FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 179–186, 2024
DOI: 10.2478/forma-2024-0015
e-ISSN: 1898–9934

sciendo.com/journal/forma

Classical Isoperimetric Theorem[1]

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Summary. In this article we formalize in Mizar [1], [2] the isoperimetric theorem, inspired by Peter D. Lax's "A Short Path to the Shortest Path" [12]. Notably, Lax's proof is remarkably concise, spanning just one page, demonstrating the elegance of his approach.

Our formalization begins by establishing fundamental properties of continuous and differentiable functions, including theorems on integrals and differentiation rules. Building upon these, it progresses to the proof of the isoperimetric theorem, addressing the following question: Among all curves of fixed length connecting two points on the x-axis, which curve maximizes the area between the curve and the x-axis? The formalization proves that for parametric curves $(x(t), y(t))$ with fixed length and endpoints on the x-axis, the integral $\int_0^\pi y(t)x'(t)\,dt$ is maximized when the curve is a semicircle.

This work represents the 99th problem solved in Freek Wiedijk's "Formalizing 100 Theorems" project, underscoring the significance of this effort in the context of formalization of mathematics. The historical background on the isoperimetric theorem is detailed in [9] and [3].

MSC: 26B15 49Q20 68V20

Keywords: isoperimetric theorem; calculus of variations; parametric curves

MML identifier: PDLAX, version: 8.1.14 5.86.1479

1. Foundations of Continuity and Integration

From now on a, b, r denote real numbers, A denotes a non empty set, X, x denote sets, f, g, F, G denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n denotes an element of $\mathbb{N}$.

Now we state the propositions:

[1]This work was supported by JSPS KAKENHI Grant Number 24K14897.

(1) Let us consider real numbers a, b, C, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} u$ and u is continuous and for every real number t such that $t \in]a, b[$ holds $u(t) = C$. Let us consider a real number t. If $t \in [a, b]$, then $u(t) = C$.

PROOF: Define $\mathcal{M}(\text{natural number}) = \frac{\frac{b-a}{2}}{\$_1+1}(\in \mathbb{R})$. Consider S_4 being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element x of $\mathbb{N}$, $S_4(x) = \mathcal{M}(x)$ from [6, Sch. 4]. For every natural number n, $S_4(n) = \frac{\frac{b-a}{2}}{n+1}$. Consider S_2 being a constant function from $\mathbb{N}$ into $\mathbb{R}$ such that for every natural number x, $S_2(x) = a$. Set $S_0 = S_2 + S_4$. $\operatorname{rng} S_0 \subseteq]a, b[$ by [10, (7)]. For every natural number n, $(u_* S_0)(n) = C$ by [6, (108), (112)]. For every objects x, y such that x, $y \in \operatorname{dom}(u_* S_0)$ holds $(u_* S_0)(x) = (u_* S_0)(y)$. Consider S_3 being a constant function from $\mathbb{N}$ into $\mathbb{R}$ such that for every natural number x, $S_3(x) = b$. Set $S_1 = S_3 - S_4$. $\operatorname{rng} S_1 \subseteq]a, b[$ by [10, (7), (10)]. For every natural number n, $(u_* S_1)(n) = C$ by [6, (108), (112)]. For every objects x, y such that x, $y \in \operatorname{dom}(u_* S_1)$ holds $(u_* S_1)(x) = (u_* S_1)(y)$. For every real number t such that $t \in [a, b]$ holds $u(t) = C$ by [11, (25)]. $\square$

(2) Let us consider real numbers a, b, c, d, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $c \leqslant d$ and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$ and $f{\restriction}[a, b]$ is continuous and for every real number t such that $t \in [c, d]$ holds $0 \leqslant f(t)$. Then $0 \leqslant \int\limits_{c}^{d} f(x)dx$.

PROOF: For every object x such that $x \in \operatorname{dom}(f \restriction [c, d])$ holds $(f \restriction [c, d])(x) = (|f| \restriction [c, d])(x)$ by [5, (47)], [15, (57)], [4, (43)]. $\square$

(3) Let us consider real numbers a, b, c, d, and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $c \leqslant d$ and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$ and c, $d \in [a, b]$ and $f{\restriction}[a, b]$ is continuous and $g{\restriction}[a, b]$ is continuous and for every real number t such that $t \in [c, d]$ holds $f(t) \leqslant g(t)$. Then $\int\limits_{c}^{d} f(x)dx \leqslant \int\limits_{c}^{d} g(x)dx$. The theorem is a consequence of (2).

(4) Let us consider real numbers a, b, c, d, e, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $c \leqslant d$ and c, $d \in [a, b]$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous and for every real number t such that $t \in [c, d]$ holds $e \leqslant f(t)$. Then $e \cdot (d - c) \leqslant \int\limits_{c}^{d} f(x)dx$.

