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Elementary Number Theory Problems. Part XVII[1]

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Summary. This paper furthers the formalization of problems from Wacław Sierpiński book "250 Problems in Elementary Number Theory" in the Mizar system. The selected twelve items are 37, 101, 115, 117, 145, 157, 159, 161–163, 165, and 169.

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Introduction

In this paper we continue formalizing the proofs of selected problems from Sierpiński's book "250 Problems in Elementary Number Theory" [29] using the Mizar formalism [15, 26]. The paper is a part of the project *Formalization of Elementary Number Theory in Mizar* [23].

In the preliminary section, we provided some valuable facts about the monotonicity of functions and some properties of finite sequences. Next, as a sequel to Problem 36, credited to [18], with its formal solution presented in [22], we show that for every positive integer s there exists a positive integer n with the sum of digits (in decimal system) equal to s which is divisible by s, which constitutes the solution to Problem 37. Later, with the example of the number 200, we

[1]The Mizar processing has been performed using the infrastructure of the University of Białystok High Performance Computing Center.

falsify the claim from Problem 101 that one can obtain a prime out of every positive integer by changing only one of its digits [11], [16]. The formalized proof of Problem 115 follows a method attributed to A. Schinzel to show that for $k \neq 1$ there exist infinitely many positive integers n such that the number $2^{2^n} + k$ is composite. The assertion $k \neq 1$ is important, since it is still not known if there exist infinitely many composite Fermat numbers [17]. However, there are known examples, e.g. $n = 5$, such that $2^{2^n} + 1$ is composite [28], [27].

Next we provide a proof of Problem 117 showing that specific forms of numbers, namely $2^{2^{2n+1}} + 3$, $2^{2^{4n+1}} + 7$, $2^{2^{6n+2}} + 13$, $2^{2^{10n+1}} + 19$, and $2^{2^{6n+2}} + 21$ are all composite for $n = 1, 2, \ldots$. The only (ordered) solution of the equation in Problem 145 is evidently given by the prime divisors of the number $F_5 = 641 \cdot 6700417$ (famous discovery by Euler which falsified Fermat's conjecture about primality of Fermat numbers; see Proth primes [10]). Then we show the proof of Problem 157 as the equivalence of two theorems: theorem T_1 asserting that there are no positive integers x, y, z for which $x/y + y/z = z/x$ and theorem T_2 asserting that there are no solutions in positive integers u, v, w of the equation $u^3 + v^3 = w^3$ (also attributed to A. Schinzel). In order to prove Problem 159, instead of using the lemma found in Sierpiński's proof, we applied the general property of geometric and arithmetic means already proved in the article [14]. The next three problems formalized in our paper (161, 162 and 163) deal with finding solutions of the same equation, $1/x_1 + 1/x_2 + \ldots + 1/x_s = 1$, for every positive integer s, when $s > 2$ and when $s \neq 2$, respectively. It should be noted that the original proof of Problem 162 contained a typo ($t_{s+2} = 2x_s$ instead of $t_{s+1} = 2x_s$). Then we show that the numbers $1, 4$, and all integers $s \geqslant 6$ form the solution of Problem 165. Finally, we use the inductive argument to justify that since Problem 169 is solvable for $s = 3$ and $s = 4$, it is also solvable for every $s \geqslant 3$.

1. Preliminaries

From now on A denotes a set, a, k, m, n denote natural numbers, s denotes a positive natural number, i, j denote integers, r denotes a real number, and c, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 denote complex numbers.

Let A be a finite set. One can check that $\mathbb{Z}_A$ is finite.

Now we state the propositions:

(1) Let us consider objects a, b, c, d. Then $\operatorname{dom}\langle a, b, c, d\rangle = \{1, 2, 3, 4\}$.

(2) Let us consider objects a, b, c, d, e. Then $\operatorname{dom}\langle a, b, c, d, e\rangle = \{1, 2, 3, 4, 5\}$.

(3) Let us consider an object a, and a set A. If $\langle a\rangle$ is A-valued, then $a \in A$.

Let s be a positive natural number and f be an s-element finite sequence. Observe that $f{\restriction}(s-1)$ is $(s-1)$-element.

Let f be a non empty, positive yielding, real-valued finite sequence. Let us note that $f(1)$ is positive.

Now we state the propositions:

(4) Let us consider an s-element, complex-valued finite sequence f. Then $\sum(f \restriction (s-1)) = \sum f - f(s)$.

(5) Let us consider a non empty set D. Then $D^5 =$ the set of all $\langle d_1, d_2, d_3, d_4, d_5 \rangle$ where d_1, d_2, d_3, d_4, d_5 are elements of D.

(6) Let us consider a non empty set D, and a 5-tuple z of D. Then there exist elements d_1, d_2, d_3, d_4, d_5 of D such that $z = \langle d_1, d_2, d_3, d_4, d_5 \rangle$. The theorem is a consequence of (5).

(7) $\langle c \rangle^{\mathbf{2}} = \langle c^{\mathbf{2}} \rangle$.

(8) $\langle c_1, c_2 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}} \rangle$. The theorem is a consequence of (7).

(9) $\langle c_1, c_2, c_3 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}} \rangle$. The theorem is a consequence of (8) and (7).

(10) $\langle c_1, c_2, c_3, c_4 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}}, {c_4}^{\mathbf{2}} \rangle$. The theorem is a consequence of (9) and (7).

(11) $\langle c_1, c_2, c_3, c_4, c_5 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}}, {c_4}^{\mathbf{2}}, {c_5}^{\mathbf{2}} \rangle$. The theorem is a consequence of (9) and (8).

(12) $\langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}}, {c_4}^{\mathbf{2}}, {c_5}^{\mathbf{2}}, {c_6}^{\mathbf{2}} \rangle$. The theorem is a consequence of (11) and (7).

(13) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}}, {c_4}^{\mathbf{2}}, {c_5}^{\mathbf{2}}, {c_6}^{\mathbf{2}}, {c_7}^{\mathbf{2}} \rangle$. The theorem is a consequence of (11) and (8).

(14) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle^{\mathbf{2}} = \langle {c_1}^{\mathbf{2}}, {c_2}^{\mathbf{2}}, {c_3}^{\mathbf{2}}, {c_4}^{\mathbf{2}}, {c_5}^{\mathbf{2}}, {c_6}^{\mathbf{2}}, {c_7}^{\mathbf{2}}, {c_8}^{\mathbf{2}} \rangle$. The theorem is a consequence of (11) and (9).

