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Elementary Number Theory Problems. Part XVI

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Summary. In this paper, we continue the work on formalizing problems from Sierpiński's book. We present a detailed formalization of selected problems verified using the Mizar system, focusing on properties of arithmetic progressions and specific characteristics related to the occurrence of prime numbers, particularly in the context of Chebyshev's theorem.

The formalization encompasses problems 63, 65, 66, 67, 68, 93, 95, 96, 102, and 103.

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1. Problem 63

From now on a, b, d, n, k, i, j, x, s denote natural numbers.

Now we state the propositions:

(1) Let us consider finite 0-sequences f, g of $\mathbb{N}$. Then $\mathrm{value}(f \frown g, b) = \mathrm{value}(f, b) + (\mathrm{value}(g, b)) \cdot b^{\mathrm{len}\, f}$.

PROOF: Consider f_2 being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, f_2 = \mathrm{dom}\, f$ and for every natural number i such that $i \in \mathrm{dom}\, f_2$ holds $f_2(i) = f(i) \cdot b^i$ and $\mathrm{value}(f, b) = \sum f_2$. Consider g_1 being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, g_1 = \mathrm{dom}\, g$ and for every natural number i such that $i \in \mathrm{dom}\, g_1$ holds $g_1(i) = g(i) \cdot b^i$ and $\mathrm{value}(g, b) = \sum g_1$. Consider f_1 being a finite 0-sequence of $\mathbb{N}$ such that $\mathrm{dom}\, f_1 = \mathrm{dom}(f \frown g)$ and for every natural number i such that $i \in \operatorname{dom} f_1$ holds $f_1(i) = (f \frown g)(i) \cdot b^i$ and $\text{value}(f \frown g, b) = \sum f_1$. Consider F_1, G_1 being finite 0-sequences such that $\operatorname{len} F_1 = \operatorname{len} f$ and $\operatorname{len} G_1 = \operatorname{len} g$ and $f_1 = F_1 \frown G_1$. For every natural number k such that $k \in \operatorname{dom} f_2$ holds $f_2(k) = F_1(k)$ by [26, (21)]. Set $B = b^{\operatorname{len} f}$. For every natural number k such that $k \in \operatorname{dom} g_1$ holds $(B \cdot g_1)(k) = G_1(k)$ by [26, (23)], [19, (8)]. □

(2) If $b > 1$ and $n > 0$ and $n \cdot b^k \leqslant x < (n+1) \cdot b^k$, then $\text{digits}(n, b) = (\text{digits}(x, b))_{\lfloor k}$. The theorem is a consequence of (1).

(3) If $b > 0$ and $d > 1$ and $s > 0$, then there exist natural numbers m, i such that $(\text{digits}((\text{ArProg}(a, b))(m), d))_{\lfloor i} = \text{digits}(s, d)$. The theorem is a consequence of (2).

Now we state the proposition:

(4) Problem 63:
If $b > 0$ and $s > 0$, then there exist natural numbers m, i such that $(\text{digits}((\text{ArProg}(a, b))(m), 10))_{\lfloor i} = \text{digits}(s, 10)$.

2. Problem 67

Now we state the proposition:

