and there exists a rational number which is non integer.

Now we state the proposition:

(4) If $j \neq 0$ and $\frac{i}{i}$ is integer, then $j \mid i$.

Let q be a non integer rational number. One can verify that q^2 is non integer. Now we state the proposition:

(5) If $\frac{a}{b} \cdot c$ is natural and $b \neq 0$ and a and b are relatively prime, then there exists a natural number d such that $c = b \cdot d$.

2. Problem 41

Let us consider an integer k. Now we state the propositions:

- (6) $2 \cdot k + 1$ and $9 \cdot k + 4$ are relatively prime.
- (7) $gcd(2 \cdot k 1, 9 \cdot k + 4) = gcd(k + 8, 17).$

3. Problem 92

Now we state the proposition:

(8) If m > 1 and n > 1 and m and n are relatively prime, then there exist prime numbers p, q such that $p \mid m$ and $p \nmid n$ and $q \mid n$ and $q \nmid m$ and $p \neq q$.

Let us consider k. The functor primesFinS(k) yielding a finite sequence of elements of \mathbb{N} is defined by

(Def. 1) len it = k and for every natural number i such that i < k holds it(i+1) = pr(i).

Let us observe that primesFinS(0) is empty. Now we state the propositions:

- (9) primesFinS(1) = $\langle 2 \rangle$.
- (10) primesFinS(2) = $\langle 2, 3 \rangle$.
- (11) primesFinS(3) = $\langle 2, 3, 5 \rangle$.
- (12) p < pr(k) if and only if primeindex(p) < k.
- (13) If $\operatorname{primeindex}(p) < k$, then $1 + \operatorname{primeindex}(p) \in \operatorname{dom}(\operatorname{primesFinS}(k))$.
- (14) If $\operatorname{primeindex}(p) < k$, then $(\operatorname{primesFinS}(k))(1 + \operatorname{primeindex}(p)) = p$.
- (15) If p < pr(k), then $p \in rng primesFinS(k)$. The theorem is a consequence of (13), (12), and (14).
- (16) If p and $\prod \text{ primesFinS}(k)$ are relatively prime, then $\text{pr}(k) \leq p$. The theorem is a consequence of (15).

Let us consider k. Let us note that primesFinS(k) is positive yielding and primesFinS(k) is increasing.

Let R be an extended real-valued binary relation. We say that R has values greater or equal one if and only if

(Def. 2) for every extended real r such that $r \in \operatorname{rng} R$ holds $r \ge 1$.

Observe that $\langle 1 \rangle$ has values greater or equal one and there exists a naturalvalued finite sequence which has values greater or equal one.

Let f be an extended real-valued function. Let us observe that f has values greater or equal one if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every object x such that $x \in \text{dom } f$ holds $f(x) \ge 1$.

Let f be an extended real-valued finite sequence. One can verify that f has values greater or equal one if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n such that $1 \leq n \leq \text{len } f$ holds $f(n) \geq 1$.

One can verify that every extended real-valued binary relation which is empty has also values greater or equal one and every extended real-valued binary relation which has values greater or equal one is also positive yielding.

Now we state the propositions:

- (17) If $m \leq n$, then primesFinS(n) $\upharpoonright m = \text{primesFinS}(m)$.
- (18) Let us consider extended real-valued binary relations P, R. Suppose rng $P \subseteq$ rng R and R has values greater or equal one. Then P has values greater or equal one.
- (19) Let us consider extended real-valued finite sequences f, g. Suppose $f \cap g$ has values greater or equal one. Then
 - (i) f has values greater or equal one, and
 - (ii) g has values greater or equal one.
- (20) Let us consider an extended real r. If $\langle r \rangle$ has values greater or equal one, then $r \ge 1$.

Let us consider a real-valued finite sequence f with values greater or equal one. Now we state the propositions:

(21) $\prod f \ge 1.$

PROOF: Define $\mathcal{P}[\text{finite sequence of elements of } \mathbb{R}] \equiv \text{for every real-valued}$ finite sequence g with values greater or equal one such that $g = \$_1$ holds $\prod \$_1 \ge 1$. For every finite sequence p of elements of \mathbb{R} and for every element x of \mathbb{R} such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \cap \langle x \rangle]$. For every finite sequence p of elements of \mathbb{R} , $\mathcal{P}[p]$. \Box

(22) $\prod (f \upharpoonright n) \leq \prod f$. The theorem is a consequence of (19) and (20).

Let us consider k. One can verify that $\operatorname{primesFinS}(k)$ has values greater or equal one.

Now we state the proposition:

(23) If $3 \leq k$, then $\operatorname{pr}(k) + \operatorname{pr}(k+1) \leq \prod \operatorname{primesFinS}(k)$. The theorem is a consequence of (8) and (16).