(15) $\langle c_1, c_2, c_3, c_4 \rangle^{-1} = \langle {c_1}^{-1}, {c_2}^{-1}, {c_3}^{-1}, {c_4}^{-1} \rangle$.

(16) $\langle c_1, c_2, c_3, c_4, c_5 \rangle^{-1} = \langle {c_1}^{-1}, {c_2}^{-1}, {c_3}^{-1}, {c_4}^{-1}, {c_5}^{-1} \rangle$.

(17) $\langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle^{-1} = \langle {c_1}^{-1}, {c_2}^{-1}, {c_3}^{-1}, {c_4}^{-1}, {c_5}^{-1}, {c_6}^{-1} \rangle$. The theorem is a consequence of (16).

(18) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7 \rangle^{-1} = \langle {c_1}^{-1}, {c_2}^{-1}, {c_3}^{-1}, {c_4}^{-1}, {c_5}^{-1}, {c_6}^{-1}, {c_7}^{-1} \rangle$. The theorem is a consequence of (16).

(19) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle^{-1} = \langle {c_1}^{-1}, {c_2}^{-1}, {c_3}^{-1}, {c_4}^{-1}, {c_5}^{-1}, {c_6}^{-1}, {c_7}^{-1}, {c_8}^{-1} \rangle$. The theorem is a consequence of (16).

(20) $\sum \langle c_1, c_2, c_3, c_4 \rangle = c_1 + c_2 + c_3 + c_4$.

(21) $\sum \langle c_1, c_2, c_3, c_4, c_5 \rangle = c_1 + c_2 + c_3 + c_4 + c_5$.

(22) $\sum \langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6$. The theorem is a consequence of (21).

(23) $\sum\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7\rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7$. The theorem is a consequence of (21).

(24) $\sum\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8$. The theorem is a consequence of (21).

(25) If $m > 1$, then there exists a positive natural number n such that $m^{m^n} > r$.

(26) Let us consider a non zero integer n. Then there exists a natural number k and there exists an odd integer j such that $n = j \cdot 2^k$.

(27) If $i \equiv j \pmod j$, then $i \equiv 0 \pmod j$.

(28) Let us consider natural numbers x, y, n. Suppose $n > 0$ and $x \geqslant y$. Then $x \equiv y \pmod n$ if and only if there exists a natural number k such that $x = k \cdot n + y$.

(29) If $m \leqslant n$ and $i^m \equiv i^{m+k} \pmod a$, then $i^n \equiv i^{n+k} \pmod a$.

(30) $3 \mid 2^{2 \cdot n} - 1$.

(31) $5 \mid 2^{4 \cdot n} - 1$.

(32) $7 \mid 2^{6 \cdot n} - 1$.

(33) $11 \mid 2^{10 \cdot n} - 1$.

(34) $9 \mid 2^{6 \cdot n} - 1$.

Let F be a rational-valued finite sequence. Note that $\sum F$ is rational and $\prod F$ is rational.

Let n be a natural number and f be an $(n+1)$-element finite sequence. One can check that $f_{\restriction 1}$ is n-element.

Let D be a set and f be a D-valued finite sequence. Observe that $f_{\restriction n}$ is D-valued.

Let f be an increasing, real-valued finite sequence. Observe that $f_{\restriction n}$ is increasing.

Let f be a decreasing, real-valued finite sequence. One can verify that $f_{\restriction n}$ is decreasing.

Let f be a non-increasing, real-valued finite sequence. Note that $f_{\restriction n}$ is non-increasing.

Let f be a non-decreasing, real-valued finite sequence. Observe that $f_{\restriction n}$ is non-decreasing.

Let m, n be natural numbers and f be a positive yielding, n-element finite sequence. Observe that $f_{\restriction m}$ is positive yielding.

Let us consider extended real-valued finite sequences f, g. Now we state the propositions:

(35) If $f \frown g$ is decreasing, then f is decreasing and g is decreasing.

Proof: f is decreasing. g is decreasing by [4, (27)]. $\square$

(36) If $f \frown g$ is non-increasing, then f is non-increasing and g is non-increasing.
PROOF: f is non-increasing. g is non-increasing by [4, (27)]. $\square$

(37) If $f \frown g$ is non-decreasing, then f is non-decreasing and g is non-decreasing.
PROOF: f is non-decreasing. g is non-decreasing by [4, (27)]. $\square$

Now we state the propositions:

(38) Let us consider increasing, real-valued finite sequences p, q. If $p(\operatorname{len} p) < q(1)$, then $p \frown q$ is increasing.

(39) Let us consider real numbers r_1, r_2. If $r_1 < r_2$, then $\langle r_1, r_2 \rangle$ is increasing. The theorem is a consequence of (38).

(40) Let us consider real numbers r_1, r_2, r_3. Suppose $r_1 < r_2 < r_3$. Then $\langle r_1, r_2, r_3 \rangle$ is increasing. The theorem is a consequence of (39) and (38).

(41) Let us consider real numbers r_1, r_2, r_3, r_4. Suppose $r_1 < r_2 < r_3 < r_4$. Then $\langle r_1, r_2, r_3, r_4 \rangle$ is increasing. The theorem is a consequence of (40) and (38).

Let n be a natural number. Let us note that there exists a natural-valued finite sequence which is increasing, positive yielding, and n-element.

Now we state the propositions:

(42) Let us consider a positive yielding, real-valued finite sequence f. Suppose $\operatorname{len} f \geqslant 2$. Let us consider a natural number k. If $k \in \operatorname{dom} f$, then $f(k) < \sum f$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \geqslant 2$, then for every positive yielding, real-valued finite sequence f such that $\operatorname{len} f = \$_1$ for every natural number k such that $k \in \operatorname{dom} f$ holds $f(k) < \sum f$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [31, (25)], [20, (7)], [9, (74)], [3, (11)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. $\square$

(43) $n \mapsto c^{-1} = (n \mapsto c)^{-1}$.

Let a be a non positive real number and b be a real number. One can check that $\langle a, b \rangle$ is non positive yielding.

Let a be a real number and b be a non positive real number. Let us observe that $\langle a, b \rangle$ is non positive yielding.

Let a be a non positive real number and b, c be real numbers. Note that $\langle a, b, c \rangle$ is non positive yielding.

Let b be a non positive real number and a, c be real numbers. Let us note that $\langle a, b, c \rangle$ is non positive yielding.

Let c be a non positive real number and a, b be real numbers. Observe that $\langle a, b, c \rangle$ is non positive yielding.

Let a be a non positive real number and b, c, d be real numbers. Observe that $\langle a, b, c, d \rangle$ is non positive yielding.