(5) Problem 67:
Let us consider natural numbers a, b. Suppose $a > 0$ and a and b are relatively prime. Then there exists an infinite subset N of $\mathbb{N}$ such that for every natural numbers n, m such that $n, m \in N$ and $n \neq m$ holds $(\text{ArProg}(b, a))(n)$ and $(\text{ArProg}(b, a))(m)$ are relatively prime.
Proof: Define $\mathcal{X}[\text{set}] \equiv \$_1$ is finite and $0 \notin \$_1$ and for every natural numbers n, m such that $n, m \in \$_1$ and $n \neq m$ holds $(\text{ArProg}(b, a))(n)$ and $(\text{ArProg}(b, a))(m)$ are relatively prime. Define $\mathcal{G}[\text{object}, \text{object}] \equiv$ for every set Y such that $Y = \$_1$ and $\mathcal{X}[Y]$ there exists a natural number k such that $k \notin Y$ and $\$_2 = Y \cup \{k\}$ and $\mathcal{X}[Y \cup \{k\}]$. For every object x such that $x \in 2^{\mathbb{N}}$ there exists an object y such that $y \in 2^{\mathbb{N}}$ and $\mathcal{G}[x, y]$ by [7, (103)], [12, (7)], [14, (17)], [23, (4)]. Consider g being a function such that $\operatorname{dom} g = 2^{\mathbb{N}}$ and $\operatorname{rng} g \subseteq 2^{\mathbb{N}}$ and for every object x such that $x \in 2^{\mathbb{N}}$ holds $\mathcal{G}[x, g(x)]$ from [6, Sch. 6]. Define $\mathcal{G}(\text{object}, \text{object}) = g(\$_2)$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \emptyset$ and for every natural number n, $f(n+1) = \mathcal{G}(n, f(n))$ from [2, Sch. 11]. Define $\mathcal{F}[\text{natural number}] \equiv f(\$_1)$ is finite and $f(\$_1) \in 2^{\mathbb{N}}$ and $\mathcal{X}[f(\$_1)]$ and for every finite set X such that $X = f(\$_1)$ holds $\overline{\overline{X}} = \$_1$. If $\mathcal{F}[n]$, then $\mathcal{F}[n+1]$ by [8, (137)], [1, (41)]. $\mathcal{F}[n]$ from [2, Sch. 2]. $\bigcup \operatorname{rng} f$ is infinite by [2, (43)], [8, (74)], [2, (13)]. $\bigcup \operatorname{rng} f \subseteq \mathbb{N}$. Reconsider $N = \bigcup \operatorname{rng} f$ as an infinite

subset of $\mathbb{N}$. Define $\mathcal{G}$[natural number] $\equiv$ for every natural number n, $f(n) \subseteq f(n + \$_1)$. For every k such that $\mathcal{G}[k]$ holds $\mathcal{G}[k+1]$. $\mathcal{G}[n]$ from [2, Sch. 2]. Consider N being a set such that $n \in N$ and $N \in \operatorname{rng} f$. Consider x_4 being an object such that $x_4 \in \operatorname{dom} f$ and $f(x_4) = N$. Consider M being a set such that $m \in M$ and $M \in \operatorname{rng} f$. Consider x_3 being an object such that $x_3 \in \operatorname{dom} f$ and $f(x_3) = M$. □

3. PROBLEM 68

Now we state the proposition:

(6) PROBLEM 68:
Suppose $a > 0$ and $b > 0$. Then there exists an infinite subset N of $\mathbb{N}$ such that for every natural numbers n, m for every prime number p such that $n, m \in N$ holds $p \mid (\text{ArProg}(b, a))(n)$ iff $p \mid (\text{ArProg}(b, a))(m)$.
PROOF: Set $d = \gcd(a, a + b)$. Consider a_1, c being natural numbers such that $a = d \cdot a_1$ and $a + b = d \cdot c$ and a_1 and c are relatively prime. $c > 1$ by [2, (14)]. For every natural number n, $a_1 \mid (c^{\text{Euler}\, a_1})^{n+1} - 1$ by [2, (14)], [20, (12)], [9, (18)], [28, (15)]. Define $\mathcal{F}$(natural number) $= c \cdot \frac{(c^{\text{Euler}\, a_1})^{\$_1+1} - 1}{a_1} + 1$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and for every element x of $\mathbb{N}$, $f(x) = \mathcal{F}(x)$ from [6, Sch. 4]. $\operatorname{rng} f \subseteq \mathbb{N}$. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$ by [10, (30)]. Reconsider $N = \operatorname{rng} f$ as an infinite subset of $\mathbb{N}$. For every natural number n and for every prime number p such that $n \in N$ holds $p \mid (\text{ArProg}(b, a))(n)$ iff $p \mid d$ or $p \mid c$ by [12, (7)], [19, (9), (6)], [27, (7)]. □

4. PROBLEM 65

Now we state the propositions:

(7) (i) $\text{Fib}(6) = 8$, and
(ii) $\text{Fib}(7) = 13$, and
(iii) $\text{Fib}(8) = 21$, and
(iv) $\text{Fib}(9) = 34$, and
(v) $\text{Fib}(10) = 55$, and
(vi) $\text{Fib}(11) = 89$, and
(vii) $\text{Fib}(12) = 144$, and
(viii) $\text{Fib}(13) = 233$, and