4. Problem 121

Let k, n be natural numbers. We say that n satisfies Sierpiński Problem 121 for k if and only if

(Def. 5) $k \cdot 2^{2^n} + 1$ is composite and for every positive natural number m such that m < n holds $k \cdot 2^{2^m} + 1$ is not composite.

Now we state the propositions:

- (24) 5 satisfies Sierpiński Problem 121 for 1. The theorem is a consequence of (3).
- (25) 1 satisfies Sierpiński Problem 121 for 2.
- (26) 2 satisfies Sierpiński Problem 121 for 3.
- (27) 2 satisfies Sierpiński Problem 121 for 4.
- (28) 1 satisfies Sierpiński Problem 121 for 5.
- (29) 1 satisfies Sierpiński Problem 121 for 6.
- (30) 3 satisfies Sierpiński Problem 121 for 7. The theorem is a consequence of (1).
- (31) 1 satisfies Sierpiński Problem 121 for 8.
- (32) 2 satisfies Sierpiński Problem 121 for 9.
- (33) 2 satisfies Sierpiński Problem 121 for 10.

5. Problem 122

Let us consider a positive natural number n.

Now we state the propositions:

- $(34) \quad 3 \mid (3 \cdot a + 2) \cdot 2^{2^n} + 1.$
- (35) $2 \cdot 2^{2^n} + 1$ is composite.
- (36) $5 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).
- (37) $8 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).
- (38) Let us consider a positive natural number k. Then $k \leq 10$ and for every positive natural number $n, k \cdot 2^{2^n} + 1$ is composite if and only if $k \in \{2, 5, 8\}$. The theorem is a consequence of (24), (26), (27), (30), (32), (33), (35), (36), and (37).

6. Problem 123

Now we state the propositions:

- $(39) \quad 2^{2^{n+1}} + 2^{2^n} + 1 \ge 7.$
- (40) If n > 0, then $2^{2^{n+1}} + 2^{2^n} + 1 \ge 21$.
- (41) If n > 1, then $2^{2^{n+1}} + 2^{2^n} + 1 \ge 273$.
- (42) If m is even or $m = 2 \cdot n$, then $2^m \mod 3 = 1$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2^{2 \cdot \$_1} \mod 3 = 1$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. \Box
- (43) If m is odd or $m = 2 \cdot n + 1$, then $2^m \mod 3 = 2$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2^{2 \cdot \$_1 + 1} \mod 3 = 2$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. \Box
- (44) Let us consider a non zero natural number n. Then $3 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42).
- (45) $7 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42) and (43). Let *n* be a non zero natural number. Note that $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is natural. Now we state the proposition:
- (46) Let us consider a non zero natural number n. If n > 1, then $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is composite. The theorem is a consequence of (39), (45), (44), and (41).

7. Problem 172

Now we state the proposition:

(47) Let us consider positive natural numbers n, x, y, z. Then $n^x + n^y = n^z$ if and only if n = 2 and y = x and z = x + 1.

8. Problem 182

Now we state the proposition:

(48) Let us consider real numbers a, b, c. If c > 1 and $c^a = c^b$, then a = b.

Let us consider positive natural numbers n, x, y, z, t. Now we state the propositions:

(49) If $x \leq y \leq z$, then $n^x + n^y + n^z = n^t$ iff n = 2 and y = x and z = x + 1and t = x + 2 or n = 3 and y = x and z = x and t = x + 1. (50) $n^x + n^y + n^z = n^t$ if and only if n = 2 and y = x and z = x + 1 and t = x + 2 or n = 2 and y = x + 1 and z = x and t = x + 2 or n = 2 and z = y and x = y + 1 and t = y + 2 or n = 3 and y = x and z = x and t = x + 1. The theorem is a consequence of (49).

9. Problem 183

Now we state the proposition:

(51) Let us consider positive natural numbers x, y, z, t. Then $4^x + 4^y + 4^z \neq 4^t$.

10. Problem 191

Now we state the proposition:

(52) Let us consider non zero integers
$$x, y, z, t$$
. Then

- (i) $x^2 + 5 \cdot y^2 \neq z^2$, or
- (ii) $5 \cdot x^2 + y^2 \neq t^2$.

11. Problem 192

Now we state the propositions:

(53) Let us consider non zero integers x, y, z, t. Then

(i)
$$x^2 + 6 \cdot y^2 \neq z^2$$
, or

(ii)
$$6 \cdot x^2 + y^2 \neq t^2$$
.

(54) (i) $3^2 + 7 \cdot 1^2 = 4^2$, and (ii) $7 \cdot 3^2 + 1^2 = 8^2$.

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