PROOF: Set $g = \mathbb{R} \longmapsto e$. For every real number t such that $t \in [c, d]$ holds

$g(t) \leqslant f(t)$ by [14, (7)]. $\int\limits_{c}^{d} g(x)dx \leqslant \int\limits_{c}^{d} f(x)dx$. $\square$

(5) Let us consider real numbers a, b, c, d, e, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $0 < e$ and $a \leqslant b$ and $c < d$ and $c, d \in [a, b]$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous and for every real number t such that $t \in [a, b]$ holds $0 \leqslant f(t)$ and for every real number t such that $t \in [c, d]$ holds $e \leqslant f(t)$. Then $0 < e \cdot (d - c) \leqslant \int\limits_{a}^{b} f(x)dx$. The theorem is a consequence of (2) and (4).

(6) Let us consider real numbers a, b, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous and for every real number t such that $t \in [a, b]$ holds $0 \leqslant f(t)$ and there exists a real number t_0 such that $t_0 \in]a, b[$ and $0 < f(t_0)$. Then there exist real numbers d, c, e such that

(i) $0 < e$, and

(ii) $c < d$, and

(iii) c, $d \in [a, b]$, and

(iv) $0 < e \cdot (d - c) \leqslant \int\limits_{a}^{b} f(x)dx$.

Proof: Consider t_0 being a real number such that $t_0 \in]a, b[$ and $0 < f(t_0)$. Set $e = \frac{f(t_0)}{2}$. Consider s_0 being a real number such that $0 < s_0$ and for every real number t such that $t \in [a, b]$ and $|t - t_0| < s_0$ holds $|f(t) - f(t_0)| < e$. Set $s = \frac{s_0}{2}$. Reconsider $s_2 = \min(t_0 - a, b - t_0)$ as a real number. Reconsider $s_3 = \min(s, s_2)$ as a real number. Set $c = t_0 - s_3$. Set $d = t_0 + s_3$. Set $e_0 = \frac{f(t_0)}{2}$. For every real number t such that $t \in [c, d]$ holds $e_0 \leqslant f(t)$ by [13, (5)]. $\square$

(7) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous. Then there exists a sequence I of real numbers such that

(i) for every natural number n, $I(n) = \int\limits_{a+\frac{1}{n+1}}^{b-\frac{1}{n+1}} f(x)dx$, and

(ii) I is convergent, and

(iii) $\lim I = \int\limits_{a}^{b} f(x)dx$.

Proof: Define $\mathcal{M}(\text{natural number}) = (\int\limits_{a+\frac{1}{\$_1+1}}^{b-\frac{1}{\$_1+1}} f(x)dx)(\in \mathbb{R})$. Consider I being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element x of $\mathbb{N}$, $I(x) = \mathcal{M}(x)$ from [6, Sch. 4]. For every natural number n, $I(n) = \int\limits_{a+\frac{1}{n+1}}^{b-\frac{1}{n+1}} f(x)dx$. Set $X = [a, b]$. Consider t_1, t_2 being real numbers such that $t_1, t_2 \in \operatorname{dom}(|f|\restriction X)$ and $(|f|\restriction X)(t_1) = \sup \operatorname{rng}(|f|\restriction X)$ and $(|f|\restriction X)(t_2) = \inf \operatorname{rng}(|f|\restriction X)$. Set $K = (|f|\restriction X)(t_1)$. For every real number t such that $t \in X$ holds $|f(t)| \leqslant K$ by [5, (3), (49)]. Set $L = \int\limits_{a}^{b} f(x)dx$. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|I(m) - L| < p$ by [11, (3)], [8, (17)], [15, (74)], [7, (10), (11)]. □

2. Differentiation Rules and Properties

Now we state the propositions:

(8) Let us consider an open subset Z of $\mathbb{R}$. Then

  (i) the function sin is differentiable on Z, and

  (ii) (the function sin)$'_{\restriction Z}$ = (the function cos)$\restriction Z$, and

  (iii) the function cos is differentiable on Z, and

  (iv) (the function cos)$'_{\restriction Z}$ = $-$(the function sin)$\restriction Z$.

(9) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Then $f + f = 2 \cdot f$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a subset Z of $\mathbb{R}$, and a real number x. Now we state the propositions:

(10) If Z is open and $x \in Z$ and $Z \subseteq \operatorname{dom} f$, then $f\restriction Z$ is differentiable in x iff f is differentiable in x.

(11) If Z is open and $x \in Z$ and $Z \subseteq \operatorname{dom} f$ and f is differentiable in x, then $f'(x) = (f\restriction Z)'(x)$. The theorem is a consequence of (10).