(ix) Fib(14) = 377, and

(x) Fib(15) = 610, and

(xi) Fib(16) = 987, and

(xii) Fib(17) = 1597, and

(xiii) Fib(18) = 2584, and

(xiv) Fib(19) = 4181, and

(xv) Fib(20) = 6765, and

(xvi) Fib(21) = 10946, and

(xvii) Fib(22) = 17711, and

(xviii) Fib(23) = 28657, and

(xix) Fib(24) = 46368, and

(xx) Fib(25) = 75025.

(8) $\mathrm{Fib}(n+2) \geqslant n$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{Fib}(\$_1 + 2) \geqslant \$_1$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [13, (44)], [2, (13)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(9) If $k < n \leqslant 7$, then there exists i such that $\mathrm{Fib}(i) \bmod n = k$. The theorem is a consequence of (7).

(10) Let us consider a natural number j. Suppose $0 < j \leqslant 7$. Then there exists a natural number i such that

(i) $i > 0$, and

(ii) $\mathrm{Fib}(0) \equiv \mathrm{Fib}(i) \pmod{j}$, and

(iii) $\mathrm{Fib}(1) \equiv \mathrm{Fib}(i+1) \pmod{j}$.

The theorem is a consequence of (7).

(11) Suppose $\mathrm{Fib}(n) \equiv \mathrm{Fib}(n+i) \pmod{j}$ and $\mathrm{Fib}(n+1) \equiv \mathrm{Fib}(n+i+1) \pmod{j}$. Let us consider natural numbers x, y. Suppose $x \equiv y \pmod{i}$. Then $\mathrm{Fib}(x) \equiv \mathrm{Fib}(y) \pmod{j}$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{Fib}(\$_1) \equiv \mathrm{Fib}(\$_1 + i) \pmod{j}$ and $\mathrm{Fib}(\$_1+1) \equiv \mathrm{Fib}(\$_1+i+1) \pmod{j}$. Define $\mathcal{Q}$[natural number] $\equiv \mathcal{P}[n+\$_1]$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [24, (16)], [4, (1)]. For every natural number k, $\mathcal{Q}[k]$ from [2, Sch. 2]. Define $\mathcal{R}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then for every natural number i such that $i = n - \$_1$ holds $\mathcal{P}[i]$. For every natural number k such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$ by [4, (1)], [2, (13)], [24, (17)]. For every natural number k, $\mathcal{R}[k]$ from [2, Sch. 2]. For every natural number k, $\mathrm{Fib}(k) \equiv \mathrm{Fib}(k+i) \pmod{j}$ by [2, (21)]. □

(12) Let us consider natural numbers i, j, k. Suppose $0 < j$ and $k < i$ and for every natural numbers x, y such that $x \equiv y \pmod{j}$ holds $\mathrm{Fib}(x) \equiv \mathrm{Fib}(y) \pmod{i}$ and for every natural number x such that $x < j$ holds $\mathrm{Fib}(x) \bmod i \neq k$. Let us consider a natural number m. Then $(\mathrm{ArProg}(k, i))(m)$ is not Fibonacci.

(13) (i) $\mathrm{Fib}(0) \equiv \mathrm{Fib}(12) \pmod{8}$, and

(ii) $\mathrm{Fib}(1) \equiv \mathrm{Fib}(12 + 1) \pmod{8}$, and

(iii) for every natural number x such that $x < 12$ holds $\mathrm{Fib}(x)$ mod $8 \neq 4$ and $\mathrm{Fib}(x)$ mod $8 \neq 6$.

The theorem is a consequence of (7).

Now we state the proposition:

(14) PROBLEM 65:

(i) for every i and j such that $0 < i \leqslant 7$ there exists k such that $(\mathrm{ArProg}(j, i))(k)$ is Fibonacci, and

(ii) for every k, $(\mathrm{ArProg}(4, 8))(k)$ is not Fibonacci.

