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Formalization of Separable Version of Banach–Alaoglu Theorem[1]

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Summary. The Banach–Alaoglu Theorem is a foundational result in functional analysis that addresses the compactness properties of the dual space of a normed vector space. Specifically, it states that the closed unit ball in the dual space is compact when equipped with the *weak** topology. Historically, Stefan Banach proved a version of this theorem for separable normed spaces, while Leonidas Alaoglu later extended the result to the general case.

In this article, using the Mizar [3], [2] system, we first formalize the *weak** sequentially compactness in dual normed spaces. Then we formalize the separable version of Banach–Alaoglu theorem. We referred to [15], [19], [9] in the formalization.

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1. *weak** Sequentially Compactness in Dual Normed Spaces

Now we state the propositions:

(1) Let us consider a real normed space X, a sequence v_1 of DualSp X, and points x, y of X. Suppose $v_1 \# x$ is convergent and $v_1 \# y$ is convergent. Then

(i) $v_1 \#(x + y)$ is convergent, and

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(ii) $\lim(v_1\#(x+y)) = \lim(v_1\#x) + \lim(v_1\#y)$.

(2) Let us consider a real normed space X, a sequence v_1 of DualSp X, a point x of X, and a real number a. Suppose $v_1\#x$ is convergent. Then

(i) $v_1\#a \cdot x$ is convergent, and

(ii) $\lim(v_1\#a \cdot x) = a \cdot (\lim(v_1\#x))$.

(3) Let us consider a real normed space X, a subset X_0 of X, and a sequence v_1 of DualSp X. Suppose for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent. Let us consider a point x of X. If $x \in \mathrm{Lin}(X_0)$, then $v_1\#x$ is convergent.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every point x of X for every subset X_1 of X for every linear combination L of X_1 such that $X_1 \subseteq X_0$ and $\overline{\overline{\text{the support of } L}} \leqslant \$_1$ and $x \in \mathrm{Lin}(X_1)$ and $x = \sum L$ holds $v_1\#x$ is convergent. $\mathcal{P}[0]$ by [16, (34)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [16, (34)], [6, (31)], [10, (8)], [17, (1)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. Consider l being a linear combination of X_0 such that $x = \sum l$. $\square$

(4) Let us consider a real Banach space X, subsets X_0, X_1 of X, and a sequence v_1 of DualSp X. Suppose X_1 = the carrier of $\mathrm{Lin}(X_0)$ and X_1 is dense and $\|v_1\|$ is bounded and for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent. Then v_1 is weakly* convergent. The theorem is a consequence of (3).

(5) Let us consider a real linear space V, a vector v of V, and a real number a. Suppose $a \neq 0$. Then there exists a linear combination l of V such that

(i) $l(v) = a$, and

(ii) the support of $l = \{v\}$.

PROOF: Reconsider $z_1 = 0a$ as an element of $\mathbb{R}$. Define $\mathcal{F}$(vector of V) $= z_1$. Consider f being a function from the carrier of V into $\mathbb{R}$ such that $f(v) = a$ and for every vector u of V such that $u \neq v$ holds $f(u) = \mathcal{F}(u)$ from [5, Sch. 6]. $\{v\} \subseteq$ the support of f. The support of $f \subseteq \{v\}$ by [16, (19)]. $\square$

(6) Let us consider a real linear space X. Then $\Omega_{\mathrm{Lin}(\Omega_X)} = \Omega_X$.

PROOF: For every object x, $x \in \Omega_{\mathrm{Lin}(\Omega_X)}$ iff $x \in \Omega_X$ by [17, (14)], (5), [16, (35)]. $\square$

(7) Let us consider a real Banach space X, and a sequence f of DualSp X. Then f is weakly* convergent if and only if $\|f\|$ is bounded and there exist subsets X_0, X_1 of X such that X_1 = the carrier of $\mathrm{Lin}(X_0)$ and X_1 is dense and for every point x of X such that $x \in X_0$ holds $f\#x$ is convergent. The theorem is a consequence of (6) and (4).

Weak* Sequentially Compactness in Separable Dual Normed Spaces

Let X be a real normed space and X_0 be a non empty subset of DualSp X. We say that X_0 is weakly* sequentially compact if and only if

(Def. 1) for every sequence s_1 of X_0, there exists a sequence s_2 of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in X_0$.

2. Separable Version of Banach–Alaoglu Theorem

Now we state the proposition:

(8) Let us consider a real Banach space X, a real number M, and a non empty subset X_0 of DualSp X. Suppose X is separable and $0 \leqslant M$ and $\overline{\mathrm{Ball}}(0_{\mathrm{DualSp}\,X}, M) = X_0$. Then X_0 is weakly* sequentially compact.

