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Inverse Element for Surreal Number

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Summary. Conway's surreal numbers have a fascinating algebraic structure, which we try to formalise in the Mizar system. In this article, building on our previous work establishing that the surreal numbers fulfil the ring properties, we construct the inverse element for any non-zero number. For that purpose, we formalise the definition of the inverse element formulated in Section *Properties of Division* of Conway's book. In this way we show formally in the Mizar system that surreal numbers satisfy all nine properties of a field.

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Introduction

In our previous work [19] realized in the Mizar system [5], [6], we have formally defined and justified a list of properties of subtraction, addition and multiplication of surreal [12] numbers. The definition of division, which has been missing so far, is, however, significantly more complicated than the other operations. For a number $x = \{L_x \mid R_x\}$ to be a positive surreal number where $0 \in L_x$ and all other members of L_x are positive, Conway [11] defines y as follows:

$$y = \left\{0, \frac{1+(x^R-x)y^L}{x^R}, \frac{1+(x^L-x)y^R}{x^L} \mid \frac{1+(x^L-x)y^L}{x^L}, \frac{1+(x^R-x)y^R}{x^R}\right\} \tag{I.1}$$

where x^L, x^R ranges over all positive members of set L_x, R_x, respectively [11]. The definition, like most of Conway's, is rather confusing and seems to be based more on the property of the inverse element than on a typical mathematical definition. In fact, $y = \{L_y \mid R_y\}$ is defined by a kind of hidden recursion since y^L, y^R which appear on the RHS of the equation (I.1) are members of L_y, R_y. As an illustration of this definition, Conway gave the example $3 = \{0, 2 \mid \}$, where there is only $x^L = 2$, $y = \{0, \ldots \mid \ldots\}$ is given as an initial value, so we can put $y^L = 0$. Then $\frac{1+(2-3)0}{2} = \frac{1}{2}$ is a new y^R, and $\frac{1+(2-3)\frac{1}{2}}{2} = \frac{1}{4}$ is a new y^L and so on, an infinite number of times. Finally $y = \{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \ldots \mid \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \frac{43}{128}, \ldots\}$.

This definition with a double recursion is a challenge to the formal approach. Mamane, in his formalisation in the Coq system [8], [7], considered the construction of the inverse element as a future work [13]. Obua formalised the surreal numbers in the Isabelle/HOLZF [15], [22], [14], by covering only the additive group [16]. Schleicher and Stoll [21] proposed a reasonably precise informal proof that we adapt in our approach.

To formalise such a concept, we first introduce a restriction that limits the members of the sets L_x, R_x to those that are positive with special exception in L_x, where we added 0. Let x be a positive surreal number. We define $\|x\|$ (see Def. 9) to be

$$\{0, \{x^L \in L_x \mid x^L > 0\} \mid \{x^R \in R_x \mid x^R > 0\}\} \tag{I.2}$$

and we prove that $x \approx \|x\|$ (see theorem (18)). Then we focus on the fact that the definition of the inverse element x^{-1} has looping, i.e. the definition uses the values of the inverse element of every positive member of the sets L_x, R_x, but they have to be born before x. Suppose I is a function, which in context, will be the corresponding inverse function defined on all the positive surreal numbers that were born before x, in particular on $L_x \cup R_x$. We define a subset of surreal numbers as follows (see Def. 2, Def. 3):

Definition 1 *Let x be a surreal number, X, P be sets of surreal numbers, and I be a function from the surreal numbers to the surreal numbers such that X is a subset of the domain of I. We define a subset of surreal numbers as follows:*

$$\mathtt{d}(P, x, X, I) = \bigcup_{p \in P} \{(1 + (a - x) \cdot p) \cdot I(a) \mid a \in X\}. \tag{I.3}$$

We define also a sequence of sets of surreal numbers $\mathtt{d}_n^L(x, I)$, $\mathtt{d}_n^R(x, I)$ for a given surreal number x and a function I recursively as follows:

$$\begin{aligned}
\mathtt{d}^L_0(x, I) &= \{0\},\\
\mathtt{d}^R_0(x, I) &= \emptyset,\\
\mathtt{d}^L_{n+1}(x, I) &= \mathtt{d}^L_n(x, I) \cup \mathtt{d}(\mathtt{d}^L_n(x, I), x, R_x, I) \cup \mathtt{d}(\mathtt{d}^R_n(x, I), x, L_x, I),\\
\mathtt{d}^R_{n+1}(x, I) &= \mathtt{d}^R_{n+1}(x, I) \cup \mathtt{d}(\mathtt{d}^L_n(x, I), x, L_x, I) \cup \mathtt{d}(\mathtt{d}^R_n(x, I), x, R_x, I).
\end{aligned} \tag{I.4}$$

We show that $\bigcup_{n\in\mathbb{N}} \mathtt{d}^L_n(x, I)$, $\bigcup_{n\in\mathbb{N}} \mathtt{d}^R_n(x, I)$ are sets of surreal numbers if I is a surreal-valued function on $L_x \cup R_x$. Next we restrict our consideration to the positive surreal number x, where we have $x \approx \|x\|$. Note that $\mathtt{born}\,\|x\| \leqslant \mathtt{born}\,x$ (see theorem (22)). Without loss of generality we can assume $\mathtt{born}\,\|x\| = \mathtt{born}\,x$. Then exploring the assumption that $I(a) \cdot a \approx 1$ for all the positive surreal numbers a that were born before a, we can prove the following key step that

$$y = \{ \bigcup_{n\in\mathbb{N}} \mathtt{d}^L_n(\|x\|, I) \mid \bigcup_{n\in\mathbb{N}} \mathtt{d}^R_n(\|x\|, I)\} \tag{I.5}$$

is a surreal number (see theorem (31)) and $x \cdot y \approx 1$ (see theorem (32)).