Let b be a non positive real number and a, c, d be real numbers. Note that $\langle a, b, c, d \rangle$ is non positive yielding.

Let c be a non positive real number and a, b, d be real numbers. Observe that $\langle a, b, c, d \rangle$ is non positive yielding.

Let d be a non positive real number and a, b, c be real numbers. One can verify that $\langle a, b, c, d \rangle$ is non positive yielding.

Let a be a non positive real number and b, c, d, e be real numbers. One can check that $\langle a, b, c, d, e \rangle$ is non positive yielding.

Let b be a non positive real number and a, c, d, e be real numbers. Note that $\langle a, b, c, d, e \rangle$ is non positive yielding.

Let c be a non positive real number and a, b, d, e be real numbers. Let us note that $\langle a, b, c, d, e \rangle$ is non positive yielding.

Let d be a non positive real number and a, b, c, e be real numbers. Observe that $\langle a, b, c, d, e \rangle$ is non positive yielding.

Let e be a non positive real number and a, b, c, d be real numbers. Let us observe that $\langle a, b, c, d, e \rangle$ is non positive yielding.

Now we state the propositions:

(44) Let us consider a positive, natural-valued, 1-element finite sequence f. Then there exists a positive natural number a such that $f = \langle a \rangle$.

(45) Let us consider a positive, natural-valued, 2-element finite sequence f. Then there exist positive natural numbers a, b such that $f = \langle a, b \rangle$.

(46) Let us consider an increasing, positive, natural-valued, 2-element finite sequence f. Then there exist positive natural numbers a, b such that

  (i) $f = \langle a, b \rangle$, and

  (ii) $a < b$.

  The theorem is a consequence of (45).

(47) Let us consider a positive, natural-valued, 3-element finite sequence f. Then there exist positive natural numbers a, b, c such that $f = \langle a, b, c \rangle$.

(48) Let us consider an increasing, positive, natural-valued, 3-element finite sequence f. Then there exist positive natural numbers a, b, c such that

  (i) $f = \langle a, b, c \rangle$, and

  (ii) $a < b < c$.

  The theorem is a consequence of (47).

Let us consider complex-valued functions f, g. Now we state the propositions:

(49) If $c \cdot f = c \cdot g$ and $c \neq 0$, then $f = g$.

(50) $(f \cdot g)^{-1} = f^{-1} \cdot g^{-1}$.

(51) $(c_1 \cdot f) \cdot (c_2 \cdot g) = c_1 \cdot c_2 \cdot (f \cdot g)$.

Now we state the propositions:

(52) Let us consider complex-valued finite sequences f_1, f_2, g_1, g_2, h_1, h_2. Suppose $\operatorname{len} f_1 = \operatorname{len} g_1 = \operatorname{len} h_1$. Then $((f_1 \frown f_2) \cdot (g_1 \frown g_2)) \cdot (h_1 \frown h_2) = ((f_1 \cdot g_1) \cdot h_1) \frown ((f_2 \cdot g_2) \cdot h_2)$.

(53) Let us consider a natural number n, and an object e. Then $\mathbb{Z}_{n+1} \longmapsto e = \langle e \rangle \frown (\mathbb{Z}_n \longmapsto e)$.
PROOF: Set $p = \mathbb{Z}_n \longmapsto e$. Set $q = \mathbb{Z}_{n+1} \longmapsto e$. For every natural number k such that $k \in \operatorname{dom}\langle e \rangle$ holds $q(k) = \langle e \rangle(k)$ by [2, (49)], [30, (7)], [3, (44)]. For every natural number k such that $k \in \operatorname{dom} p$ holds $q(\operatorname{len}\langle e \rangle + k) = p(k)$ by [3, (44)], [30, (7)], [32, (34)]. $\square$

2. PROBLEM 37

Now we state the propositions:

(54) Let us consider natural numbers n, b. Suppose $n > 0$ and $b > 1$. Then $\operatorname{digits}(n \cdot b, b) = \langle 0 \rangle \frown \operatorname{digits}(n, b)$.
PROOF: Set $d = \operatorname{digits}(n, b)$. Consider d' being a finite 0-sequence of $\mathbb{N}$ such that $\operatorname{dom} d' = \operatorname{dom} d$ and for every natural number i such that $i \in \operatorname{dom} d'$ holds $d'(i) = d(i) \cdot b^i$ and $n = \sum d'$. Define $\mathcal{A} = \operatorname{len} d'$. Define $\mathcal{F}(\text{natural number}) = b \cdot d'(\$_1)$. Consider d_1 being a finite 0-sequence such that $\operatorname{len} d_1 = \mathcal{A}$ and for every natural number n such that $n \in \mathcal{A}$ holds $d_1(n) = \mathcal{F}(n)$ from [32, Sch. 2]. $\operatorname{rng} d_1 \subseteq \mathbb{N}$. $\square$

(55) Let us consider natural numbers n, b. Suppose $n > 0$ and $b > 1$. Let us consider a natural number i. Then $\operatorname{digits}(n \cdot b^i, b) = (i \longmapsto 0) \frown \operatorname{digits}(n, b)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \operatorname{digits}(n \cdot b^{\$_1}, b) = (\$_1 \longmapsto 0) \frown \operatorname{digits}(n, b)$. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. $\square$

(56) Let us consider a natural number s. Then there exists a natural number n such that

(i) $\sum \operatorname{digits}(n, 10) = s$, and

(ii) $s \mid n$.

The theorem is a consequence of (55).

3. PROBLEM 101

Now we state the proposition:

(57) Let us consider sets a, b. Then $\operatorname{Replace}(\langle a \rangle, 0, b) = \langle b \rangle$.

Let us consider sets a, b, c. Now we state the propositions:

(58) $\mathrm{Replace}(\langle a, b\rangle, 0, c) = \langle c, b\rangle$.

(59) $\mathrm{Replace}(\langle a, b\rangle, 1, c) = \langle a, c\rangle$.

Let us consider sets a, b, c, d. Now we state the propositions:

(60) $\mathrm{Replace}(\langle a, b, c\rangle, 0, d) = \langle d, b, c\rangle$.

(61) $\mathrm{Replace}(\langle a, b, c\rangle, 1, d) = \langle a, d, c\rangle$.

(62) $\mathrm{Replace}(\langle a, b, c\rangle, 2, d) = \langle a, b, d\rangle$.

Now we state the propositions:

(63) $\mathrm{digits}(201, 10) = \langle 1, 0, 2\rangle$.

(64) $\mathrm{digits}(202, 10) = \langle 2, 0, 2\rangle$.