PROOF: For every i and j such that $0 < i \leqslant 7$ there exists k such that $(\mathrm{ArProg}(j, i))(k)$ is Fibonacci by (10), [24, (58)], (9), [15, (5)]. $\mathrm{Fib}(0) \equiv \mathrm{Fib}(0+12) \pmod{8}$ and $\mathrm{Fib}(0+1) \equiv \mathrm{Fib}(0+12+1) \pmod{8}$. For every natural numbers x, y such that $x \equiv y \pmod{12}$ holds $\mathrm{Fib}(x) \equiv \mathrm{Fib}(y) \pmod{8}$. For every natural number x such that $x < 12$ holds $\mathrm{Fib}(x)$ mod $8 \neq 4$. □

5. PROBLEM 66

Now we state the proposition:

(15) PROBLEM 66:

(i) 4 and 11 are relatively prime, and

(ii) for every natural number m, $(\mathrm{ArProg}(4, 11))(m)$ is not Fibonacci.

PROOF: $\mathrm{Fib}(0) \equiv \mathrm{Fib}(0 + 10) \pmod{11}$ and $\mathrm{Fib}(0 + 1) \equiv \mathrm{Fib}(0 + 10 + 1) \pmod{11}$. For every natural numbers x, y such that $x \equiv y \pmod{10}$ holds $\mathrm{Fib}(x) \equiv \mathrm{Fib}(y) \pmod{11}$. For every natural number x such that $x < 10$ holds $\mathrm{Fib}(x)$ mod $11 \neq 4$ by [17, (16)], (7), [13, (21), (22), (23)]. □

6. Problem 96

Now we state the propositions:

(16) $\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10) = \frac{10^{n+1}-7}{3}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv 3 \cdot (\text{value}(\langle 1\rangle \frown (\$_1 \longmapsto 3), 10)) = 10^{\$_1+1} - 7$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [26, (87), (27)], (1), [26, (34), (17)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(17) There exists a natural number k such that $17 \mid k = \frac{10^{16\cdot n+9}-7}{3}$. The theorem is a consequence of (16).

(18) 33331 is prime.

(19) 333331 is prime.

Now we state the proposition:

(20) Problem 96:

(i) for every non zero natural number n such that $n < 6$ holds $\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10)$ is prime, and

(ii) $\text{value}(\langle 1\rangle \frown (8 \longmapsto 3), 10)$ is not prime, and

(iii) $\{\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10)$, where n is a natural number : $\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10)$ is non prime$\}$ is infinite.

Proof: Consider v being a natural number such that $17 \mid v = \frac{10^{16\cdot 0+9}-7}{3}$. $\text{value}(\langle 1\rangle \frown (8 \longmapsto 3), 10) = \frac{10^{8+1}-7}{3}$. Set $V = \{\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10)$, where n is a natural number : $\text{value}(\langle 1\rangle \frown (n \longmapsto 3), 10)$ is not prime $\}$. Define $\mathcal{F}(\text{natural number}) = \frac{10^{16\cdot\$_1+9}-7}{3}$. Consider f being a function such that $\text{dom} f = \mathbb{N}$ and for every element d of $\mathbb{N}$, $f(d) = \mathcal{F}(d)$ from [6, Sch. 4]. For every objects x_1, x_2 such that $x_1, x_2 \in \text{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$ by [10, (30)]. $\text{rng} f \subseteq V$. □

7. Product of Different Primes Selected Properties

Now we state the proposition:

(21) Let us consider a non zero natural number n, and a prime number p. Suppose $\text{support}\,\text{PFExp}(n) = \{p\}$. Then $n = p^{(\text{PFExp}(n))(p)}$.

Let us consider a non zero natural number n. Now we state the propositions:

(22) $\text{rng}\,\text{PFExp}(n) \subseteq \{0, 1\}$ and $\overline{\overline{\text{support}\,\text{PFExp}(n)}} = 1$ if and only if n is prime.

Proof: $\text{rng}\,\text{PFExp}(n) \subseteq \{0, 1\}$ by [16, (41)]. □

(23) $0 \in \text{rng}\,\text{PFExp}(n)$.

