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A Formal Proof of Stirling's Formula

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Summary. In this article, we formalized the proof of the Stirling's formula, which is considered an essential item in the field of statistics, as shown below:

$$\lim_{n\to\infty}\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}=\sqrt{2\pi}$$

using the Mizar formalism [3]. We referred to [9, 6].

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Introduction

The formalization of the proof of this Stirling's formula was based on the proof by Prof. S. Kurokawa [9], which is constructed using elementary concepts. The proof is divided into two parts.

In the first part, we formalized the proof of the following lemma which is essential to compute the integral using the Riemann sum over an equal partition.

Lemma 1(`STIRLIN1:11`) Let $f(x)$ be a C^1 function on $[0,1]$ (i.e., $f'(x)$ exists and is continuous). Then the following holds:

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-n\int_0^1 f(x)\,dx\right\}=\frac{f(1)-f(0)}{2}.$$

For this lemma, we defined that the function defined on the open half-interval $]-1,\infty[$ is of class C^1 as `STIRLIN1:def 5`.

In the second part, we apply the lemma to $\log(1+x)$, and obtained the result:

$$\sum_{k=1}^{n}\left\{\log\left(1+\frac{k}{n}\right)-n\int_0^1 \log(1+x)\,dx\right\}=\log\left(\frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n\right).$$

From Lemma 1, the left-hand side limit is evaluated as $\frac{\log 2}{2}$, and thus:

$$\lim_{n\to\infty}\frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n=\sqrt{2}. \qquad \texttt{(STIRLIN1:19,22)}.$$

Considering the ratio between $n!$ and $n^{n+\frac{1}{2}}e^{-n}$, then the ratio can be transformed as follows:

$$\frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}=\frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n\Bigg/\frac{\sqrt{n}(2n)!}{4^n(n!)^2} \qquad \texttt{(STIRLIN1:25)}.$$

The limit left hand side can be calculated, and the denominator on the right-hand side equals the square root of the Wallis product sequence. It is known that this limit equals $\frac{1}{\sqrt{\pi}}$ (see WALLIS01:52), so the final limit is $\sqrt{2\pi}$.

1. Lemma on the Riemann Sum over an Equal Partition

Now we state the proposition:

(1) Let us consider a natural number n, and a natural number k. Suppose $k\in\mathrm{Seg}(n+1)$. Then $\mathrm{divset}(\mathrm{EqDiv}([0,1],n+1),k)=[\frac{k-'1}{n+1},\frac{k}{n+1}]$.

The functor $\boxed{\mathrm{D[01]}}$ yielding a sequence of divs$[0,1]$ is defined by

(Def. 1) for every element n of $\mathbb{N}$, $\mathit{it}(n)=\mathrm{EqDiv}([0,1],n+1)$.

Now we state the propositions:

(2) Let us consider a natural number n. Then $\delta_{\mathrm{EqDiv}([0,1],n+1)}=\frac{1}{n+1}$.
PROOF: Set $A=[0,1]$. Set $D=\mathrm{EqDiv}([0,1],n+1)$. For every natural number i such that $i\in\mathrm{dom}\,D$ holds $(\mathrm{upper\_volume}(\chi_{A,A},D))(i)=\frac{1}{n+1}$ by [8, (20)], [7, (15)]. $\mathrm{rng}\,\mathrm{upper\_volume}(\chi_{A,A},D)=\{\frac{1}{n+1}\}$ by [12, (29)], [13, (104)], [4, (3)], [5, (31)]. □

(3) $\delta_{\mathrm{D[01]}}$ is a 0-convergent, non-zero sequence of real numbers. The theorem is a consequence of (2).

Let a, b be real numbers. The functor $\boxed{\mathrm{ax\_+\_b}(a,b)}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by the term

(Def. 2) $\mathrm{AffineMap}(a,b)$.

Let r be a real number. The functor $\boxed{\mathrm{const}\,r}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by the term

(Def. 3) $\mathrm{AffineMap}(0, r)$.

Observe that $\mathrm{const}\, r$ is constant.

The functor $\boxed{\mathrm{rohl}}$ yielding a subset of $\mathbb{R}$ is defined by the term

(Def. 4) $]-1, +\infty[$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a non empty, closed interval subset of $\mathbb{R}$. We say that $\boxed{f \text{ is } \mathrm{C}^1\text{-Class on } A}$ if and only if

(Def. 5) f is differentiable on $]-1, +\infty[$ and $f'_{\restriction]-1,+\infty[}$ is continuous and $A \subseteq\ ]-1, +\infty[\ \subseteq \mathrm{dom}\, f$.