(65) $\mathrm{digits}(203, 10) = \langle 3, 0, 2\rangle$.

(66) $\mathrm{digits}(205, 10) = \langle 5, 0, 2\rangle$.

(67) $\mathrm{digits}(206, 10) = \langle 6, 0, 2\rangle$.

(68) Let us consider elements i, j of $\mathbb{N}$. Suppose $0 \leqslant j \leqslant 9$. Then value(Replace(digits(20 is not prime. The theorem is a consequence of (60), (63), (64), (65), (66), (67), and (62).

4. Problem 115

Now we state the propositions:

(69) A consequence of Euler's theorem:
Let us consider natural numbers a, n, x, y. Suppose a and n are relatively prime and $a \neq 0$ and $n \neq 0$ and $x \equiv y \pmod{\mathrm{Euler}\, n}$. Then $a^x \equiv a^y \pmod{n}$. The theorem is a consequence of (28).

(70) Let us consider a natural number a, and an integer k. Suppose $k \neq 1$. Then there exists a positive natural number n such that

(i) $a < 2^{2^n} + k$, and

(ii) $2^{2^n} + k$ is composite.

The theorem is a consequence of (26), (25), (29), and (69).

Let us consider an integer k. Now we state the propositions:

(71) Suppose $k \neq 1$. Then $\{n$, where n is a positive natural number : $2^{2^n} + k$ is composite$\} \approx \{2^{2^n} + k$, where n is a positive natural number : $2^{2^n} + k$ is composite$\}$.
Proof: Set $X = \{n$, where n is a positive natural number : $2^{2^n}+k$ is composite$\}$. Set $Y = \{2^{2^n}+k$, where n is a positive natural number : $2^{2^n}+k$ is composite$\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every m such that $m = \$_1$ holds $\$_2 = 2^{2^m}+k$.

For every object x such that $x \in X$ there exists an object y such that $y \in Y$ and $\mathcal{P}[x, y]$. Consider f being a function from X into Y such that for every object x such that $x \in X$ holds $\mathcal{P}[x, f(x)]$ from [8, Sch. 1]. f is one-to-one by [12, (30)]. Consider n being a positive natural number such that $y = 2^{2^n} + k$ and $2^{2^n} + k$ is composite. $\square$

(72) If $k \neq 1$, then $\{2^{2^n} + k$, where n is a positive natural number : $2^{2^n} + k$ is composite$\}$ is infinite. The theorem is a consequence of (70).

(73) If $k \neq 1$, then $\{n$, where n is a positive natural number : $2^{2^n} + k$ is composite$\}$ is infinite. The theorem is a consequence of (71) and (72).

5. Problem 117

Now we state the propositions:

(74) $7 \mid 2^{2^{2 \cdot s+1}} + 3$. The theorem is a consequence of (30) and (27).

(75) $2^{2^{2 \cdot s+1}} + 3$ is composite. The theorem is a consequence of (74).

(76) $11 \mid 2^{2^{4 \cdot s+1}} + 7$. The theorem is a consequence of (31) and (27).

(77) $2^{2^{4 \cdot s+1}} + 7$ is composite. The theorem is a consequence of (76).

(78) $29 \mid 2^{2^{6 \cdot s+2}} + 13$. The theorem is a consequence of (32) and (27).

(79) $2^{2^{6 \cdot s+2}} + 13$ is composite. The theorem is a consequence of (78).

(80) $23 \mid 2^{2^{10 \cdot s+1}} + 19$. The theorem is a consequence of (33) and (27).

(81) $2^{2^{10 \cdot s+1}} + 19$ is composite. The theorem is a consequence of (80).

(82) $37 \mid 2^{2^{6 \cdot s+2}} + 21$. The theorem is a consequence of (34) and (27).

(83) $2^{2^{6 \cdot s+2}} + 21$ is composite. The theorem is a consequence of (82).

6. Problem 145

Now we state the propositions:

(84) $\mathrm{Fermat}\,5 = 641 \cdot 6700417$.

(85) Let us consider positive natural numbers x, y. Then $x \cdot y + x + y = 2^{32}$ if and only if $x = 640$ and $y = 6700416$ or $x = 6700416$ and $y = 640$.

Proof: If $x \cdot y + x + y = 2^{32}$, then $x = 640$ and $y = 6700416$ or $x = 6700416$ and $y = 640$ by (84), [19, (59)], [**?** , (3)]. $\square$

7. Problem 157

Let a be a non positive integer and n be an odd natural number. One can check that a^n is non positive.

Now we state the propositions:

(86) Let us consider natural numbers a, b, c, d, n. Suppose a, b, c are mutually coprime and $a \cdot b \cdot c = d^n$. Then there exists a natural number k such that $a = k^n$.

(87) for every positive natural numbers x, y, z, $\frac{x}{y} + \frac{y}{z} \neq \frac{z}{x}$ if and only if for every positive natural numbers u, v, w, $u^3 + v^3 \neq w^3$.

Proof: If for every positive natural numbers x, y, z, $\frac{x}{y} + \frac{y}{z} \neq \frac{z}{x}$, then for every positive natural numbers u, v, w, $u^3 + v^3 \neq w^3$ by [25, (2)]. Set $a = x^{\mathbf{2}} \cdot z$. Set $b = y^{\mathbf{2}} \cdot x$. Set $d = \gcd(a, b)$. Consider a_1, b_1 being natural numbers such that $a = d \cdot a_1$ and $b = d \cdot b_1$ and a_1 and b_1 are relatively prime. Consider t being a natural number such that $x \cdot y \cdot z = d \cdot t$. Consider u being a natural number such that $u^3 = a_1$. Consider v being a natural number such that $v^3 = a_1 + b_1$. Consider w being a natural number such that $w^3 = b_1$. □

8. Problem 159

Let a, b, c be positive real numbers. One can check that $\langle a, b, c\rangle$ is positive yielding.

Let a, b, c, d be positive real numbers. One can check that $\langle a, b, c, d\rangle$ is positive yielding.

Now we state the proposition:

(88) Let us consider complex numbers a, b, c, d. Then $\prod\langle a, b, c, d\rangle = a \cdot b \cdot c \cdot d$.

Let us consider a positive real number a. Now we state the propositions:

(89) $\sqrt[2]{a} = \sqrt{a}$.

(90) $\mathrm{GMean}\langle a\rangle = a$.

Now we state the propositions:

(91) Let us consider positive real numbers a, b. Then $(\mathrm{GMean}\langle a, b\rangle)^2 = a \cdot b$.

(92) Let us consider positive real numbers a, b, c. Then $(\mathrm{GMean}\langle a, b, c\rangle)^3 = a \cdot b \cdot c$.