It is easy to see that, based on this step, we can extend the domain of the function I, which covers all surreal numbers created in days before α, by all positive surreal numbers born on day α, where α is an ordinal number. Consequently, using second-order schemes formulated in [19] which are a consequence of transfinite induction, we construct a unique sequence of $\{I_\alpha\}$ functions, where I_α is the inverse function defined on day α. Finally, we define x^{-1} as $I_\alpha(x)$ (see Def. 13, Def. 14), where α is a day where a given positive x is born and $-I_\alpha(-x)$ in the negative case.

Our formal construction of the inverse element seems to differ from the definition (I.1) proposed by Conway. This difficulty can be avoided by directly using transfinite induction-recursion, which is not available in the Mizar system. We test our approach by proving that our concept of an inverse element satisfies the property formulated by Conway (see theorems (31), (32)):

Theorem 1 *Let x be a positive surreal number. We define $\mathtt{d}(A, x, B) = \{(1 + (a-x) \cdot b) \cdot (a^{-1}) | a \in A \wedge b \in B \wedge 0 < a\}$. Then*

$$x^{-1} \approx \{\{0\} \cup \mathtt{d}(R_x, x, L_{x^{-1}}) \cup \mathtt{d}(L_x, x, R_{x^{-1}}), \mathtt{d}(L_x, x, L_{x^{-1}}) \cup \mathtt{d}(R_x, x, R_{x^{-1}})\}. \tag{I.6}$$

The formalization follows [11], [21], selected fragments have been described in [20].

1. Construction of the Inverse Element for Surreal Numbers

From now on A, B, O denote ordinal numbers, n, m denote natural numbers, a, b, o denote objects, x, y, z denote surreal numbers, X, Y, Z denote sets, and Inv, I_1, I_2 denote functions.

Let x, y be objects. Assume x is surreal and y is surreal. The functor $x * y$ yielding a surreal number is defined by

(Def. 1) for every surreal numbers x_1, y_1 such that $x_1 = x$ and $y_1 = y$ holds $it = x_1 \cdot y_1$.

Let λ, x be objects, X be a set, and Inv be a function. The functor $\mathrm{divs}(\lambda, x, X, Inv)$ yielding a set is defined by

(Def. 2) $o \in it$ iff there exists an object x_3 such that $x_3 \in X$ and $x_3 \neq \mathbf{0_{No}}$ and $o = (\mathbf{1_{No}} +' (x_3 +' -'x) * \lambda) * Inv(x_3)$.

Let Λ be a set and x be an object. The functor $\mathrm{divset}(\Lambda, x, X, Inv)$ yielding a set is defined by

(Def. 3) $o \in it$ iff there exists an object λ such that $\lambda \in \Lambda$ and $o \in \mathrm{divs}(\lambda, x, X, Inv)$.

The functor $\mathrm{Transitions}(x, Inv)$ yielding a function is defined by

(Def. 4) $\mathrm{dom}\, it = \mathbb{N}$ and $it(0) = \mathbf{1_{No}}$ and for every natural number k, $it(k)$ is pair and $(it(k+1))_{\mathbf{1}} = (\mathrm{L}_{it(k)} \cup \mathrm{divset}(\mathrm{L}_{it(k)}, x, \mathrm{R}_x, Inv)) \cup \mathrm{divset}(\mathrm{R}_{it(k)}, x, \mathrm{L}_x, Inv)$ and $(it(k+1))_{\mathbf{2}} = (\mathrm{R}_{it(k)} \cup \mathrm{divset}(\mathrm{L}_{it(k)}, x, \mathrm{L}_x, Inv)) \cup \mathrm{divset}(\mathrm{R}_{it(k)}, x, \mathrm{R}_x, Inv)$.

The functor $\mathrm{d}^{\mathrm{L}}(x, Inv)$ yielding a function is defined by

(Def. 5) $\mathrm{dom}\, it = \mathbb{N}$ and for every natural number k, $it(k) = ((\mathrm{Transitions}(x, Inv))(k))_{\mathbf{1}}$.

The functor $\mathrm{d}^{\mathrm{R}}(x, Inv)$ yielding a function is defined by

(Def. 6) $\mathrm{dom}\, it = \mathbb{N}$ and for every natural number k, $it(k) = ((\mathrm{Transitions}(x, Inv))(k))_{\mathbf{2}}$.