Assume f is C^1-Class on A. The functor $\boxed{@'(f, A)}$ yielding a function from A into $\mathbb{R}$ is defined by the term

(Def. 6) $f'_{\restriction]-1,+\infty[} \restriction A$.

Assume $A \subseteq \mathrm{dom}\, f$. The functor ${}^{@}(f, A)$ yielding a function from A into $\mathbb{R}$ is defined by the term

(Def. 7) $f \restriction A$.

From now on Z denotes an open subset of $\mathbb{R}$.

Now we state the proposition:

(4) Let us consider a natural number n, a natural number k, a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $k \in \mathrm{Seg}(n + 1)$ and $x_0 \in [\frac{k -' 1}{n+1}, \frac{k}{n+1}]$ and f is C^1-Class on $[0, 1]$. Then $\displaystyle\int\limits_{[x_0, \frac{k}{n+1}]} f'_{\restriction \mathrm{rohl}}(x)dx = f(\sup[x_0, \frac{k}{n+1}]) - f(\inf[x_0, \frac{k}{n+1}])$.

Let n be a natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\boxed{\mathrm{UPBND\_rng\_d}(f, n)}$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 8) $\mathrm{len}\, it = n + 1$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \sup \mathrm{rng}(f'_{\restriction \mathrm{rohl}} \restriction [\frac{i -' 1}{n+1}, \frac{i}{n+1}])$.

The functor $\boxed{\mathrm{LWBND\_rng\_d}(f, n)}$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 9) $\mathrm{len}\, it = n + 1$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \inf \mathrm{rng}(f'_{\restriction \mathrm{rohl}} \restriction [\frac{i -' 1}{n+1}, \frac{i}{n+1}])$.

The functor $\boxed{\mathrm{Sum\_UPBND\_rng\_d}\, f}$ yielding a sequence of $\mathbb{R}$ is defined by

(Def. 10) for every natural number i, $it(i) = \frac{\sum \mathrm{UPBND\_rng\_d}(f,i)}{i+1}$.

The functor $\boxed{\mathrm{Sum\_LWBND\_rng\_d}\, f}$ yielding a sequence of $\mathbb{R}$ is defined by

(Def. 11) for every natural number i, $it(i) = \frac{\sum \mathrm{LWBND\_rng\_d}(f,i)}{i+1}$.

Let us consider a natural number n, a natural number k, a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(5) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is C^1-Class on $[0,1]$. Then

(i) $\int\limits_{x_0}^{\frac{k}{n+1}} \mathrm{const}(\mathrm{UPBND\_rng\_d}(f,n))(k)(x)dx = (\mathrm{UPBND\_rng\_d}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$, and

(ii) $\int\limits_{x_0}^{\frac{k}{n+1}} \mathrm{const}(\mathrm{LWBND\_rng\_d}(f,n))(k)(x)dx = (\mathrm{LWBND\_rng\_d}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$.

Proof: For every real number t such that $t \in [x_0, \frac{k}{n+1}]$ holds $(\mathrm{const}(\mathrm{UPBND\_rng\_d}(f$ $(\mathrm{UPBND\_rng\_d}(f,n))(k)$ by [4, (3)]. For every real number t such that $t \in [x_0, \frac{k}{n+1}]$ holds $(\mathrm{const}(\mathrm{LWBND\_rng\_d}(f,n))(k))(t) = (\mathrm{LWBND\_rng\_d}(f,n))(k)$ by [4, (3)]. $\square$

(6) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is C^1-Class on $[0,1]$. Then $\int\limits_{x_0}^{\frac{k}{n+1}} f'_{\restriction \mathrm{rohl}}(x)dx \leqslant \int\limits_{x_0}^{\frac{k}{n+1}} \mathrm{const}(\mathrm{UPBND\_rng\_d}(f,n))(k)(x)dx$.

(7) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is C^1-Class on $[0,1]$. Then $\int\limits_{x_0}^{\frac{k}{n+1}} \mathrm{const}(\mathrm{LWBND\_rng\_d}(f,n))(k)(x)dx \leqslant \int\limits_{x_0}^{\frac{k}{n+1}} f'_{\restriction \mathrm{rohl}}(x)dx$.

(8) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is C^1-Class on $[0,1]$. Then

(i) $f(\frac{k}{n+1}) - f(x_0) \leqslant (\mathrm{UPBND\_rng\_d}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$, and

(ii) $(\mathrm{LWBND\_rng\_d}(f,n))(k) \cdot (\frac{k}{n+1} - x_0) \leqslant f(\frac{k}{n+1}) - f(x_0)$.

