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Extensions of Languages in Polish Notation[1]

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Summary. A collection of basic tools for adding new operators to a language in Polish notation. A need for this arose while attempting to formalize some extensions of Roman Suszko's basic non-Fregean logic SCI, namely WB and WH, described in [3]. The formalizations have been submitted for publication in *Formalized Mathematics.*

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From now on a, b denote objects, k, l, m, n denote natural numbers, p, q, r, s denote finite sequences, P denotes a non empty, finite sequence-membered set, S, T denote Polish languages, V denotes a Polish language of T, K denotes a non trivial Polish language, and E denotes a Polish arity-function of K.

Let F be a function. We say that F is Polish-arity-like if and only if

(Def. 1) there exists T such that F is a Polish arity-function of T.

Let us consider T. Note that every Polish arity-function of T is Polish-arity-like.

Let F be a function. Let us observe that F is Polish-arity-like if and only if the condition (Def. 2) is satisfied.

(Def. 2) $\operatorname{dom} F$ is a Polish language and F is natural-valued and has zero.

Note that every function which is Polish-arity-like is also natural-valued and has also zero and there exists a function which is Polish-arity-like.

A Polish-arity-function is a Polish-arity-like function. From now on B denotes a Polish-arity-function.

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Let B, C be Polish-arity-functions. We say that C is B-extending if and only if

(Def. 3) $B \subseteq C$.

Let B be a Polish-arity-function. Note that the functor $\operatorname{dom} B$ yields a Polish language. Now we state the proposition:

(1) B is a Polish arity-function of T if and only if $T = \operatorname{dom} B$.

In the sequel A denotes a Polish arity-function of T.

Let B be a Polish-arity-function. The functor $\boxed{\text{Polish-WFF-set}(B)}$ yielding a Polish language of $\operatorname{dom} B$ is defined by

(Def. 4) there exists T and there exists A such that $A = B$ and $it = \text{Polish-WFF-set}(T, A)$.

Let us consider T and A. Let us note that A is empty yielding if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every a such that $a \in T$ holds $A(a) = 0$.

Let us consider P. Observe that P is A-closed if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every p, n, and q such that $p \in T$ and $n = A(p)$ and $q \in P \frown n$ holds $p \frown q \in P$.

Note that there exists a Polish arity-function of T which is empty yielding.

Let us consider V. We say that V is full if and only if

(Def. 7) there exists A such that $V = \text{Polish-WFF-set}(T, A)$.

Let us observe that there exists a Polish language of T which is full.

Let us consider A. Let us observe that the functor Polish-WFF-set(T, A) yields a full Polish language of T. Let T, U be Polish languages. We say that U is T-extending if and only if

(Def. 8) $T \subseteq U$.

Let us consider T. One can verify that there exists a Polish language which is T-extending.

Let us consider K. One can check that every Polish language which is K-extending is also non trivial.

Let us consider B and S. We say that S is B-compatible if and only if

(Def. 9) for every s such that $s \in S$ and s is $(\operatorname{dom} B)$-headed there exists n such that $n = B(\operatorname{dom} B\text{-head}(s))$ and $\operatorname{dom} B\text{-tail}(s) \in S \frown n$.

Let us consider T and A. Let us observe that S is A-compatible if and only if the condition (Def. 10) is satisfied.

(Def. 10) for every s such that $s \in S$ and s is T-headed holds $T\text{-tail}(s) \in S \frown A(T\text{-head}(s))$.

Let us observe that Polish-WFF-set(T, A) is A-compatible and there exists a Polish language which is A-closed and A-compatible.

Let us consider B. Let us note that there exists a Polish-arity-function which is B-extending and there exists a Polish language which is B-closed and B-compatible.

A Polish-ext-set of B is a B-closed, B-compatible Polish language.

An extension of B is a B-extending Polish-arity-function. From now on C denotes an extension of B.

Let us consider B and C. Let us note that every Polish-arity-function which is C-extending is also B-extending and every Polish language which is C-closed is also B-closed and every Polish language which is C-compatible is also B-compatible.