Let a, b be surreal numbers and x, y be objects. We identify $x * y$ with $a \cdot b$.

Now we state the propositions:

(1) (i) $(\mathrm{d}^{\mathrm{L}}(o, Inv))(0) = \{\mathbf{0_{No}}\}$, and

(ii) $(\mathrm{d}^{\mathrm{R}}(o, Inv))(0) = \emptyset$.

(2) If $n \leqslant m$, then $(\mathrm{d}^{\mathrm{L}}(o, Inv))(n) \subseteq (\mathrm{d}^{\mathrm{L}}(o, Inv))(m)$ and $(\mathrm{d}^{\mathrm{R}}(o, Inv))(n) \subseteq (\mathrm{d}^{\mathrm{R}}(o, Inv))(m)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{d}^{\mathrm{L}}(o, Inv))(n) \subseteq (\mathrm{d}^{\mathrm{L}}(o, Inv))(n + \$_1)$ and $(\mathrm{d}^{\mathrm{R}}(o, Inv))(n) \subseteq (\mathrm{d}^{\mathrm{R}}(o, Inv))(n + \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

Let X be a set and f be a function. We say that f is X-surreal-valued if and only if

(Def. 7) if $o \in X$, then $f(o)$ is a surreal number.

Now we state the propositions:

(3) If Inv is Y-surreal-valued and $X \subseteq Y$, then Inv is X-surreal-valued.

(4) $\mathrm{divs}(y, x, X, Inv)$ is surreal-membered.

(5) If Y is surreal-membered and X is surreal-membered and Inv is X-surreal-valued, then $\mathrm{divset}(Y, x, X, Inv)$ is surreal-membered. The theorem is a consequence of (4).

(6) (i) $(\mathrm{d}^{\mathrm{L}}(o, Inv))(n{+}1) = ((\mathrm{d}^{\mathrm{L}}(o, Inv))(n) \cup \mathrm{divset}((\mathrm{d}^{\mathrm{L}}(o, Inv))(n), o, \mathrm{R}_o, Inv)) \cup \mathrm{divset}((\mathrm{d}^{\mathrm{R}}(o, Inv))(n), o, \mathrm{L}_o, Inv)$, and

(ii) $(\mathrm{d}^{\mathrm{R}}(o, Inv))(n{+}1) = ((\mathrm{d}^{\mathrm{R}}(o, Inv))(n) \cup \mathrm{divset}((\mathrm{d}^{\mathrm{L}}(o, Inv))(n), o, \mathrm{L}_o, Inv)) \cup \mathrm{divset}((\mathrm{d}^{\mathrm{R}}(o, Inv))(n), o, \mathrm{R}_o, Inv)$.

(7) $\mathrm{divs}(o, x, X, Inv) = \mathrm{divs}(o, x, X \setminus \{\mathbf{0}_{\mathbf{No}}\}, Inv)$.
PROOF: $\mathrm{divs}(o, x, X, Inv) \subseteq \mathrm{divs}(o, x, X \setminus \{\mathbf{0}_{\mathbf{No}}\}, Inv)$ by [10, (56)]. Consider x_3 being an object such that $x_3 \in X \setminus \{\mathbf{0}_{\mathbf{No}}\}$ and $x_3 \neq \mathbf{0}_{\mathbf{No}}$ and $a = (\mathbf{1}_{\mathbf{No}} +' (x_3 +' -'x) * o) * Inv(x_3)$. $\square$

(8) $\mathrm{divset}(Y, x, X, Inv) = \mathrm{divset}(Y, x, X \setminus \{\mathbf{0}_{\mathbf{No}}\}, Inv)$. The theorem is a consequence of (7).

(9) Suppose Inv is $((\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0}_{\mathbf{No}}\})$-surreal-valued. Then

(i) $(\mathrm{d}^{\mathrm{L}}(x, Inv))(n)$ is surreal-membered, and

(ii) $(\mathrm{d}^{\mathrm{R}}(x, Inv))(n)$ is surreal-membered.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{d}^{\mathrm{L}}(x, Inv))(\$_1)$ is surreal-membered and $(\mathrm{d}^{\mathrm{R}}(x, Inv))(\$_1)$ is surreal-membered. $\mathcal{P}[0]$. If $\mathcal{P}[m]$, then $\mathcal{P}[m+1]$. $\mathcal{P}[m]$ from [3, Sch. 2]. $\square$

(10) Suppose Inv is $((\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0}_{\mathbf{No}}\})$-surreal-valued. Then

(i) $\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv)$ is surreal-membered, and

(ii) $\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv)$ is surreal-membered.

PROOF: $\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv)$ is surreal-membered by [2, (2)], (9). Consider n being an object such that $n \in \mathrm{dom}(\mathrm{d}^{\mathrm{R}}(x, Inv))$ and $o \in (\mathrm{d}^{\mathrm{R}}(x, Inv))(n)$. $(\mathrm{d}^{\mathrm{R}}(x, Inv))(n)$ is surreal-membered. $\square$

(11) If $Y \subseteq Z$, then $\mathrm{divset}(Y, x, X, Inv) \subseteq \mathrm{divset}(Z, x, X, Inv)$.