(93) Let us consider positive real numbers a, b, c, d. Then $(\mathrm{GMean}\langle a, b, c, d\rangle)^4 = a \cdot b \cdot c \cdot d$. The theorem is a consequence of (88).

(94) Let us consider a real number a. Then $\mathrm{Mean}\langle a\rangle = a$.

(95) Let us consider real numbers a, b. Then $\mathrm{Mean}\langle a, b\rangle = \frac{a+b}{2}$.

(96) Let us consider real numbers a, b, c. Then $\mathrm{Mean}\langle a, b, c\rangle = \frac{a+b+c}{3}$.

(97) Let us consider real numbers a, b, c, d. Then $\mathrm{Mean}\langle a, b, c, d\rangle = \frac{a+b+c+d}{4}$. The theorem is a consequence of (20).

(98) Let us consider objects w, x, y, z. Suppose $\langle w, x, y, z\rangle$ is constant. Then $w = x = y = z$.

(99) Let us consider positive real numbers a, b, c, d. If $a \cdot b \cdot c \cdot d \geqslant (\frac{a+b+c+d}{4})^4$, then $a = b = c = d$. The theorem is a consequence of (88), (93), (97), and (98).

(100) Let us consider a real number m. Suppose $0 \leqslant m < 4$. Then there exist no positive integers x, y, z, t such that $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = m$.

(101) $\{\langle x, y, z, t\rangle$, where x, y, z, t are positive integers : $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4\} = \{\langle x, y, z, t\rangle$, where x, y, z, t are positive integers : $x = y = z = t\}$. The theorem is a consequence of (99).

Let a, b be natural numbers. Let us note that $\langle a, b\rangle (\in \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b\rangle$.

Let a, b, c be natural numbers. One can verify that $\langle a, b, c\rangle (\in \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b, c\rangle$.

Let a, b, c, d be natural numbers. Let us observe that $\langle a, b, c, d\rangle (\in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b, c, d\rangle$.

Let a be a positive natural number. Observe that $a (\in \mathbb{N}_+)$ reduces to a.

Let a, b be positive natural numbers. Let us observe that $\langle a, b\rangle (\in \mathbb{N}_+ \times \mathbb{N}_+)$ reduces to $\langle a, b\rangle$.

Let a, b, c be positive natural numbers. Note that $\langle a, b, c\rangle (\in \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+)$ reduces to $\langle a, b, c\rangle$.

Let a, b, c, d be positive natural numbers. One can check that $\langle a, b, c, d\rangle (\in \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+)$ reduces to $\langle a, b, c, d\rangle$.

The functor Cart4Id yielding a function from $\mathbb{N}_+$ into $\mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+$ is defined by

(Def. 1) for every positive natural number n, $it(n) = \langle n, n, n, n\rangle$.

Observe that Cart4Id is one-to-one.

Now we state the propositions:

(102) $\mathrm{rng}\,\mathrm{Cart4Id} \subseteq \{\langle x, y, z, t\rangle$, where x, y, z, t are positive integers : $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4\}$.

(103) $\{\langle x, y, z, t\rangle$, where x, y, z, t are positive integers : $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4\}$ is infinite. The theorem is a consequence of (102).

9. Problem 161

Let R be a binary relation. The functor $\boxed{\text{Permutations}(R)}$ yielding a set is defined by the term

(Def. 2) the set of all $p \cdot R$ where p is a permutation of dom R.

Let f be a function. One can check that the functor Permutations(f) yields a set and is defined by the term

(Def. 3) the set of all $f \cdot p$ where p is a permutation of dom f.

Let A be a set. We say that A is relational if and only if

(Def. 4) for every object a such that $a \in A$ holds a is a binary relation.

Let a, b be objects. One can check that $\{\{\langle a, b \rangle\}\}$ is relational and there exists a set which is non empty, finite, and relational.

Let A be a relational set. Let us note that every element of A is relation-like and every set which is functional is also relational.

Let R be a finite binary relation. One can check that Permutations(R) is finite.

Let A be a relational set. The functor $\boxed{\text{Permutations}(A)}$ yielding a set is defined by the term

(Def. 5) $\{\text{Permutations}(f)$, where f is an element of $A : f \in A\}$.

Let A be a finite, relational set. One can check that Permutations(A) is finite.

Let f be a complex-valued finite sequence. We say that $\boxed{f \text{ has sum of reciprocals equal}}$ if and only if

(Def. 6) $\sum f^{-1} = 1$.

Observe that $\langle 1 \rangle$ has sum of reciprocals equal 1 and there exists a complex-valued finite sequence which has sum of reciprocals equal 1.

Let s be a natural number. The functor $\boxed{\text{SolutionsOfReciprocalsSum}(s)}$ yielding a set is defined by the term

(Def. 7) $\{f$, where f is a positive yielding, s-element, natural-valued finite sequence : f has sum of reciprocals equal $1\}$.

Let s be a zero natural number. Note that SolutionsOfReciprocalsSum(s) is empty.

Now we state the proposition:

(104) $\sum(s \mapsto s^{-1}) = 1$.

Let s be a positive natural number. Let us note that $s \mapsto s$ has sum of reciprocals equal 1.

Now we state the proposition:

(105) $s \mapsto s \in \text{SolutionsOfReciprocalsSum}(s)$.

Let s be a positive natural number. Let us observe that SolutionsOfReciprocalsSum(s) is non empty.

Let A, B be finite sequence-membered sets. The functor $A \frown B$ yielding a set is defined by the term

(Def. 8) $\{p \frown q$, where p is an element of A, q is an element of $B : p \in A$ and $q \in B\}$.

One can verify that $A \frown B$ is finite sequence-membered.

Let us consider a finite sequence-membered set A. Now we state the propositions:

(106) $A \frown \emptyset \subseteq A$.

(107) $\emptyset \frown A \subseteq A$.

Let A, B be finite, finite sequence-membered sets. Let us note that $A \frown B$ is finite.

Let D, A be sets. We say that A is D-FinSequence-membered if and only if

(Def. 9) for every object f such that $f \in A$ holds f is a D-valued finite sequence.

Let D be a non empty set and d be an element of D. Let us note that $\{\langle d \rangle\}$ is D-FinSequence-membered and there exists a set which is D-FinSequence-membered.

Let A be a D-FinSequence-membered set. The functor decomp A yielding a binary relation is defined by

(Def. 10) for every objects x, y, $\langle x, y \rangle \in it$ iff $x \in D$ and there exists a finite sequence f of elements of D such that $y = f$ and $\langle x \rangle \frown f \in A$.