Now we state the propositions:

(12) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and subsets X, Z of $\mathbb{R}$. Suppose Z is open and $Z \subseteq X$ and f is differentiable on X. Then $f'_{\restriction Z} = f'_{\restriction X}\restriction Z$.

Proof: For every object x such that $x \in \operatorname{dom}(f'_{\restriction X}\restriction Z)$ holds $(f'_{\restriction X}\restriction Z)(x) = f'_{\restriction Z}(x)$ by [5, (49)]. □

(13) Let us consider real numbers a, b, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and u is differentiable on $]a, b[$ and $\mathrm{dom}\, u = [a, b]$ and u is continuous and for every real number t such that $t \in\ ]a, b[$ holds $u'_{\restriction]a,b[}(t) = 0$. Then there exists a real number C such that for every real number t such that $t \in [a, b]$ holds $u(t) = C$. The theorem is a consequence of (1).

3. Properties of Parametric Curves and Area Calculations

Now we state the proposition:

(14) Let us consider partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$, and an open subset Z of $\mathbb{R}$. Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq Z \subseteq \mathrm{dom}\, x$ and $Z \subseteq \mathrm{dom}\, y$ and $y'_{\restriction Z}$ is continuous and $x'_{\restriction Z}$ is continuous and for every real number t such that $t \in Z$ holds $x'_{\restriction Z}(t)^{\mathbf{2}} + y'_{\restriction Z}(t)^{\mathbf{2}} = 1$ and $y(0) = 0$ and $y(\pi) = 0$. Then there exists a partial function u from $\mathbb{R}$ to $\mathbb{R}$ and there exists a sequence F of real numbers such that u is differentiable on $]0, \pi[$ and $u'_{\restriction]0,\pi[}$ is continuous and $\mathrm{dom}\, u = [0, \pi]$ and u is continuous and $y \restriction [0, \pi] = (u \cdot (\text{the function sin})) \restriction [0, \pi]$ and for every real number t such that $t \in\ ]0, \pi[$ holds $y'(t) = u'(t) \cdot (\text{the function sin})(t) + u(t) \cdot (\text{the function cos})(t)$ and for every natural number n, $F(n) =$

$$\int\limits_{\frac{1}{n+1}}^{\pi-\frac{1}{n+1}} ((\mathrm{AffineMap}(0,1)) - ((u'_{\restriction]0,\pi[} \cdot u'_{\restriction]0,\pi[}) \cdot (\text{the function sin})) \cdot (\text{the function sin}))($$

and F is convergent and $\int\limits_0^\pi (y \cdot x'_{\restriction Z})(x)dx \leqslant \frac{1}{2} \cdot (\int\limits_0^\pi (y \cdot y + x'_{\restriction Z} \cdot x'_{\restriction Z})(x)dx)$ and $y \cdot y + x'_{\restriction Z} \cdot x'_{\restriction Z} = y \cdot y + (\mathrm{AffineMap}(0,1)) - y'_{\restriction Z} \cdot y'_{\restriction Z}$ and $\int\limits_0^\pi (y \cdot y + x'_{\restriction Z} \cdot x'_{\restriction Z})(x)dx =$ $\int\limits_0^\pi (y \cdot y + (\mathrm{AffineMap}(0,1)) - y'_{\restriction Z} \cdot y'_{\restriction Z})(x)dx$ and $\int\limits_0^\pi (y \cdot y + (\mathrm{AffineMap}(0,1)) - y'_{\restriction Z}$ $\lim F$.

4. Formalization of the Isoperimetric Theorem

Now we state the propositions:

(15) Let us consider partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$, and an open subset Z of $\mathbb{R}$. Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq Z \subseteq$

$\mathrm{dom}\, x$ and $Z \subseteq \mathrm{dom}\, y$ and $x'_{\restriction Z}$ is continuous and $y'_{\restriction Z}$ is continuous and for every real number t such that $t \in Z$ holds $x'_{\restriction Z}(t)^{\mathbf{2}} + y'_{\restriction Z}(t)^{\mathbf{2}} = 1$ and $y(0) = 0$ and $y(\pi) = 0$. Then

(i) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction Z})(x)dx \leqslant \frac{1}{2} \cdot \pi$, and

(ii) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction Z})(x)dx = \frac{1}{2} \cdot \pi$ iff for every real number t such that $t \in [0, \pi]$ holds $y(t) =$ (the function sin)(t) and $x(t) = -$(the function cos)$(t)+$ (the function cos)$(0) + x(0)$ or for every real number t such that $t \in [0, \pi]$ holds $y(t) = -$(the function sin)(t) and $x(t) =$ (the function cos)$(t) -$ (the function cos)$(0) + x(0)$.