In the sequel Z denotes a B-closed Polish language.

Now we state the propositions:

(2) If $p \in \operatorname{dom} B$ and $B(p) = 1$ and $q \in Z$, then $p \frown q \in Z$.

(3) If $p \in \operatorname{dom} B$ and $B(p) = 2$ and q, $r \in Z$, then $p \frown (q \frown r) \in Z$.

Let us consider B, Z, and p. Assume $p \in \operatorname{dom} B$ and $B(p) = 1$. The functor Polish-unOp(B, Z, p) yielding a unary operation on Z is defined by

(Def. 11) for every q such that $q \in Z$ holds $it(q) = p \frown q$.

Assume $p \in \operatorname{dom} B$ and $B(p) = 2$. The functor Polish-binOp(B, Z, p) yielding a binary operation on Z is defined by

(Def. 12) for every q and r such that q, $r \in Z$ holds $it(q, r) = p \frown (q \frown r)$.

From now on J denotes a Polish-ext-set of B.

Let us consider B and J. The functor Polish-ext-complement(B, J) yielding a subset of J is defined by the term

(Def. 13) $\{p$, where p is an element of $J : p$ is not $(\operatorname{dom} B)$-headed$\}$.

The functor Polish-ext-domain(B, J) yielding a $(\operatorname{dom} B)$-extending Polish language is defined by the term

(Def. 14) $\operatorname{dom} B \cup$ Polish-ext-complement(B, J).

Now we state the proposition:

(4) Polish-WFF-set$(B) \subseteq J$. The theorem is a consequence of (1).

Let us consider P and n. The functor $P \restriction n$ yielding a subset of P is defined by the term

(Def. 15) $\{p$, where p is an element of $P : \operatorname{len} p \leqslant n\}$.

Now we state the proposition:

(5) T is a full Polish language of T.

In the sequel V denotes a full Polish language of T, U denotes a T-extending Polish language, and W denotes a full Polish language of U.

Let us consider T and V. The Polish arity V yielding a Polish arity-function of T is defined by

(Def. 16) $V = \text{Polish-WFF-set}(T, it)$.

Let us consider A. One can check that the Polish arity Polish-WFF-set(T, A) reduces to A.

Let us consider V. Note that Polish-WFF-set$(T, \text{the Polish arity } V)$ reduces to V.

Let us consider B and J. The functor Polish-ext-arity(B, J) yielding a Polish arity-function of Polish-ext-domain(B, J) is defined by

(Def. 17) $J = \text{Polish-WFF-set}(\text{Polish-ext-domain}(B, J), it)$.

Let us consider T.

A formula of T is an element of T. Let us consider B and J. Let F be a formula of J. The functor Polish-ext-head(F) yielding an element of Polish-ext-domain(B, J) is defined by the term

(Def. 18) Polish-ext-domain(B, J)-head(F).

Now we state the proposition:

(6) If S-head$(p) \in T$, then p is T-headed and S-head$(p) = T$-head(p).

Let us consider B, J, and a formula F of J. Now we state the propositions:

(7) If Polish-ext-head$(F) \in \operatorname{dom} B$, then Polish-ext-head$(F) = \operatorname{dom} B$-head$(F)$.

(8) If F is $(\operatorname{dom} B)$-headed, then Polish-ext-head$(F) = \operatorname{dom} B$-head$(F)$.

From now on M denotes a Polish-ext-set of C, e denotes an element of $\operatorname{dom} C$, and F, G, H denote formulae of M.

Now we state the propositions:

(9) If $C(e) = 1$ and Polish-ext-head$(F) = e$, then there exists G such that $F = (\text{Polish-unOp}(C, M, e))(G)$. The theorem is a consequence of (6).

(10) If $C(e) = 1$, then Polish-ext-head$((\text{Polish-unOp}(C, M, e))(G)) = e$. The theorem is a consequence of (8).