(12) $\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv) = (\{\mathbf{0}_{\mathbf{No}}\} \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{R}_x, Inv)) \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{L}$
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{d}^{\mathrm{L}}(x, Inv))(\$_1) \subseteq (\{\mathbf{0}_{\mathbf{No}}\} \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv)$ $\mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{L}_x, Inv)$. $(\mathrm{d}^{\mathrm{L}}(x, Inv))(0) = \{\mathbf{0}_{\mathbf{No}}\}$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (6), [1, (1)], (11). $\mathcal{P}[n]$ from [3, Sch. 2]. $\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv) \subseteq (\{\mathbf{0}_{\mathbf{No}}\} \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{R}_x, Inv)) \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{L}_x, Inv)$ by [2, (2)]. $\mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{R}_x, Inv) \subseteq \bigcup \mathrm{d}^{\mathrm{L}}(x, Inv)$ by [2, (2)], (6). $\mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{L}_x, Inv) \subseteq \bigcup \mathrm{d}^{\mathrm{L}}(x, Inv)$ by [2, (2)], (6). $(\mathrm{d}^{\mathrm{L}}(x, Inv))(0) = \{\mathbf{0}_{\mathbf{No}}\}$. $\square$

(13) $\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv) = \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{L}_x, Inv) \cup \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{R}_x, Inv)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv (\mathrm{d}^{\mathrm{R}}(x, Inv))(\$_1) \subseteq \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{L}_x, I$ divset$(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{R}_x, Inv)$. $(\mathrm{d}^{\mathrm{R}}(x, Inv))(0) = \emptyset$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (6), (11), [1, (1)]. $\mathcal{P}[n]$ from [3, Sch. 2]. $\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv) \subseteq \mathrm{divset}(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{L}_x, I$ divset$(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{R}_x, Inv)$ by [2, (2)]. divset$(\bigcup \mathrm{d}^{\mathrm{L}}(x, Inv), x, \mathrm{L}_x, Inv) \subseteq \bigcup \mathrm{d}^{\mathrm{R}}(x, Inv)$ by [2, (2)], (6). divset$(\bigcup \mathrm{d}^{\mathrm{R}}(x, Inv), x, \mathrm{R}_x, Inv) \subseteq \bigcup \mathrm{d}^{\mathrm{R}}(x, Inv)$ by [2, (2)], (6). $\square$

(14) Suppose $X \setminus \{\mathbf{0_{No}}\} \subseteq Z$ and $I_1 \restriction Z = I_2 \restriction Z$. Then $\mathrm{divs}(a, b, X, I_1) = \mathrm{divs}(a, b, X, I_2)$.
PROOF: $\mathrm{divs}(a, b, X, I_1) \subseteq \mathrm{divs}(a, b, X, I_2)$ by [10, (56)], [9, (49)]. Consider x_3 being an object such that $x_3 \in X$ and $x_3 \neq \mathbf{0_{No}}$ and $o = (\mathbf{1_{No}} +' (x_3 +' -'b) * a) * I_2(x_3)$. $\square$

(15) Suppose $X \setminus \{\mathbf{0_{No}}\} \subseteq Z$ and $I_1 \restriction Z = I_2 \restriction Z$. Then $\mathrm{divset}(Y, o, X, I_1) = \mathrm{divset}(Y, o, X, I_2)$. The theorem is a consequence of (14).

Let us consider an object x. Now we state the propositions:

(16) Suppose $(\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0_{No}}\} \subseteq Z$ and $I_1 \restriction Z = I_2 \restriction Z$. Then $\mathrm{Transitions}(x, I_1) = \mathrm{Transitions}(x, I_2)$.
PROOF: Set $T_1 = \mathrm{Transitions}(x, I_1)$. Set $T_2 = \mathrm{Transitions}(x, I_2)$. Define $\mathcal{P}$[natural number] $\equiv T_1(\$_1) = T_2(\$_1)$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$ from [3, Sch. 2]. $\square$

(17) Suppose $(\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0_{No}}\} \subseteq Z$ and $I_1 \restriction Z = I_2 \restriction Z$. Then
(i) $\mathrm{d}^{\mathrm{L}}(x, I_1) = \mathrm{d}^{\mathrm{L}}(x, I_2)$, and
(ii) $\mathrm{d}^{\mathrm{R}}(x, I_1) = \mathrm{d}^{\mathrm{R}}(x, I_2)$.

The theorem is a consequence of (16).

2. The Concept of Positive Options in Conway's Sense

Let x be a surreal number. We say that x is positive if and only if

(Def. 8) $\mathbf{0_{No}} < x$.

One can verify that $\mathbf{1_{No}}$ is positive and there exists a surreal number which is positive.

Let x, y be positive surreal numbers. Let us note that $x + y$ is positive and $x \cdot y$ is positive.