Let D be a set. Let us note that every set which is D-FinSequence-membered is also finite sequence-membered.

Let A be a set. The functor OneFS(A) yielding a many sorted set indexed by A is defined by

(Def. 11) for every object a such that $a \in A$ holds $it(a) = \langle a \rangle$.

Let us observe that OneFS(A) is finite sequence-yielding.

Let f be a finite sequence-yielding function. One can check that rng f is finite sequence-membered.

Let us consider n. Let f be an n-element, real-valued finite sequence. One can verify that $\mathrm{sort}_{\mathrm{a}}\, f$ is n-element and $\mathrm{sort}_{\mathrm{d}}\, f$ is n-element.

Now we state the propositions:

(108) Let us consider complex-valued finite sequences f, g, and a permutation P of dom g. Suppose $f = g \cdot P$ and len $g \geqslant 1$. Then $f^{-1} = g^{-1} \cdot P$.
PROOF: Reconsider $k = g^{-1} \cdot P$ as a finite sequence of elements of $\mathbb{C}$. For every natural number i such that $i \in \mathrm{dom}(f^{-1})$ holds $(f^{-1})(i) = k(i)$ by

[7, (12)]. □

(109) Let us consider complex-valued finite sequences f, g. Suppose f and g are fiberwise equipotent. Then f^{-1} and g^{-1} are fiberwise equipotent. The theorem is a consequence of (108).

(110) Let us consider a natural number s, and a non-decreasing, positive, $(s+1)$-element, natural-valued finite sequence f. Then

(i) $f(1) \leqslant \frac{s+1}{\sum f^{-1}}$, and

(ii) $\sum (f{\restriction}1)^{-1} = \sum f^{-1} - \frac{1}{f(1)}$.

Proof: Define $\mathcal{Q}$[finite sequence of elements of $\mathbb{N}$] $\equiv$ for every non-decreasing, positive, natural-valued finite sequence F such that $F = \$_1$ holds $\sum \$_1{}^{-1} \leqslant \frac{\operatorname{len} \$_1}{\$_1(1)}$. For every finite sequence p of elements of $\mathbb{N}$ and for every element x of $\mathbb{N}$ such that $\mathcal{Q}[p]$ holds $\mathcal{Q}[p \mathbin{^\frown} \langle x \rangle]$ by [31, (154)], [24, (63)], [3, (14)], [4, (34), (39)]. For every finite sequence p of elements of $\mathbb{N}$, $\mathcal{Q}[p]$ from [5, Sch. 2]. □

Let us consider a natural number s and a rational number w. Now we state the propositions:

(111) $\{f$, where f is a non-decreasing, positive, s-element, natural-valued finite sequence : $\sum f^{-1} = w\}$ is finite.

Proof: Define $\mathcal{A}$(natural number, rational number) $= \{f$, where f is a non-decreasing, positive, $\$_1$-element, natural-valued finite sequence : $\sum f^{-1} = \$_2\}$. Define $\mathcal{P}$[natural number] $\equiv$ for every rational number q, $\mathcal{A}(\$_1, q)$ is finite. $\mathcal{P}[0]$. For every natural number s such that $\mathcal{P}[s]$ holds $\mathcal{P}[s+1]$ by [6, (86)], [4, (103)], (110), [1, (7)]. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(112) $\{f$, where f is a positive yielding, s-element, natural-valued finite sequence : $\sum f^{-1} = w\}$ is finite.

Proof: Set $A = \{f$, where f is a positive , s-element , natural-valued finite sequence : $\sum f^{-1} = w\}$. Set $I = \{f$, where f is a non-decreasing , positive , s-elemen sequence : $\sum f^{-1} = w\}$. I is functional. Reconsider $I_1 = I$ as a functional set. I is finite. For every set X such that $X \in \operatorname{Permutations}(I_1)$ holds X is finite. Define $\mathcal{S}$(natural-valued finite sequence) $= \operatorname{sort_a} \$_1$. $A \subseteq \bigcup \operatorname{Permutations}(I_1)$ by [20, (4)], (109), [20, (9)]. □

Let s be a natural number. Note that SolutionsOfReciprocalsSum(s) is finite.

10. Problem 162

Observe that $\langle 2, 3 \rangle$ is increasing and $\langle 2, 3, 6 \rangle$ is increasing and $\langle 2, 3, 7, 42 \rangle$ is increasing and $\langle 2, 3, 8, 24 \rangle$ is increasing.

Let f be a complex-valued finite sequence. The functors: $\boxed{\text{Sierp162FS}(f)}$ and $\boxed{\text{Sierp162FS2}(f)}$ yielding finite sequences are defined by terms

(Def. 12) $\langle 2 \rangle \frown (2 \cdot f)$,

(Def. 13) $\langle 2, 3 \rangle \frown (6 \cdot f)$,

respectively. One can check that Sierp162FS(f) is complex-valued and Sierp162FS2(f) is complex-valued.

Let f be a real-valued finite sequence. One can verify that Sierp162FS(f) is real-valued and Sierp162FS2(f) is real-valued.

Let f be a rational-valued finite sequence. Let us observe that Sierp162FS(f) is rational-valued and Sierp162FS2(f) is rational-valued.

Let f be an integer-valued finite sequence. Observe that Sierp162FS(f) is integer-valued and Sierp162FS2(f) is integer-valued.

Let f be a natural-valued finite sequence. Let us note that Sierp162FS(f) is natural-valued and Sierp162FS2(f) is natural-valued.

Let s be a natural number and f be an s-element, complex-valued finite sequence. Let us note that Sierp162FS(f) is $(s+1)$-element and Sierp162FS2(f) is $(s+2)$-element and there exists a natural-valued finite sequence which is increasing and s-element.

Now we state the propositions:

(113) Let us consider a natural number s. Suppose $s \geqslant 2$. Let us consider a positive yielding, s-element, real-valued finite sequence f. Suppose f has sum of reciprocals equal 1. Let us consider a natural number k. If $k \in \operatorname{dom} f$, then $f(k) > 1$. The theorem is a consequence of (42).

(114) Let us consider a natural number s. Suppose $s \geqslant 2$. Let us consider an increasing, positive yielding, s-element, real-valued finite sequence f. Suppose f has sum of reciprocals equal 1. Then Sierp162FS(f) is increasing. The theorem is a consequence of (113).

(115) Let us consider an increasing, positive yielding, s-element, natural-valued finite sequence f. Then Sierp162FS2(f) is increasing. The theorem is a consequence of (38).