(11) If $C(e) = 2$ and Polish-ext-head$(F) = e$, then there exists G and there exists H such that $F = (\text{Polish-binOp}(C, M, e))(G, H)$. The theorem is a consequence of (6).

(12) If $C(e) = 2$, then Polish-ext-head$((\text{Polish-binOp}(C, M, e))(G, H)) = e$. The theorem is a consequence of (8).

Let us consider T, U, V, and W. Now we state the propositions:

(13) the Polish arity $V \subseteq$ the Polish arity W if and only if $V \subseteq W$.

PROOF: If the Polish arity $V \subseteq$ the Polish arity W, then $V \subseteq W$ by [1, (2)], [2, (22), (23), (17)]. Set $A =$ the Polish arity V. Set $B =$ the Polish arity W. For every a such that $a \in \operatorname{dom} A$ holds $A(a) = B(a)$ by [2, (32), (17), (43)]. □

(14) $V \subseteq W$ if and only if W is (the Polish arity V)-closed.

PROOF: Set $A =$ the Polish arity V. If $V \subseteq W$, then W is A-closed by (13), [1, (2)]. □

Let us consider T, U, V, and W. Let us observe that W is V-extending if and only if the condition (Def. 19) is satisfied.

(Def. 19) the Polish arity $V \subseteq$ the Polish arity W.

Let us note that there exists a full Polish language of U which is V-extending.

An extension of V is a Polish-ext-set of the Polish arity V. Observe that every extension of V is (the Polish arity V)-closed and (the Polish arity V)-compatible.

In the sequel Q denotes an extension of V.

Now we state the proposition:

(15) $V \subseteq Q$.

Let us consider T, V, and Q. The functor Polish-ext-complement(Q) yielding a subset of Q is defined by the term

(Def. 20) Polish-ext-complement(the Polish arity V, Q).

The functor Polish-ext-domain(Q) yielding a T-extending Polish language is defined by the term

(Def. 21) Polish-ext-domain(the Polish arity V, Q).

The functor Polish-ext-arity(Q) yielding a Polish arity-function of Polish-ext-domain(
is defined by the term

(Def. 22) Polish-ext-arity(the Polish arity V, Q).

Let F be a formula of Q. The functor Polish-ext-head(F) yielding an element of Polish-ext-domain(Q) is defined by the term

(Def. 23) Polish-ext-domain(Q)-head(F).

Let us consider T, V, Q, and a formula F of Q. Now we state the propositions:

(16) If Polish-ext-head(F) $\in T$, then Polish-ext-head(F) $= T$-head(F).

(17) If F is T-headed, then Polish-ext-head(F) $= T$-head(F).

From now on M denotes an extension of Polish-WFF-set(K, E), e denotes an element of K, and F, G, H denote formulae of M.

Let us consider T, V, and Q. Let t be an element of T. Assume (the Polish arity V)(t) $= 1$. The functor Polish-unOp(T, Q, t) yielding a unary operation on

Q is defined by

(Def. 24) for every p such that $p \in Q$ holds $it(p) = t \frown p$.

Assume (the Polish arity V)$(t) = 2$. The functor Polish-binOp(T, Q, t) yielding a binary operation on Q is defined by

(Def. 25) for every p and q such that p, $q \in Q$ holds $it(p, q) = t \frown (p \frown q)$.

Now we state the propositions:

(18) If $E(e) = 1$ and Polish-ext-head$(F) = e$, then there exists G such that $F =$ (Polish-unOp(K, M, e))(G). The theorem is a consequence of (6).

(19) If $E(e) = 1$, then Polish-ext-head((Polish-unOp(K, M, e))(G)) $= e$. The theorem is a consequence of (17).

(20) If $E(e) = 2$ and Polish-ext-head$(F) = e$, then there exists G and there exists H such that $F =$ (Polish-binOp(K, M, e))(G, H). The theorem is a consequence of (6).

(21) If $E(e) = 2$, then Polish-ext-head((Polish-binOp(K, M, e))(G, H)) $= e$. The theorem is a consequence of (17).

References

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