Let x be an object. Assume x is a positive surreal number. The functor $\|x\|$ yielding a positive surreal number is defined by

(Def. 9) ($y \in \mathrm{L}_{it}$ iff $y = \mathbf{0_{No}}$ or $y \in \mathrm{L}_x$ and y is positive) and ($y \in \mathrm{R}_{it}$ iff $y \in \mathrm{R}_x$ and y is positive).

Now we state the propositions:

(18) If x is positive, then $x \approx \|x\|$.

(19) If x is positive, then $\|\|x\|\| = \|x\|$.

(20) Suppose x is positive. Then $(\mathrm{L}_{\|x\|} \cup \mathrm{R}_{\|x\|}) \setminus \{\mathbf{0}_{\mathbf{No}}\} \subseteq \mathrm{L}_x \cup \mathrm{R}_x$.

(21) Suppose x is positive and $y \in (\mathrm{L}_{\|x\|} \cup \mathrm{R}_{\|x\|}) \setminus \{\mathbf{0}_{\mathbf{No}}\}$. Then y is positive.

(22) If x is positive, then $\mathfrak{born}\,\|x\| \subseteq \mathfrak{born}\,x$.

Proof: Set $N_2 = \|x\|$. For every object o such that $o \in \mathrm{L}_{N_2} \cup \mathrm{R}_{N_2}$ there exists O such that $O \in \mathfrak{born}\,x$ and $o \in \mathrm{Day}O$ by [10, (56)], (20), [18, (1)]. □

Let A be an ordinal number. The functor Positives(A) yielding a subset of Day$A$ is defined by

(Def. 10) $x \in it$ iff $x \in \mathrm{Day}A$ and $\mathbf{0}_{\mathbf{No}} < x$.

Now we state the propositions:

(23) If $A \subseteq B$, then Positives(A) $\subseteq$ Positives(B).

(24) Suppose x is positive. Then $(\mathrm{L}_{\|x\|} \cup \mathrm{R}_{\|x\|}) \setminus \{\mathbf{0}_{\mathbf{No}}\} \subseteq$ Positives($\mathfrak{born}\,x$).

The theorem is a consequence of (20) and (21).

3. The Inverse Element for Surreal Numbers

Let A be an ordinal number. The functor $\mathrm{inverse}_{\mathbf{No}}(A)$ yielding a many sorted set indexed by Positives(A) is defined by

(Def. 11) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\,S = \mathrm{succ}\,A$ and $it = S(A)$ and for every ordinal number B such that $B \in \mathrm{succ}\,A$ there exists a many sorted set S_4 indexed by Positives(B) such that $S(B) = S_4$ and for every object x such that $x \in$ Positives(B) holds $S_4(x) = \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|x\|, \bigcup \mathrm{rng}(S{\restriction}B)), \bigcup \mathrm{d}^{\mathrm{R}}(\|x\|, \bigcup \mathrm{rng}(S{\restriction}B)) \rangle$.

Now we state the proposition:

(25) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every B such that $B \in \mathrm{dom}\,S$ there exists a many sorted set S_4 indexed by Positives(B) such that $S(B) = S_4$ and for every o such that $o \in$ Positives(B) holds $S_4(o) = \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|o\|, \bigcup \mathrm{rng}(S{\restriction}B)), \bigcup \mathrm{d}^{\mathrm{R}}(\|o\|, \bigcup \mathrm{rng}(S{\restriction}B)) \rangle$. If $A \in \mathrm{dom}\,S$, then $\mathrm{inverse}_{\mathbf{No}}(A) = S(A)$.

Proof: Define $\mathcal{D}$(ordinal number) $=$ Positives($\$_1$). Define $\mathcal{H}$(object, $\subseteq$-monotone , fu yielding transfinite sequence) $= \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|\$_1\|, \bigcup \mathrm{rng}\,\$_2), \bigcup \mathrm{d}^{\mathrm{R}}(\|\$_1\|, \bigcup \mathrm{rng}\,\$_2) \rangle$. Consider S_2 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\,S_2 = \mathrm{succ}\,A$ and $S_2(A) = \mathrm{inverse}_{\mathbf{No}}(A)$ and for every ordinal number B such that $B \in \mathrm{succ}\,A$ there exists a many sorted set S_4 indexed by $\mathcal{D}(B)$ such that $S_2(B) = S_4$ and for every object x such that

$x \in \mathcal{D}(B)$ holds $S_4(x) = \mathcal{H}(x, S_2 \restriction B)$. $S1 \restriction \operatorname{succ} A = S_2 \restriction \operatorname{succ} A$ from [19, Sch. 2]. □

Let x be a surreal number. The functor $\operatorname{inv} x$ yielding an object is defined by the term

(Def. 12) $(\text{inverse}_{\mathbf{No}}(\mathfrak{born}\, x))(x)$.

The functor $\text{inverses}_{\mathbf{No}}(x)$ yielding a function is defined by

(Def. 13) $\operatorname{dom} it = (\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0_{No}}\}$ and for every y such that $y \in (\mathrm{L}_x \cup \mathrm{R}_x) \setminus \{\mathbf{0_{No}}\}$ holds $it(y) = \operatorname{inv} y$.