Let us consider a complex-valued finite sequence f. Now we state the propositions:

(116) If f has sum of reciprocals equal 1, then Sierp162FS(f) has sum of reciprocals equal 1.

(117) If f has sum of reciprocals equal 1, then Sierp162FS2(f) has sum of reciprocals equal 1.

Now we state the propositions:

(118) $\langle 2, 3, 6 \rangle$ has sum of reciprocals equal 1.

(119) $\langle 2, 3, 7, 42\rangle$ has sum of reciprocals equal 1. The theorem is a consequence of (20).

(120) $\langle 2, 3, 8, 24\rangle$ has sum of reciprocals equal 1. The theorem is a consequence of (20).

(121) Let us consider a natural number s. Suppose $s > 2$. Then there exists an increasing, positive yielding, s-element, natural-valued finite sequence f such that f has sum of reciprocals equal 1.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ there exists an increasing, positive, $\$_1$-element, natural-valued finite sequence f such that f has sum of reciprocals equal 1. $\mathcal{P}[3]$. For every natural number j such that $3 \leqslant j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$. For every natural number i such that $3 \leqslant i$ holds $\mathcal{P}[i]$ from [3, Sch. 8]. $\square$

Let s be a natural number. The functor $\boxed{\text{IncreasingSolutionsOfReciprocalsSum}(s)}$ yielding a set is defined by the term

(Def. 14) $\{f$, where f is an increasing, positive yielding, s-element, natural-valued finite sequence : f has sum of reciprocals equal $1\}$.

Now we state the proposition:

(122) Let us consider a natural number s. Then IncreasingSolutionsOfReciprocalsSum(s)
SolutionsOfReciprocalsSum(s).

Let s be a positive natural number. Let us observe that IncreasingSolutionsOfReciproca
is finite.

Now we state the propositions:

(123) IncreasingSolutionsOfReciprocalsSum$(3) = \{\langle 2, 3, 6\rangle\}$.
PROOF: Set $S =$ IncreasingSolutionsOfReciprocalsSum(3). $S \subseteq \{\langle 2, 3, 6\rangle\}$ by (48), [24, (65)], [21, (35)], [3, (13)]. $\square$

(124) $\{\langle 2, 3, 7, 42\rangle, \langle 2, 3, 8, 24\rangle\} \subseteq$ IncreasingSolutionsOfReciprocalsSum(4). The theorem is a consequence of (119) and (120).

(125) Let us consider a positive natural number s. Suppose $s > 2$. Then $\overline{\overline{\text{IncreasingSolutionsOfReciprocalsSum}(s)}} < \overline{\overline{\text{IncreasingSolutionsOfReciprocalsSum}(}}$
The theorem is a consequence of (123), (124), (121), (114), (116), (115), (117), and (49).

11. Problem 163

Let n be a natural number. One can verify that Triangle n is triangular.

Let a be a triangular natural number. One can check that $\langle a\rangle$ is triangular-valued and there exists a finite sequence which is triangular-valued.

Let f, g be triangular-valued finite sequences. One can check that $f \frown g$ is triangular-valued.

Let a, b be triangular natural numbers. One can verify that $\langle a, b\rangle$ is triangular-valued.

Let a, b, c be triangular natural numbers. Let us note that $\langle a, b, c\rangle$ is triangular-valued.

Let a, b, c, d be triangular natural numbers. Let us note that $\langle a, b, c, d\rangle$ is triangular-valued.

Let a, b, c, d, e be triangular natural numbers. One can verify that $\langle a, b, c, d, e\rangle$ is triangular-valued.

Let a, b, c, d, e, f be triangular natural numbers. One can verify that $\langle a, b, c, d, e, f\rangle$ is triangular-valued.

Now we state the propositions:

(126) $\langle \mathrm{Triangle}\,1\rangle$ has sum of reciprocals equal 1.

(127) $\langle \mathrm{Triangle}\,2, \mathrm{Triangle}\,2, \mathrm{Triangle}\,2\rangle$ has sum of reciprocals equal 1.

(128) $\langle \mathrm{Triangle}\,2, \mathrm{Triangle}\,2, \mathrm{Triangle}\,3, \mathrm{Triangle}\,3\rangle$ has sum of reciprocals equal 1. The theorem is a consequence of (15) and (20).

(129) $\langle \mathrm{Triangle}\,3, \mathrm{Triangle}\,3, \mathrm{Triangle}\,3, \mathrm{Triangle}\,3, \mathrm{Triangle}\,3, \mathrm{Triangle}\,3\rangle$ has sum of reciprocals equal 1. The theorem is a consequence of (17) and (22).

Let n, a be natural numbers. Assume $n \geqslant a+1$. The functor $\boxed{\mathrm{TriangleShift2}(n, a)}$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by

(Def. 15) $\mathrm{len}\, it = n - a - 1$ and for every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, it$ holds $it(i) = \mathrm{Triangle}(a + i)$.

One can verify that $\mathrm{TriangleShift2}(3, 2)$ is empty.

Now we state the propositions:

(130) $\mathrm{TriangleShift2}(4, 2) = \langle \mathrm{Triangle}\,3\rangle$.

(131) If $n \geqslant a + 1$, then $\mathrm{TriangleShift2}(n + 1, a) = \mathrm{TriangleShift2}(n, a) \frown \langle \mathrm{Triangle}\, n\rangle$.

(132) If $a + 1 \leqslant n$, then $\mathrm{TriangleShift2}(n, a)$ is triangular-valued.

Let us consider m and n. Note that $m \mapsto \mathrm{Triangle}\, n$ is triangular-valued.

Now we state the propositions:

(133) If $3 \leqslant n$, then $\sum(\mathrm{TriangleShift2}(n, 2))^{-1} = \frac{2}{3} - \frac{n+1}{\mathrm{Triangle}\, n}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \sum(\mathrm{TriangleShift2}(\$_1, 2))^{-1} = \frac{2}{3} - \frac{\$_1+1}{\mathcal{T}(\$_1)}$. For every natural number j such that $3 \leqslant j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j + 1]$ by [13, (10), (19)], [24, (63)], (131). For every natural number i such that $3 \leqslant i$ holds $\mathcal{P}[i]$ from [3, Sch. 8]. $\square$

(134) If $s \neq 2$, then there exists a natural-valued, s-element finite sequence f such that f is triangular-valued and has sum of reciprocals equal 1. The

theorem is a consequence of (127), (128), (132), (133), (43), and (129).

12. Problem 165

Let f be a complex-valued finite sequence. We say that f has sum of squares of recipro
if and only if

(Def. 16) $\sum (f^{-1})^{\mathbf{2}} = 1$.