(16) Let us consider partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$. Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq \mathrm{dom}\, x$ and $[0, \pi] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}} = 1$ and $y(0) = 0$ and $y(\pi) = 0$. Then

(i) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx \leqslant \frac{1}{2} \cdot \pi$, and

(ii) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx = \frac{1}{2} \cdot \pi$ iff for every real number t such that $t \in [0, \pi]$ holds $y(t) =$ (the function sin)(t) and $x(t) = -$(the function cos)$(t)+$ (the function cos)$(0) + x(0)$ or for every real number t such that $t \in [0, \pi]$ holds $y(t) = -$(the function sin)(t) and $x(t) =$ (the function cos)$(t) -$ (the function cos)$(0) + x(0)$.

The theorem is a consequence of (12).

(17) Let us consider partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$, and a real number L. Suppose $0 < L$ and x is differentiable and y is differentiable and $[0, \pi] \subseteq \mathrm{dom}\, x$ and $[0, \pi] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}} = \frac{L}{\pi}$ and $y(0) = 0$ and $y(\pi) = 0$. Then

(i) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx \leqslant \frac{1}{2} \cdot L$, and

(ii) $\int\limits_{0}^{\pi} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x)dx = \frac{1}{2} \cdot L$ iff for every real number t such that $t \in$

$[0,\pi]$ holds $y(t) = \frac{(\text{the function }\sin)(t)}{\sqrt{\frac{\pi}{L}}}$ and $x(t) = -\frac{(\text{the function }\cos)(t)}{\sqrt{\frac{\pi}{L}}} + \frac{(\text{the function }\cos)(0)}{\sqrt{\frac{\pi}{L}}} + x(0)$ or for every real number t such that $t \in [0,\pi]$ holds $y(t) = -\frac{(\text{the function }\sin)(t)}{\sqrt{\frac{\pi}{L}}}$ and $x(t) = \frac{(\text{the function }\cos)(t)}{\sqrt{\frac{\pi}{L}}} - \frac{(\text{the function }\cos)(0)}{\sqrt{\frac{\pi}{L}}} + x(0)$.

Proof: Set $k = \sqrt{\frac{\pi}{L}}$. Set $x_1 = k \cdot x$. Set $y_1 = k \cdot y$. For every real number t such that $t \in \operatorname{dom} x_1 \cap \operatorname{dom} y_1$ holds ${x_1}'(t)^{\mathbf{2}} + {y_1}'(t)^{\mathbf{2}} = 1$. $\int\limits_0^\pi (y_1 \cdot {x_1}'_{\restriction \operatorname{dom} x_1})(x)dx \leqslant \frac{1}{2} \cdot \pi$ and $(\int\limits_0^\pi (y_1 \cdot {x_1}'_{\restriction \operatorname{dom} x_1})(x)dx = \frac{1}{2} \cdot \pi$ iff for every real number t such that $t \in [0,\pi]$ holds $y_1(t) = (\text{the function }\sin)(t)$ and $x_1(t) = -(\text{the function }\cos)(t) + (\text{the function }\cos)(0) + x_1(0)$ or for every real number t such that $t \in [0,\pi]$ holds $y_1(t) = -(\text{the function }\sin)(t)$ and $x_1(t) = (\text{the function }\cos)(t) - (\text{the function }\cos)(0) + x_1(0))$. $\int\limits_0^\pi (y_1 \cdot {x_1}'_{\restriction \operatorname{dom} x_1})(x)dx = \frac{1}{2} \cdot \pi$ iff $\int\limits_0^\pi (y \cdot x'_{\restriction \operatorname{dom} x})(x)dx = \frac{1}{2} \cdot L$ by [8, (10)]. for every real number t such that $t \in [0,\pi]$ holds $y_1(t) = (\text{the function }\sin)(t)$ and $x_1(t) = -(\text{the function }\cos)(t) + (\text{the function }\cos)(0) + x_1(0)$ iff for every real number t such that $t \in [0,\pi]$ holds $y(t) = \frac{(\text{the function }\sin)(t)}{k}$ and $x(t) = -\frac{(\text{the function }\cos)(t)}{k} + \frac{(\text{the function }\cos)(0)}{k} + x(0)$. for every real number t such that $t \in [0,\pi]$ holds $y_1(t) = -(\text{the function }\sin)(t)$ and $x_1(t) = (\text{the function }\cos)(t) - (\text{the function }\cos)(0) + x_1(0)$ iff for every real number t such that $t \in [0,\pi]$ holds $y(t) = -\frac{(\text{the function }\sin)(t)}{k}$ and $x(t) = \frac{(\text{the function }\cos)(t)}{k} - \frac{(\text{the function }\cos)(0)}{k} + x(0)$. $\square$

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Accepted December 14, 2024