Now we state the propositions:

(26) Suppose x is positive and $\text{inverses}_{\mathbf{No}}(\|x\|) \subseteq \mathit{Inv}$. Then $\operatorname{inv} x = \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|x\|, \mathit{Inv}), \bigcup \mathrm{d}^{\mathrm{R}}(\|x\|, \mathit{Inv}) \rangle$.

Proof: Set $A = \mathfrak{born}\, x$. Set $N_2 = \|x\|$. Consider S being a $\subseteq$-monotone, function yielding transfinite sequence such that $\operatorname{dom} S = \operatorname{succ} A$ and $\text{inverse}_{\mathbf{No}}(A) = S(A)$ and for every ordinal number B such that $B \in \operatorname{succ} A$ there exists a many sorted set S_4 indexed by $\text{Positives}(B)$ such that $S(B) = S_4$ and for every object o such that $o \in \text{Positives}(B)$ holds $S_4(o) = \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|o\|, \bigcup \operatorname{rng}(S \restriction B)), \bigcup \mathrm{d}^{\mathrm{R}}(\|o\|, \bigcup \operatorname{rng}(S \restriction B)) \rangle$. Consider S_4 being a many sorted set indexed by $\text{Positives}(A)$ such that $S(A) = S_4$ and for every object o such that $o \in \text{Positives}(A)$ holds $S_4(o) = \langle \bigcup \mathrm{d}^{\mathrm{L}}(\|o\|, \bigcup \operatorname{rng}(S \restriction A)), \bigcup \mathrm{d}^{\mathrm{R}}(\|o\|, \bigcup \operatorname{rng}(S \restriction A)) \rangle$. Set $U_1 = \bigcup \operatorname{rng}(S \restriction A)$. Set $X_8 = (\mathrm{L}_{N_2} \cup \mathrm{R}_{N_2}) \setminus \{\mathbf{0_{No}}\}$. $X_8 \subseteq \text{Positives}(A)$. $X_8 \subseteq \mathrm{L}_x \cup \mathrm{R}_x$. $X_8 \subseteq \operatorname{dom} U_1$ by [18, (1)], [4, (8)], [19, (5)]. If $a \in X_8$, then $(U_1 \restriction X_8)(a) = (\mathit{Inv} \restriction X_8)(a)$ by [9, (49)], [18, (1)], [4, (8)], [19, (5)]. $\mathrm{d}^{\mathrm{L}}(N_2, U_1) = \mathrm{d}^{\mathrm{L}}(N_2, \mathit{Inv})$ and $\mathrm{d}^{\mathrm{R}}(N_2, U_1) = \mathrm{d}^{\mathrm{R}}(N_2, \mathit{Inv})$. □

(27) Let us consider a function f. Suppose $\operatorname{dom} f = \mathbb{N}$ and $y \in \bigcup f$. Then there exists n such that

(i) $y \in f(n)$, and

(ii) for every m such that $y \in f(m)$ holds $n \leqslant m$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv y \in f(\$_1)$. Consider n being an object such that $n \in \operatorname{dom} f$ and $y \in f(n)$. There exists a natural number k such that $\mathcal{P}[k]$ and for every natural number n such that $\mathcal{P}[n]$ holds $k \leqslant n$ from [3, Sch. 5]. □

(28) Let us consider surreal numbers x_1, x_1^R, y_1, y_1^R. Suppose $\mathbf{0_{No}} < x_1$ and $x_1 \cdot x_1^R \approx \mathbf{1_{No}}$ and $\mathbf{0_{No}} < y_1$ and $y_1 \cdot y_1^R \approx \mathbf{1_{No}}$ and $x \cdot y_1 < y \cdot x_1$. Then $x \cdot x_1^R < y \cdot y_1^R$.

(29) Let us consider surreal numbers x, x_1, x_2, y_1, y_2. Then

(i) $(\mathbf{1_{No}} + (x_2 - x) \cdot y_2) \cdot x_1 + -(\mathbf{1_{No}} + (x_1 - x) \cdot y_1) \cdot x_2 \approx (x_1 - x_2) \cdot (\mathbf{1_{No}} - x \cdot y_1) + (y_1 - y_2) \cdot x_1 \cdot (x - x_2)$, and

(ii) $(\mathbf{1_{No}} + (x_2 - x) \cdot y_2) \cdot x_1 - (\mathbf{1_{No}} + (x_1 - x) \cdot y_1) \cdot x_2 \approx (x_1 - x_2) \cdot (\mathbf{1_{No}} - x \cdot y_2) + (y_2 - y_1) \cdot x_2 \cdot (x_1 - x)$.

(30) Let us consider surreal numbers x_1, y_1, I_4. Suppose $x_1 \cdot I_4 \approx \mathbf{1_{No}}$. Then $x_1 \cdot y + x \cdot y_1 - x_1 \cdot y_1 \approx \mathbf{1_{No}} + x_1 \cdot (y - (\mathbf{1_{No}} + (x_1 - x) \cdot y_1) \cdot I_4)$.

Let x be a positive surreal number. Note that $\operatorname{inv} x$ is surreal.