The theorem is a consequence of (4), (5), (7), and (6).

Let n be a natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $'\mathrm{F}'(f,n)$ yielding a finite sequence of elements of $\mathbb{R}_\mathrm{F}$ is defined by

(Def. 12) dom $it = \mathrm{Seg}(n+1)$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \int\limits_{\frac{i-'1}{n+1}}^{\frac{i}{n+1}} \mathrm{const}\, f(\frac{i}{n+1})(x)dx$.

The functor $\boxed{'\mathrm{G}'(f,n)}$ yielding a finite sequence of elements of $\mathbb{R}_\mathrm{F}$ is defined by

(Def. 13) $\operatorname{dom} it = \operatorname{Seg}(n+1)$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = \displaystyle\int\limits_{\frac{i-'1}{n+1}}^{\frac{i}{n+1}} (-f)(x)dx$.

The functor $\boxed{\mathrm{step\_}(f,n)}$ yielding a finite sequence of elements of $\mathbb{R}_\mathrm{F}$ is defined by

(Def. 14) $\operatorname{dom} it = \operatorname{Seg}(n+1)$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = f(\frac{i}{n+1})$.

Now we state the proposition:

(9) Let us consider a natural number n, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Then $\displaystyle\int\limits_{0}^{0} (-f)(x)dx = 0$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\boxed{\mathrm{Sum\_step}\, f}$ yielding a sequence of real numbers is defined by

(Def. 15) for every natural number n, $it(n) = \sum \mathrm{step\_}(f,n)$.

The functor $\boxed{\mathrm{n\_Integral}\, f}$ yielding a sequence of real numbers is defined by

(Def. 16) for every natural number n, $it(n) = (n+1) \cdot (\displaystyle\int\limits_{0}^{1} (-f)(x)dx)$.

The functor $\boxed{\mathrm{w\_seq}\, f}$ yielding a sequence of real numbers is defined by the term

(Def. 17) $\mathrm{Sum\_step}\, f + \mathrm{n\_Integral}\, f$.

Now we state the proposition:

(10) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f$is C^1-Class on $[0,1]$. Then $\lim \mathrm{w\_seq}\, f = \frac{1}{2} \cdot (\mathrm{integral}\, f'_{\restriction \mathrm{rohl}} \restriction [0,1])$.

2. Apply the Lemma to log(1+x)

Let c be a real number. The functor $\boxed{\mathrm{x\_} + \mathrm{\_}c}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by the term

(Def. 18) $\mathrm{AffineMap}(1,c)$.

Now we state the propositions:

(11) (The function ln) $\cdot$ (x_ + _1$\restriction$rohl) is differentiable on $]-1, +\infty[$.

Proof: Set $Z =]-1, +\infty[$. For every real number x_0 such that $x_0 \in Z$ holds (the function ln) $\cdot$ (x_ + _1$\restriction$rohl) is differentiable in x_0 by [10, (20)]. $\square$

(12) Let us consider an open subset Z of $\mathbb{R}$. Suppose $Z \subseteq$ dom((the function ln) $\cdot$ (x_ + _1$\restriction$rohl)). Then

(i) (the function ln) $\cdot$ (x_ + _1$\restriction$rohl) is differentiable on Z, and

(ii) for every real number x such that $x \in Z$ holds ((the function ln) $\cdot$ (x_ + _1$\restriction$rohl))$'_{\restriction Z}(x) = \frac{1}{1+x}$.

Proof: Set $f =$ x_ + _1$\restriction$rohl. For every real number x such that $x \in Z$ holds $f(x) = 1 + x$ and $f(x) > 0$ by [4, (49)]. $\square$

(13) ((The function ln) $\cdot$ (x_ + _1$\restriction$rohl))$'_{\restriction \mathrm{rohl}} = \frac{1}{\text{x_+_1}}\restriction$rohl. The theorem is a consequence of (12).

In the sequel x, x_0, x_1, x_2 denote real numbers.

Now we state the propositions:

(14) (x_ + _1$\cdot$(the function ln)$\cdot$(x_ + _1$\restriction$rohl)$+-1\cdot$ x_ + _0)$'_{\restriction \mathrm{rohl}}$ = (the function ln) $\cdot$ (x_ + _1$\restriction$rohl).