PROOF: For every sequence s_1 of X_0, there exists a sequence s_2 of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in X_0$ by [5, (6)], [11, (18)], [5, (4)], [13, (26), (15)]. □

Weakly* sequentially continuous mappings on Normed Spaces

Let X be a real normed space, f be a partial function from X to $\mathbb{R}$, and x_0 be a point of X. We say that f is weakly continuous in x_0 if and only if

(Def. 2) $x_0 \in \mathrm{dom}\, f$ and for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of DualSp X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of X such that $x \in \mathrm{dom}\, f$ and for every point y of DualSp X such that $y \in Y$ holds $|y(x - x_0)| < d$ holds $|f(x) - f(x_0)| < e$.

Let X_0 be a subset of X. We say that f is weakly continuous on X_0 if and only if

(Def. 3) $X_0 \subseteq \mathrm{dom}\, f$ and for every point x_0 of X such that $x_0 \in X_0$ for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of DualSp X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of X such that $x \in X_0$ and for every point y of DualSp X such that $y \in Y$ holds $|y(x - x_0)| < d$ holds $|f(x) - f(x_0)| < e$.

Now we state the proposition:

(9) Let us consider a real normed space X, a partial function f from X to $\mathbb{R}$, and a subset X_0 of X. Then f is weakly continuous on X_0 if and only if $X_0 \subseteq \mathrm{dom}\, f$ and for every point x_0 of X such that $x_0 \in X_0$ holds $f{\restriction}X_0$ is weakly continuous in x_0.

Let X be a real normed space, f be a partial function from DualSp X to $\mathbb{R}$, and x_0 be a point of DualSp X. We say that f is weakly* continuous in x_0 if and only if

(Def. 4) $x_0 \in \operatorname{dom} f$ and for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of DualSp X such that $x \in \operatorname{dom} f$ and for every point y of X such that $y \in Y$ holds $|(x - x_0)(y)| < d$ holds $|f(x) - f(x_0)| < e$.

Let X_0 be a subset of DualSp X. We say that

f. (5)(10)(11) $f_* x$ is convergent, and

$\lim(f_* x) = f(x_0)$.

Proof: For every real number e such that $0 < e$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|(f_* x)(m) - f(x_0)| < e$ by [5, (15)], [14, (19)], [5, (4), (108)]. □

Let us consider a real normed space X, a partial function f from DualSp X to $\mathbb{R}$, a point x_0 of DualSp X, and a sequence x of DualSp X. Suppose f is weakly* continuous in x_0 and $\operatorname{rng} x \subseteq \operatorname{dom} f$ and x is weakly* convergent and w*-lim$(x) = x_0$. Then

(i) $f_* x$ is convergent, and

(ii) $\lim(f_* x) = f(x_0)$.

Proof: For every real number e such that $0 < e$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|(f_* x)(m) - f(x_0)| < e$ by [12, (33)], [5, (4), (108)]. □

Let us consider a real normed space X, a partial function f from DualSp X to $\mathbb{R}$, and a non empty subset S of DualSp X. Suppose S is weakly* sequentially compact and *weakly** continuous on S.*Then*

there exists a real number r such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$, and

there exists a point x_0 of DualSp X such that $x_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(x) \leqslant f(x_0)$, and

there exists a point v_0 of DualSp X such that $v_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(v_0) \leqslant f(x)$.

Proof: There exists a real number r such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$ by [5, (3)], (10), (12), [7, (3)]. Consider r being a real number such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$. Reconsider $Y = \operatorname{rng}(f \restriction S)$ as a non empty, real-membered set. Consider s being an object such that $s \in S$. For every real number z such that $z \in Y$ holds $|z| < r + 1$ by [18, (62)], [4, (49)]. There exists a point x_0 of DualSp X such that $x_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(x) \leqslant f(x_0)$ by [20, (1)], [18, (62)], [8, (3)], [5, (108), (3)]. Set $N = \inf Y$. Define $\mathcal{Q}$[natural number, point of DualSp X] $\equiv \$_2 \in S$ and $|(f \restriction S)(\$_2) - N| < \frac{1}{\$_1+1}$. For every element x of $\mathbb{N}$, there exists an element y of DualSp X such that $\mathcal{Q}[x, y]$ by [18, (62)], [20, (1)]. Consider s_1 being a function from $\mathbb{N}$ into DualSp X such that for every element n of $\mathbb{N}$, $\mathcal{Q}[n, s_1(n)]$ from [5, Sch. 3]. Consider s_2 being a sequence of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in S$. $f \restriction S$ is weakly* continuous in w*-lim(s_2). $f \restriction S_* s_2$ is convergent and $\lim(f \restriction S_* s_2) = (f \restriction S)(\text{w*-lim}(s_2))$. Consider K being an increasing sequence of $\mathbb{N}$ such that $s_2 = s_1 \cdot K$. □

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