Now we state the propositions:

(31) If x is positive, then $\operatorname{inv} x$ is a surreal number.

(32) If x is positive and $y = \operatorname{inv} x$, then $x \cdot y \approx \mathbf{1_{No}}$.

Let x be a surreal number. Assume $x \not\approx \mathbf{0_{No}}$. The functor x^{-1} yielding a surreal number is defined by

(Def. 14) (i) $it = \operatorname{inv} x$, **if** x is positive,

(ii) $-it = \operatorname{inv}(-x)$, **otherwise**.

4. Basic Properties of the Inverse Element

Now we state the proposition:

(33) If $x \not\approx \mathbf{0_{No}}$, then $x \cdot (x^{-1}) \approx \mathbf{1_{No}}$.

Let X, Y be sets and x be a surreal number. The functor $\operatorname{divset}(X, x, Y)$ yielding a set is defined by

(Def. 15) $o \in it$ iff there exist surreal numbers x_1, y_1 such that $\mathbf{0_{No}} < x_1$ and $x_1 \in X$ and $y_1 \in Y$ and $o = (\mathbf{1_{No}} + (x_1 - x) \cdot y_1) \cdot ({x_1}^{-1})$.

Note that $\operatorname{divset}(X, x, Y)$ is surreal-membered.

Now we state the propositions:

(34) Let us consider sets X, n_1, and a surreal-membered set Y. Suppose x is positive and ($X = \mathrm{L}_x$ and $n_1 = \mathrm{L}_{\|x\|}$ or $X = \mathrm{R}_x$ and $n_1 = \mathrm{R}_{\|x\|}$). Then $\operatorname{divset}(X, \|x\|, Y) = \operatorname{divset}(Y, \|x\|, n_1, \operatorname{inverses}_{\mathbf{No}}(\|x\|))$.

Proof: Set $N_2 = \|x\|$. Set $Inv = \operatorname{inverses}_{\mathbf{No}}(N_2)$. $\operatorname{divset}(X, N_2, Y) \subseteq \operatorname{divset}(Y, N_2, X_1, Inv)$ by [10, (56)]. Consider y_1 being an object such that $y_1 \in Y$ and $o \in \operatorname{divs}(y_1, N_2, X_1, Inv)$. Consider x_1 being an object such that $x_1 \in X_1$ and $x_1 \neq \mathbf{0_{No}}$ and $o = (\mathbf{1_{No}} +' (x_1 +' -' N_2) * y_1) * Inv(x_1)$. $\square$

(35) If $x \approx y$, then $\operatorname{divset}(X, x, Y) \lessdot \operatorname{divset}(X, y, Y)$.

(36) Suppose x is positive. Then $x^{-1} = \langle\!\langle (\{\mathbf{0_{No}}\} \cup \operatorname{divset}(\mathrm{R}_x, \|x\|, \mathrm{L}_{x^{-1}})) \cup \operatorname{divset}(\mathrm{L}_x, \|x\|, \mathrm{R}_{x^{-1}}), \operatorname{divset}(\mathrm{L}_x, \|x\|, \mathrm{L}_{x^{-1}}) \cup \operatorname{divset}(\mathrm{R}_x, \|x\|, \mathrm{R}_{x^{-1}}) \rangle\!\rangle$. The theorem is a consequence of (26), (34), (12), and (13).

(37) Let us consider surreal-membered sets X_1, X_2, Y_1, Y_2. Suppose $X_2 \lessdot X_1$ and $Y_2 \lessdot Y_1$ and $\langle\!\langle X_1, Y_1 \rangle\!\rangle$ is surreal. Then $\langle\!\langle X_2, Y_2 \rangle\!\rangle$ is surreal.

PROOF: $X_2 \ll Y_2$ by [17, (45)], [18, (4)]. Consider M being an ordinal number such that for every o such that $o \in X_2 \cup Y_2$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. □

(38) Suppose x is positive. Then $\langle\!(\{\mathbf{0_{No}}\}\cup\mathrm{divset}(\mathrm{R}_x, x, \mathrm{L}_{x^{-1}}))\cup\mathrm{divset}(\mathrm{L}_x, x, \mathrm{R}_{x^{-1}}), \mathrm{divset}(\mathrm{L}_x, x, \mathrm{L}_{x^{-1}})\cup\mathrm{divset}(\mathrm{R}_x, x, \mathrm{R}_{x^{-1}})\rangle\!$ is a surreal number. The theorem is a consequence of (18), (35), (36), and (37).

(39) Suppose x is positive and $y = \langle\!(\{\mathbf{0_{No}}\}\cup\mathrm{divset}(\mathrm{R}_x, x, \mathrm{L}_{x^{-1}}))\cup\mathrm{divset}(\mathrm{L}_x, x, \mathrm{R}_{x^{-1}}), \mathrm{divset}(\mathrm{L}_x, x, \mathrm{L}_{x^{-1}}) \cup \mathrm{divset}(\mathrm{R}_x, x, \mathrm{R}_{x^{-1}})\rangle\!$. Then $x^{-1} \approx y$. The theorem is a consequence of (18), (35), and (36).