Now we state the propositions:

(24) Let us consider non zero natural numbers n, m. Suppose n and m are relatively prime. Then $\mathrm{rng}\,\mathrm{PFExp}(n \cdot m) = \mathrm{rng}\,\mathrm{PFExp}(n) \cup \mathrm{rng}\,\mathrm{PFExp}(m)$.
PROOF: $\mathrm{rng}\,\mathrm{PFExp}(n \cdot m) \subseteq \mathrm{rng}\,\mathrm{PFExp}(n) \cup \mathrm{rng}\,\mathrm{PFExp}(m)$ by (23), [16, (44)]. $\mathrm{rng}\,\mathrm{PFExp}(n) \subseteq \mathrm{rng}\,\mathrm{PFExp}(n \cdot m)$ by (23), [16, (44)]. $\mathrm{rng}\,\mathrm{PFExp}(m) \subseteq \mathrm{rng}\,\mathrm{PFExp}(n \cdot m)$ by (23), [16, (44)]. □

(25) $\prod \mathrm{primesFinS}((n+1)) = (\prod \mathrm{primesFinS}(n)) \cdot (\mathrm{pr}(n))$.

(26) Let us consider a natural number k. Then $2^k \leqslant \prod \mathrm{primesFinS}(k)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2^{\$_1} \leqslant \prod \mathrm{primesFinS}(\$_1)$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (25), [11, (8), (21)], [19, (6)]. $\mathcal{P}[n]$ from [2, Sch. 2]. □

(27) If $2 \leqslant n$, then there exists a non zero natural number k such that $\prod \mathrm{primesFinS}(k) \leqslant n < \prod \mathrm{primesFinS}((k+1))$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv n < \prod \mathrm{primesFinS}((\$_1+1))$. Consider k being a natural number such that $2^k \leqslant n < 2^{k+1}$. $2^{k+1} \leqslant \prod \mathrm{primesFinS}((k+1))$. Consider m being a natural number such that $\mathcal{P}[m]$ and for every natural number w such that $\mathcal{P}[w]$ holds $m \leqslant w$ from [2, Sch. 5]. $\prod \mathrm{primesFinS}(m) \leqslant n$ by [2, (13)]. □

Let us consider a prime number p and a natural number k. Now we state the propositions:

(28) (i) p-count$(\prod \mathrm{primesFinS}(k)) = 1$ iff $\mathrm{primeindex}(p) < k$, and

(ii) p-count$(\prod \mathrm{primesFinS}(k)) = 0$ iff $\mathrm{primeindex}(p) \geqslant k$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every prime number p, (p-count$(\prod \mathrm{primesFin}$ 1 iff $\mathrm{primeindex}(p) < \$_1$) and ($p$-count$(\prod \mathrm{primesFinS}(\$_1)) = 0$ iff $\mathrm{primeindex}(p) \geqslant$ $\$_1$). $\mathcal{P}[0]$ by [7, (94)], [16, (21)]. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (25), [16, (28)], [2, (13)], [16, (24), (22)]. $\mathcal{P}[n]$ from [2, Sch. 2]. □

(29) $p \mid \prod \mathrm{primesFinS}(k)$ if and only if $\mathrm{primeindex}(p) < k$.
PROOF: If $p \mid \prod \mathrm{primesFinS}(k)$, then $\mathrm{primeindex}(p) < k$ by [16, (27)], (28). p-count$(\prod \mathrm{primesFinS}(k)) = 1$. □

(30) If $k \leqslant \mathrm{primeindex}(p)$, then p and $\prod \mathrm{primesFinS}(k)$ are relatively prime.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant k$, then p and $\prod \mathrm{primesFinS}(\$_1)$ are relatively prime. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [2, (13)], (25), [11, (21)], [12, (3)]. $\mathcal{P}[n]$ from [2, Sch. 2]. □

Now we state the proposition:

(31) (i) $\mathrm{rng}\,\mathrm{PrimeExponents}(\prod \mathrm{primesFinS}(n)) \subseteq \{0,1\}$, and