Proof: Set $f_1 =$ x_ + _1. Set $f_2 =$ (the function ln) $\cdot$ (x_ + _1$\restriction$rohl). f_2 is differentiable on rohl. $f_1 \cdot (\frac{1}{f_1}\restriction$rohl) = rohl $\longmapsto 1$ by [4, (49)], [11, (7)]. (rohl $\longmapsto 1) \cdot f_2 +$ (rohl $\longmapsto 1) + (-1) \cdot$ (rohl $\longmapsto 1) = f_2$ by [11, (7)]. $\square$

(15) $\int\limits_0^1$(the function ln) $\cdot$ (x_ + _1$\restriction$rohl)$(x)dx = \log_e \frac{4}{e}$. The theorem is a consequence of (12) and (14).

(16) $\int\limits_{[0,1]}$((the function ln) $\cdot$ (x_ + _1$\restriction$rohl))$'_{\restriction \mathrm{rohl}}(x)dx$ = (the function ln)(2).

(17) lim w_seq(the function ln) $\cdot$ (x_ + _1$\restriction$rohl) $= \frac{1}{2} \cdot$ (the function ln)(2). The theorem is a consequence of (10) and (16).

Let m be a non zero natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor Step_(f, m) yielding a sequence of real numbers is defined by

(Def. 19) for every natural number i, $it(i) = f(\frac{i}{m})$.

Let us consider a non zero natural number m and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(18) XFS2FS(Step_$(f, m)\restriction\mathbb{Z}_{m+1}) = \langle f(0)\rangle ^\frown$ step_$(f, m -' 1)$.

Proof: Reconsider $m_1 = m + 1$ as a natural number. Reconsider $\gtrdot_1 = m -' 1$ as a natural number. For every natural number x such that $x \in$ dom(Shift(Step_$(f, m)\restriction\mathbb{Z}_{m_1}, 1)$) holds (Shift(Step_$(f, m)\restriction\mathbb{Z}_{m_1}, 1))(x) = (\langle f(0)\rangle ^\frown$ step_$(f, \gtrdot_1))(x)$ by [2, (1)], [1, (44)], [4, (49)], [1, (14)]. $\square$

(19) Suppose f is C^1-Class on $[0,1]$. Then $\sum \mathrm{XFS2FS}(\mathrm{Step\_}(f,m){\restriction}\mathbb{Z}_{m+1}) = f(0) + \sum \mathrm{step\_}(f, m -' 1)$. The theorem is a consequence of (18).

Let us consider a non zero natural number m. Now we state the propositions:

(20) $\sum \mathrm{XFS2FS}(\mathrm{Step\_}((\text{the function ln}) \cdot (\mathrm{x\_} + \_1{\restriction}\,\mathrm{rohl}), m){\restriction}\mathbb{Z}_{m+1}) = \sum \mathrm{step\_}((\text{the funct}$ $\mathrm{ln}) \cdot (\mathrm{x\_} + \_1{\restriction}\,\mathrm{rohl}), m -' 1)$. The theorem is a consequence of (19).

(21) $(\mathrm{w\_seq}(\text{the function ln}) \cdot (\mathrm{x\_} + \_1{\restriction}\,\mathrm{rohl}))(m -' 1) = (\text{the function ln})(\frac{2 \cdot m!}{m!} \cdot (\frac{e}{4 \cdot m} \frown m))$. The theorem is a consequence of (15).

The functor $\boxed{\mathrm{W\_STIRL}}$ yielding a sequence of real numbers is defined by

(Def. 20) for every natural number n, $it(n) = \frac{2 \cdot (n+1)!}{(n+1)!} \cdot (\frac{e}{4 \cdot (n+1)} \frown (n+1))$.

Now we state the propositions:

(22) Let us consider a natural number n. Then $(\mathrm{w\_seq}(\text{the function ln}) \cdot (\mathrm{x\_} + \_1{\restriction}\,\mathrm{rohl}))(n) = (\text{the function ln})((\mathrm{W\_STIRL})(n))$. The theorem is a consequence of (15).

(23) (i) $\lim \mathrm{W\_STIRL} = \sqrt{2}$, and

(ii) W_STIRL is convergent.

The theorem is a consequence of (17).

The functor $\boxed{\mathrm{StirlSeq}}$ yielding a sequence of real numbers is defined by

(Def. 21) for every natural number n, $it(n) = \frac{n!}{(n \frown (n+\frac{1}{2})) \cdot (e \frown (-n))}$.

Now we state the proposition:

(24) $\lim \mathrm{StirlSeq} = \sqrt{2 \cdot \pi}$. The theorem is a consequence of (23).

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