5. Fundamental Properties of the Inverse Element

Now we state the proposition:

(40) If $x \not\approx \mathbf{0_{No}}$, then $\mathbf{0_{No}} < x$ iff $\mathbf{0_{No}} < x^{-1}$.

PROOF: $x \cdot (x^{-1}) \approx \mathbf{1_{No}}$. If $\mathbf{0_{No}} < x$, then $\mathbf{0_{No}} < x^{-1}$ by [19, (72)]. □

Let x be a positive surreal number. Note that x^{-1} is positive.

Now we state the propositions:

(41) $x \cdot y \approx \mathbf{0_{No}}$ if and only if $x \approx \mathbf{0_{No}}$ or $y \approx \mathbf{0_{No}}$.

(42) If $x \not\approx \mathbf{0_{No}}$ and $x \cdot y \approx \mathbf{1_{No}}$, then $y \approx x^{-1}$. The theorem is a consequence of (33).

(43) If $\mathbf{0_{No}} \not\approx x$ and $x \approx y$, then $x^{-1} \approx y^{-1}$. The theorem is a consequence of (33) and (42).

(44) If $x \not\approx \mathbf{0_{No}}$, then $(x^{-1})^{-1} \approx x$. The theorem is a consequence of (33) and (42).

(45) If $x \not\approx \mathbf{0_{No}}$ and $y \not\approx \mathbf{0_{No}}$, then $x \cdot y^{-1} \approx x^{-1} \cdot (y^{-1})$. The theorem is a consequence of (33), (41), and (42).

References

[1] Grzegorz Bancerek. Towards the construction of a model of Mizar concepts. *Formalized Mathematics*, 16(**2**):207–230, 2008. doi:10.2478/v10037-008-0027-x.

[2] Grzegorz Bancerek. On powers of cardinals. *Formalized Mathematics*, 3(**1**):89–93, 1992.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[5] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[6] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[7] Yves Bertot. A short presentation of Coq. In Otmane Aït Mohamed, César A. Muñoz, and Sofiène Tahar, editors, *Theorem Proving in Higher Order Logics (TPHOLs 2008)*, volume 5170 of *LNCS*, pages 12–16. Springer, 2008. doi:10.1007/978-3-540-71067-7_3.

[8] Yves Bertot and Pierre Casteran. *Interactive Theorem Proving and Program Development.* Springer, 2004. ISBN 3540208542.

[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[11] John Horton Conway. *On Numbers and Games.* A K Peters Ltd., Natick, MA, second edition, 2001. ISBN 1-56881-127-6.

[12] Donald E. Knuth. *Surreal Numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness.* Addison-Wesley, 1974.

[13] Lionel Elie Mamane. Surreal numbers in Coq. In Jean-Christophe Filliâtre, Christine Paulin-Mohring, and Benjamin Werner, editors, *Types for Proofs and Programs, TYPES 2004*, volume 3839 of *LNCS*, pages 170–185. Springer, 2004. doi:10.1007/11617990_11.

[14] Tobias Nipkow and Gerwin Klein. *Concrete Semantics: With Isabelle/HOL.* Springer, 2014.

[15] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL – A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer, 2002. ISBN 3-540-43376-7.

[16] Steven Obua. Partizan games in Isabelle/HOLZF. In Kamel Barkaoui, Ana Cavalcanti, and Antonio Cerone, editors, *Theoretical Aspects of Computing – ICTAC 2006*, volume 4281 of *LNCS*, pages 272–286. Springer, 2006.

[17] Karol Pąk. Conway numbers – formal introduction. *Formalized Mathematics*, 31(1): 193–203, 2023. doi:10.2478/forma-2023-0018.

[18] Karol Pąk. Integration of game theoretic and tree theoretic approaches to Conway numbers. *Formalized Mathematics*, 31(1):205–213, 2023. doi:10.2478/forma-2023-0019.

[19] Karol Pąk. The ring of Conway numbers in Mizar. *Formalized Mathematics*, 31(1): 215–228, 2023. doi:10.2478/forma-2023-0020.

[20] Karol Pąk and Cezary Kaliszyk. Conway normal form: Bridging approaches for comprehensive formalization of surreal numbers. In Yves Bertot, Temur Kutsia, and Michael Norrish, editors, *15th International Conference on Interactive Theorem Proving, ITP 2024, September 9-14, 2024, Tbilisi, Georgia*, volume 309 of *LIPIcs*, pages 29:1–29:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPICS.ITP.2024.29.

[21] Dierk Schleicher and Michael Stoll. An introduction to Conway's games and numbers. *Moscow Mathematical Journal*, 6:359–388, 2006. doi:10.17323/1609-4514-2006-6-2-359-388.

[22] Makarius Wenzel, Lawrence C. Paulson, and Tobias Nipkow. The Isabelle framework. In Otmane Ait Mohamed, César Muñoz, and Sofiène Tahar, editors, *Theorem Proving in Higher Order Logics*, pages 33–38. Springer Berlin Heidelberg, 2008.

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