(ii) $\overline{\overline{\mathrm{support}\,\mathrm{PrimeExponents}(\prod \mathrm{primesFinS}(n))}} = n$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \overline{\overline{\mathrm{support}\,\mathrm{PrimeExponents}(\prod \mathrm{primesFinS}(\$_1))}} =$ $\$_1$ and $\mathrm{rng}\,\mathrm{PrimeExponents}(\prod \mathrm{primesFinS}(\$_1)) \subseteq \{0,1\}$. For every n such

that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by (25), [16, (46)], (30), [16, (44)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

Let us consider natural numbers n, m. Now we state the propositions:

(32) If for every natural number k such that $k < m$ holds $\mathrm{pr}(k) \mid n$, then $\prod \mathrm{primesFinS}(m) \mid n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant m$, then $\prod \mathrm{primesFinS}(\$_1) \mid n$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (13)], (30), [10, (4)], (25). For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(33) $n < m$ if and only if $\prod \mathrm{primesFinS}(n) < \prod \mathrm{primesFinS}(m)$.

PROOF: If $n < m$, then $\prod \mathrm{primesFinS}(n) < \prod \mathrm{primesFinS}(m)$ by [2, (13)], [11, (8), (21)], (25). □

8. Problem 93

Now we state the proposition:

(34) Problem 93:

Let us consider a sequence r of real numbers. Suppose for every non zero natural number n, there exists a prime number q such that $r(n) = \frac{q}{n}$ and $q \nmid n$ and for every prime number p such that $p \nmid n$ holds $q \leqslant p$. Then

(i) r is convergent, and

(ii) $\lim r = 0$.

PROOF: For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|r(m) - 0| < p$ by [21, (1)], (26), [2, (14)], [22, (12)]. □

9. Problem 95

Now we state the proposition:

(35) Problem 95:

Let us consider a non zero natural number s, and a natural number n. Suppose $n > \prod \mathrm{primesFinS}(s)$. Then there exists a natural number p such that

(i) $n < p < 2 \cdot n$, and

(ii) $\mathrm{rng}\,\mathrm{PrimeExponents}(p) \subseteq \{0, 1\}$, and

(iii) $\overline{\overline{\mathrm{support}\,\mathrm{PrimeExponents}(p)}} = s$.

PROOF: Reconsider $s_1 = s-1$ as a natural number. Set $P_1 = \prod \text{primesFinS}(s_1)$. Set $k = n \operatorname{div} P_1$. Set $r = n \bmod P_1$. $k \cdot P_1 + r > P_1 \cdot (\text{pr}(s_1))$. Consider p being a prime number such that $k < p \leqslant 2 \cdot k$. $p \neq 2 \cdot k$ by [12, (2)], [2, (13)]. $s_1 < \text{primeindex}(p)$ by [18, (12)]. support PFExp(p) misses support PFExp(P_1). rng PFExp($p \cdot P_1$) = rng PFExp(p) ∪ rng PFExp(P_1). rng PFExp(p) ⊆ $\{0,1\}$. rng PFExp(P_1) ⊆ $\{0,1\}$. $\overline{\overline{\text{support PFExp}(p)}} = 1$. □

10. PROBLEM 102

Now we state the propositions:

(36) Let us consider a natural number n, and a prime number p. If $p \leqslant n$ and $p^2 \mid n!$, then $2 \cdot p \leqslant n$.
PROOF: Consider o being a natural number such that $p \cdot p \cdot o = n!$. Set $I = \text{idseq}(n)$. For every real number r such that $r \in \operatorname{rng} I$ holds $0 < r$ by [5, (49)], [25, (25)]. Consider i being a natural number such that $i \in \operatorname{dom}((I\restriction(p -' 1)) ^\frown I_{\restriction p})$ and $p \mid ((I\restriction(p -' 1)) ^\frown I_{\restriction p})(i)$. □

(37) If $0 < a < b \leqslant n$, then $a \cdot b \mid n!$.
PROOF: For every object x such that $x \in \operatorname{dom}\langle a, b\rangle$ holds $\langle a, b\rangle(x) \leqslant n$ by [3, (44), (2)]. For every natural number i such that $i \in \operatorname{dom}\langle a\rangle$ holds $\langle a\rangle(i) < b$ by [3, (38), (2)]. □

(38) Let us consider a prime number p. Suppose $2 < n$ and $n \operatorname{div} 2 < p \leqslant 2 \cdot (n \operatorname{div} 2)$. Then p-count$(n!) = 1$. The theorem is a consequence of (36).