Now we state the propositions:

(135) $\langle 1 \rangle$ has sum of squares of reciprocals equal 1.

(136) $\langle 2, 2, 2, 2 \rangle$ has sum of squares of reciprocals equal 1. The theorem is a consequence of (15) and (10).

(137) $\langle 2, 2, 2, 3, 3, 6 \rangle$ has sum of squares of reciprocals equal 1. The theorem is a consequence of (17), (12), and (22).

(138) $\langle 2, 2, 2, 4, 4, 4, 4 \rangle$ has sum of squares of reciprocals equal 1. The theorem is a consequence of (18), (13), and (23).

(139) $\langle 2, 2, 2, 3, 3, 7, 14, 21 \rangle$ has sum of squares of reciprocals equal 1. The theorem is a consequence of (19), (14), and (24).

(140) Let us consider an s-element, natural-valued finite sequence f. Suppose f has sum of squares of reciprocals equal 1. Then $(f {\restriction} (s-1)) \frown \langle 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s) \rangle$ has sum of squares of reciprocals equal 1. The theorem is a consequence of (15), (10), (20), and (4).

Let s be a positive natural number, a, b, c, d be positive real numbers, and f be an s-element, positive, real-valued finite sequence. One can check that $\langle a \cdot f(s), b \cdot f(s), c \cdot f(s), d \cdot f(s) \rangle$ is positive yielding.

Now we state the proposition:

(141) Let us consider an s-element, positive, natural-valued finite sequence f. Suppose f has sum of squares of reciprocals equal 1. Then there exists an $(s + 3 \cdot n)$-element, positive, natural-valued finite sequence g such that g has sum of squares of reciprocals equal 1.

PROOF: Set $g = (f {\restriction} (s-1)) \frown \langle 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s) \rangle$. Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an $(s+3 \cdot \$_1)$-element, positive, natural-valued finite sequence g such that g has sum of squares of reciprocals equal 1. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$ from [3, Sch. 2]. $\square$

The scheme *STEP2a* deals with a natural number $\mathcal{N}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every natural number k such that $k \geqslant \mathcal{N}$ holds $\mathcal{P}[k]$

provided

- $\mathcal{P}[\mathcal{N}]$ and
- $\mathcal{P}[\mathcal{N}+1]$ and
- for every natural number k such that $k \geqslant \mathcal{N}$ and $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k+2]$.

The scheme *STEP2* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 2) For every natural number k, $\mathcal{P}[k]$

provided

- for every natural number k, $\mathcal{P}[2 \cdot k]$ and $\mathcal{P}[2 \cdot k+1]$.

The scheme *STEP3* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 3) For every natural number k, $\mathcal{P}[k]$

provided

- for every natural number k, $\mathcal{P}[3 \cdot k]$ and $\mathcal{P}[3 \cdot k+1]$ and $\mathcal{P}[3 \cdot k+2]$.

The scheme *STEP4* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 4) For every natural number k, $\mathcal{P}[k]$

provided

- for every natural number k, $\mathcal{P}[4 \cdot k]$ and $\mathcal{P}[4 \cdot k+1]$ and $\mathcal{P}[4 \cdot k+2]$ and $\mathcal{P}[4 \cdot k+3]$.

Now we state the proposition:

(142) there exists an s-element, positive, natural-valued finite sequence f such that f has sum of squares of reciprocals equal 1 if and only if $s = 1$ or $s = 4$ or $s \geqslant 6$.

PROOF: Define $\mathcal{S}$[natural number] $\equiv$ there exists a $\$_1$-element, positive, natural-valued finite sequence f such that f has sum of squares of reciprocals equal 1. If $\mathcal{S}[s]$, then $s = 1$ or $s = 4$ or $s \geqslant 6$ by [4, (103)], [5, (100)], [24, (64)], (8). $\square$

13. PROBLEM 169

Let s be a natural number and f be an $(s+1)$-element, complex-valued finite sequence. We say that f satisfies 169th Sierpiński problem if and only if

(Def. 17) $\sum(((f\restriction s) \cdot (f\restriction s)) \cdot (f\restriction s))^{-1} = \frac{1}{f(s+1)^3}$.

Let a, b, c, d be objects. One can verify that $\langle a, b, c, d\rangle$ is $(3+1)$-element.

Now we state the proposition:

(143) $\langle 12, 15, 20, 10\rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50).

Let a, b, c, d, e be objects. Let us note that $\langle a, b, c, d, e\rangle$ is $(4+1)$-element.

Now we state the proposition:

(144) $\langle 455, 780, 1092, 5460, 420\rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50), (15), and (20).

Let s be a positive natural number, n be a natural number, c be a complex number, f be an $(s+n)$-element, complex-valued finite sequence, and g be a 4-element finite sequence. Let us note that $(c\cdot(f\restriction(s-1)))^\frown g$ is $(s+2+1)$-element.

Now we state the proposition:

(145) Let us consider an $(s+1)$-element, complex-valued finite sequence f. Suppose f satisfies 169th Sierpiński problem. Then $(10\cdot(f\restriction(s-1)))^\frown\langle 12\cdot f(s), 15\cdot f(s), 20\cdot f(s), 10\cdot f(s+1)\rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50), (52), and (51).

Let s be a positive natural number, a, b, c, d be positive real numbers, and f be an $(s+1)$-element, positive, real-valued finite sequence. One can verify that $\langle a\cdot f(s), b\cdot f(s), c\cdot f(s), d\cdot f(s+1)\rangle$ is positive yielding.

Now we state the proposition:

(146) Let us consider an $(s+1)$-element, complex-valued finite sequence f. Suppose f satisfies 169th Sierpiński problem. Then $n\cdot f$ satisfies 169th Sierpiński problem. The theorem is a consequence of (51).

Let s be a positive natural number. Assume $s \geqslant 3$.

A solution of 169th Sierpiński problem of s is an $(s+1)$-element, positive yielding, natural-valued finite sequence defined by

(Def. 18) *it* satisfies 169th Sierpiński problem.

Now we state the propositions:

(147) Suppose $s \geqslant 3$. Let us consider a solution of 169th Sierpiński problem f of s. Then $\operatorname{rng}\mathrm{SolutionsofSierp168}(f) \subseteq$ the set of all g where g is a solution of 169th Sierpiński problem of s. The theorem is a consequence of (146).

(148) Let us consider a positive natural number s. Suppose $s \geqslant 3$. Then the set of all f where f is a solution of 169th Sierpiński problem of s is infinite. The theorem is a consequence of (147).

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