Now we state the proposition:

(39) PROBLEM 102:
for every natural number n such that $n > 1$ there exists a prime number p such that $n < p < 2 \cdot n$ if and only if for every natural number n such that $n > 1$ there exists a prime number p such that p-count$(n!) = 1$.
PROOF: Consider p being a prime number such that p-count$(2 \cdot n!) = 1$. $n < p$ by (37), [19, (6)], [20, (9)], [11, (40)]. □

11. PROBLEM 103

Now we state the proposition:

(40) Suppose for every natural number n such that $n > 5$ there exist prime numbers p, q such that $n < p < q < 2 \cdot n$. Let us consider a natural number n. Suppose $n > 10$. Then there exist prime numbers p, q such that

(i) $p < q$, and

(ii) p-count$(n!) = 1$, and

(iii) q-count$(n!) = 1$.

The theorem is a consequence of (36) and (38).

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Grzegorz Bancerek and Piotr Rudnicki. Two programs for **SCM**. Part I – preliminaries. *Formalized Mathematics*, 4(**1**):69–72, 1993.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[9] Yoshinori Fujisawa, Yasushi Fuwa, and Hidetaka Shimizu. Euler's Theorem and small Fermat's Theorem. *Formalized Mathematics*, 7(**1**):123–126, 1998.

[10] Yoshinori Fujisawa, Yasushi Fuwa, and Hidetaka Shimizu. Public-key cryptography and Pepin's test for the primality of Fermat numbers. *Formalized Mathematics*, 7(**2**):317–321, 1998.

[11] Adam Grabowski. On square-free numbers. *Formalized Mathematics*, 21(**2**):153–162, 2013. doi:10.2478/forma-2013-0017.

[12] Adam Grabowski. Elementary number theory problems. Part VI. *Formalized Mathematics*, 30(**3**):235–244, 2022. doi:10.2478/forma-2022-0019.

[13] Magdalena Jastrzębska and Adam Grabowski. Some properties of Fibonacci numbers. *Formalized Mathematics*, 12(**3**):307–313, 2004.

[14] Artur Korniłowicz. Elementary number theory problems. Part IX. *Formalized Mathematics*, 31(1):161–169, 2023. doi:10.2478/forma-2023-0015.

[15] Artur Korniłowicz and Adam Naumowicz. Niven's theorem. *Formalized Mathematics*, 24 (**4**):301–308, 2016. doi:10.1515/forma-2016-0026.

[16] Artur Korniłowicz and Piotr Rudnicki. Fundamental Theorem of Arithmetic. *Formalized Mathematics*, 12(**2**):179–186, 2004.

[17] Artur Korniłowicz and Dariusz Surowik. Elementary number theory problems. Part II. *Formalized Mathematics*, 29(**1**):63–68, 2021. doi:10.2478/forma-2021-0006.

[18] Artur Korniłowicz and Rafał Ziobro. Elementary number theory problems. Part XIII. *Formalized Mathematics*, 32(1):1–8, 2024. doi:10.2478/forma-2024-0001.

[19] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[20] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relatively primes. *Formalized Mathematics*, 1(**5**):829–832, 1990.

[21] Yatsuka Nakamura and Andrzej Trybulec. Lebesgue's covering lemma, uniform continuity and segmentation of arcs. *Formalized Mathematics*, 6(**4**):525–529, 1997.

[22] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[23] Marco Riccardi. Ramsey's theorem. *Formalized Mathematics*, 16(**2**):203–205, 2008. doi:10.2478/v10037-008-0026-y.

[24] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[25] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[26] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.

[27] Li Yan, Xiquan Liang, and Junjie Zhao. Gauss lemma and law of quadratic reciprocity. *Formalized Mathematics*, 16(**1**):23–28, 2008. doi:10.2478/v10037-008-0004-4.

[28] Rafał Ziobro. Parity as a property of integers. *Formalized Mathematics*, 26(**2**):91–100, 2018. doi:10.2478/forma-2018-0008.

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